# Infinitely many solutions for Variable-order fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-Choquard equations with Hardy nonlinearity in $\mathbb{R}^N$

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**Abstract** In this paper, we discuss a class of fractional Schrödinger-Choquard equations, which involve the variable-order fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian and Hardy nonlinearity. The main innovation of this paper is the use of weighted Lebesgue spaces to overcome the difficulty with the compact embedding result for variable exponents and variable-order fractional Sobolev spaces in $\mathbb{R}^N$. In addition, the existence of infinitely many solutions for the problem are derived by utilizing the three different critical point theorems. Here the nonlinearity $h(x,\omega)$ does not satisfy the classical Ambrosetti-Rabinowitz condition.

**Keywords:** $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplace operators, Variable-order fractional, Schrödinger-Choquard equations, Hardy nonlinearity, Variational methods

**Mathematics Subject Classification (2020):** 35J50; 35R11; 35D30; 46E35.

## 1. Introduction

In the past several decades, fractional differential equations have received great attention. Fractional order differential equations are the extension of the integer order differential equations, which greatly enrich the content of differential equations. There are many kinds of fractional differential equations, the fractional Schrödinger equation is an important representative.

The classical Schrödinger equation is in the following form

$$i\hbar\frac{\partial}{\partial t}\varphi = -\frac{\hbar^2}{2m}\nabla^2\varphi + V\varphi,$$

where $V$, $\varphi$ denote the potential function and wave function, respectively, $i$, $\hbar$ are constants. The original fractional Schrödinger equation was discovered by Laskin when expanding the Feynman path integral, see [1, 2]. Laskin proposed the following model

$$i\frac{\partial}{\partial t}\phi(x,t) = (-\Delta)^{\alpha}\phi + V(x)\phi - f(x,t),\ (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where $(-\Delta)^{\alpha}$ is the fractional Laplace operator, $\alpha\in(0,1)$. Since then, several forms of the Schrödinger equation have been created, and a lot of research work appeared. Many scholars investigated the existence and multiplicity of solutions to the fractional Schrödinger equation by using the variational method [6–31]. The problem studied in these articles contains three different types of operators.

The first class is the Laplace operator [6–11]. The existence of nontrivial radially symmetric solutions for a fractional Schrödinger equation with critical nonlinear terms were studied by Zhang et al. [7]. Especially, when $s = s(\cdot)$, the Laplace operator is transformed into the variable order Laplace operator. In [8], Xiang et al. are concerned with the following equation

$$\begin{cases} (-\Delta)^{s(\cdot)}\omega + \lambda V(x)\omega = \alpha|\omega|^{p(x)-2}\omega + \beta|\omega|^{q(x)-2}\omega,\ x\in\Omega, \\ \omega = 0,\ x\in\partial\Omega, \end{cases}$$

and they proved an embedding theorem of variable-order fractional Sobolev space for the first time. With the aid of the mountain pass theorem and Ekeland's variational principle, they showed the existence of at least two distinct solutions. We also refer to [9] for related problems.

The second class is the $p$-Laplace operator [12–22]. Pucci et al. [14] investigated the following nonhomogeneous Schrödinger-Kirchhoff type problem involving the perturbation term

$$M\left(\int_{\mathbb{R}^{2N}}\frac{|\omega(x)-\omega(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s\omega + V(x)|\omega|^{p-2}\omega = f(x,\omega)+g(x),\ x\in\mathbb{R}^N.$$

They firstly established the compact embedding theorem in the whole space $\mathbb{R}^N$, which can be applied to many fractional Schrödinger with $p$-Laplacian in $\mathbb{R}^N$. Particularly, in [17] the author obtained the multiplicity result for a class of fractional

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$(p,q)$-Laplacian problem in $\mathbb{R}^N$. Moreover, the study of Schrödinger equations has already been extended to the case of the variable-order Laplace operator [18]. So far, there are only a few results involving the Hardy nonlinearity, we refer to the recent papers of existence of multiple solutions [19] using the theory of genus and [20] using the Nehari manifold approach.

The third class is the $p(\cdot)$-Laplace operator [23–31]. It's more complex than the $p$-Laplace operator, since the $p(\cdot)$-Laplace operator is not homogeneous and has no first eigenvalue. For nonlocal Choquard type equations, Biswas and Tiwari [28] gave the existence result by employing the critical point theorem. Additionally, in [29] they also considered the following Kirchhoff-Choquard type equation

$$\begin{cases} m\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega V(x)\frac{|\omega(x)|^{\overline{p}(x)}}{\overline{p}(x)}\right)\left[(-\Delta)^{s(\cdot)}_{p(\cdot)}\omega+V(x)|\omega|^{\overline{p}(x)-2}\omega\right] \\ =\left(\int_\Omega\frac{H(y,\omega(y))}{|x-y|^{\mu(x,y)}}dy\right)h(x,\omega),\ x\in\Omega, \\ \omega=0,\ x\in\mathbb{R}^N\backslash\Omega, \end{cases}$$

where $m$ is a Kirchhoff type function, $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is the variable-order fractional $p(\cdot)$-Laplace operator. Under some weaker assumptions on $h$ compared to that of [28], they proved the existence of ground solution and infinitely many solutions. We also encourage interested readers to refer to results about fractional $p(\cdot)$-Laplace operator problems [26, 31].

At present, the double operators problem is one of the active topics, but there are few researches on this kind of problem [17, 32–34]. As far as we know, there is no work devoted to the study of variable-order fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger equations in $\mathbb{R}^N$. Enlightened by the above literature, we discuss the following Schrödinger-Choquard type equation

$$\sum_{i=1}^{2}\left[(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}\omega+V(x)|\omega|^{\overline{p_i}(x)-2}\omega\right]=\frac{\xi|\omega|^{q(x)-2}\omega}{|x|^{a(x)}}+\left(\int_{\mathbb{R}^N}\frac{H(y,\omega(y))}{|x-y|^{\phi(x,y)}}dy\right)h(x,\omega(x)),\ x\in\mathbb{R}^N, \qquad (H_\xi)$$

where $p_i(x,\cdot)$, $s(x,\cdot)$, $\phi(x,y)$, $q(x)$ and $a(x)$ are continuous functions with $p_i(x,y)s(x,y)<N$ for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ and $0\le a(x)<N$. $V\in C(\mathbb{R}^N,\mathbb{R}^+)$ is the potential function, $\xi>0$ is a parameter and $h\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ is a Carathéodory function with $H(x,\omega)=\int_0^\omega h(x,s)ds$. With the help of the symmetric mountain pass theorem, dual fountain theorem and Krasnoselskii's genus theory, we obtain the existence of infinitely many solutions. The operator $(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}$ is the variable-order fractional $p_i(x,\cdot)$-Laplace operator defined on $C_0^\infty(\mathbb{R}^N)$ by

$$(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}\omega(x):=P.V.\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{p_i(x,y)-2}(\omega(x)-\omega(y))}{|x-y|^{N+p_i(x,y)s(x,y)}}dy,\ i=1,2,\ x\in\mathbb{R}^N,$$

where $P.V.$ stands for the Cauchy principal value. We first introduce some notations. For any real valued function $r$ defined on domain $\Theta$, denote

$$r^-:=\min_{x\in\Theta}r(x),\qquad r^+:=\max_{x\in\Theta}r(x).$$

Define

$$C_+(\Theta):=\{r(x):\ r(x)\in C(\Theta,\mathbb{R}),\ 1<r^-\le r\le r^+<\infty\}.$$

Through out this article, $p^*_{s(\cdot)}=\frac{N\overline{p}(x)}{N-\overline{p}(x)\overline{s}(x)}$ denotes the critical exponent, where $\overline{p}(x)=p(x,x)$ and $\overline{s}(x)=s(x,x)$. We assume that $s(x,y)$, $p(x,y)$, $\phi(x,y)$, $q(x)$, $a(x)$ and $b(x)$ satisfy the following conditions

(A1): $p_i(x,y)$, $s(x,y)$ and $\phi(x,y)$ are symmetric, i.e., $p_i(x,y)=p_i(y,x)$, $s(x,y)=s(y,x)$ and $\phi(x,y)=\phi(y,x)$ for any $(x,y)\in\mathbb{R}^{2N}$, $p_{min}(x,y)=\min\{p_1(x,y),p_2(x,y)\}$ and $p_{max}(x,y)=\max\{p_1(x,y),p_2(x,y)\}$.

(A2): $0<\phi^-<\phi(x,y)<\phi^+<N$, $0<s^-<s(x,y)<s^+<1<p_i^-<p_i(x,y)<p_i^+<p^*_{s(\cdot)}$.

(A3): $a(x),q(x)\in C(\mathbb{R}^N)$, $q^+<p_i^-$ and $0\le a^-<a^+<N$.

(B1): $b(x)\in C(\mathbb{R}^N,\mathbb{R})$. For all $x\in\mathbb{R}^N$, $b(x)\ge0$ and $b(x)\ne0$.

(B2): $b(x)\in L^{\beta(x)}(\mathbb{R}^N)$ and $\beta\in C_+(\mathbb{R}^N)$ satisfies $b(x)\ge0$.

Our work is the first consideration for the existence of infinitely many solutions of the variable-order fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-Choquard equations. It is worth noting that the equation we consider is on the whole space $\mathbb{R}^N$, which is different from the work of [29, 34]. Compared to [26, 31], the double Laplacian operator we deal with is more complex. In addition, we discuss the problem involving Hardy nonlinearity, which is more general than [17, 32, 33], and we don't need the Ambrosetti-Rabinowitz condition for nonlinearity function $h$.

Throughout this paper, we consider problems $(H_\xi)$ under the following conditions for the potential function $V$ and the nonlinearity $h$

(V1): $V(x)\in C(\mathbb{R}^N)$ and there exists $V_0$ such that $\inf_{x\in\mathbb{R}^N}V(x)=V_0>0$.

(H1): $h(x,-\omega)=-h(x,\omega)$, for any $(x,\omega)\in\mathbb{R}^N\times\mathbb{R}$.

(H2): Let $\theta(x) \in C_+(\mathbb{R}^N)$ with $\theta^- > p^+_{max}$. Suppose that $b(x)$ satisfies (B2) such that

$$|h(x,\omega)| \le b(x)|\omega|^{\theta(x)-1}, \text{ for any } (x,\omega) \in \mathbb{R}^N \times \mathbb{R},$$

with $\overline{p_i}(x) < \theta(x)m^- < \theta(x)m^+ < p^*_{s(\cdot)}$ and $m$ satisfies

$$\frac{2}{m(x,y)} + \frac{\phi(x,y)}{N} = 2, \ (x,y) \in \mathbb{R}^{2N}.$$

(H3): There exists $\kappa > 0$, for any $x \in \mathbb{R}^N$ and $\omega \in (0,\kappa]$ satisfies

$$|h(x,\omega)| \ge b'(x)|\omega|^{\theta'(x)-1},$$

where $b'(x)$ satisfies (B1) and $\theta' \in C_+(\mathbb{R}^N)$ with $q^+ < 2\theta'^- < 2\theta'^+ < p^-_{min}$.

(H4): $h(x,\omega) = o(|\omega|^{\frac{1}{2}p^+_{max}-2}\omega)$ as $|\omega| \to 0$, uniformly in $x \in \mathbb{R}^N$.

(H5): $\lim_{|\omega|\to\infty} \frac{H(x,\omega)}{|\omega|^{\frac{1}{2}p^+_{max}}} = \infty$ uniformly in $x \in \mathbb{R}^N$.

(H6): There exists $\lambda \ge 1$ such that

$$\lambda\vartheta(x,\omega) \ge \vartheta(x,\tau\omega), \text{ for any } (x,\omega) \in \mathbb{R}^N \times \mathbb{R},$$

where $0 < \tau < 1$, and

$$\vartheta(x,\omega) = 2\omega h(x,\omega) - p^+_{max}H(x,\omega).$$

**Remark 1.1.** Compared to the well-known Ambrosetti-Rabinowitz condition, the assumption (H6) is weaker.

**Remark 1.2.** From (H4) and (H6), we obtain $H(x,\omega)$ is decreasing in $\omega \le 0$ and $H(x,\omega)$ is increasing in $\omega \ge 0$ for all $x \in \mathbb{R}^N$. Moreover, we have $H(x,\omega) \ge 0$ for all $x \in \mathbb{R}^N \times \mathbb{R}$. (see [29]).

The rest of this article reads as follows. In Sect.2, we collect some necessary definitions and basic lemmas of $L^{\mu(x)}(\mathbb{R}^N)$, $W^{\overline{p}(x),p(x,\cdot),s(x,\cdot)}(\mathbb{R}^N)$ and $L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$ spaces. In Sect.3 we state the main results, i.e. Theorem 3.1, Theorem 3.2 and Theorem 3.3. Sect.4 discusses the Cerami condition related to the functional $\Phi$. In Sects. 5, 6 and 7, we give the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3, respectively.

## 2. Preliminaries

We introduce the definitions, basic properties and embedding results of some important function spaces, which will be used later.

### 2.1. *The space* $L^{\mu(x)}(\mathbb{R}^N)$.

The variable exponent Lebesgue space is defined as

$$L^{\mu(x)}(\mathbb{R}^N) := \left\{\omega : \omega \text{ is a measurable and } \int_{\mathbb{R}^N} |\omega(x)|^{\mu(x)}dx < \infty\right\},$$

which is a reflexive uniformly convex and separable Banach space (see [23, 25]) with the Luxemburg norm

$$\|\omega\|_{\mu(x)} = \|\omega\|_{L^{\mu(x)}(\mathbb{R}^N)} := \inf\left\{\chi > 0 : \int_{\mathbb{R}^N} \left|\frac{\omega(x)}{\chi}\right|^{\mu(x)} dx \le 1\right\}.$$

Define the modular $\varrho : L^{\mu(x)}(\mathbb{R}^N) \to \mathbb{R}$ as $\varrho(\omega) := \int_{\mathbb{R}^N} |\omega|^{\mu(x)}dx$.

**Lemma 2.1.** ([23]) Suppose that $\omega_n, \omega \in L^{\mu(x)}(\Omega)$. Then the following properties hold

(i) $\chi = \|\omega\|_{\mu(x)}$ if and only if $\varrho(\frac{\omega}{\chi}) = 1$;

(ii) $\|\omega\|_{\mu(x)} > 1 \Rightarrow \|\omega\|^{\mu^-}_{\mu(x)} \le \varrho(\omega) \le \|\omega\|^{\mu^+}_{\mu(x)}$;

(iii) $\|\omega\|_{\mu(x)} < 1 \Rightarrow \|\omega\|^{\mu^+}_{\mu(x)} \le \varrho(\omega) \le \|\omega\|^{\mu^-}_{\mu(x)}$;

(iv) $\|\omega\|_{\mu(x)} < 1\ (= 1; > 1) \Leftrightarrow \varrho(\omega) < 1\ (= 1; > 1)$;

(v) $\lim_{n\to\infty} \|\omega_n - \omega\|_{\mu(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \varrho(\omega_n - \omega) = 0$.

**Lemma 2.2.** ([25]) The space $(L^{\mu'(x)}(\mathbb{R}^N), \|\omega\|_{\mu'(x)})$ is conjugate space of space $(L^{\mu(x)}(\mathbb{R}^N), \|\omega\|_{\mu(x)})$, where $\mu'(x)$ is the conjugate function of $\mu(x)$. Let

$$\frac{1}{\mu'(x)} + \frac{1}{\mu(x)} = 1, \ x \in \mathbb{R}^N,$$

the Hölder type inequality

$$\left|\int_{\mathbb{R}^N} \omega\nu dx\right| \le \left(\frac{1}{(\mu')^-} + \frac{1}{\mu^-}\right)\|\omega\|_{\mu(x)}\|\nu\|_{\mu'(x)} \le 2\|\omega\|_{\mu(x)}\|\nu\|_{\mu'(x)},$$

for all $\omega \in L^{\mu(x)}(\mathbb{R}^N), \nu \in L^{\mu'(x)}(\mathbb{R}^N)$ hold.

**Lemma 2.3.** ([34]) Assume that $\mu_2(x):\mathbb{R}^N\to\mathbb{R}$ be a measurable function. If $\mu_1(x)\in L^\infty(\mathbb{R}^N)$ satisfies $\mu_1\geq 0$, $\mu_1\not\equiv 0$ and $\mu_1\mu_2\geq 1$ a.e. in $\mathbb{R}^N$, then for all $\omega\in L^{\mu_1(\cdot)\mu_2(\cdot)}(\mathbb{R}^N)$, we have

$$\||\omega|^{\mu_1(\cdot)}\|_{L^{\mu_2(\cdot)}}\leq\|\omega\|^{\mu_1^-}_{L^{\mu_1(\cdot)\mu_2(\cdot)}(\mathbb{R}^N)}+\|\omega\|^{\mu_1^+}_{L^{\mu_1(\cdot)\mu_2(\cdot)}(\mathbb{R}^N)}.$$

**2.2.** ***The space*** $W^{\overline{p}(x),p(x,\cdot),s(x,\cdot)}(\mathbb{R}^N)$**.** The variable exponents and variable-order fractional Sobolev spaces is defined by

$$W=W^{\overline{p}(x),p(x,\cdot),s(x,\cdot)}(\mathbb{R}^N):=\left\{\omega\in L^{\overline{p}(x)}(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{p(x,y)}}{\chi^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy<\infty\text{ for some }\chi>0\right\},$$

endowed with the norm

$$|\omega|_W:=[\omega]_w+\|\omega\|_{\overline{p}(x)},$$

where

$$[\omega]_w:=\inf\left\{\chi>0:\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{p(x,y)}}{\chi^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy<1\right\}.$$

Define the variable-order fractional Sobolev linear subspace $E_i$ with potential function as follows

$$E_i=\left\{\omega:\omega\in W,\int_{\mathbb{R}^N}\frac{V(x)|\omega|^{\overline{p_i}(x)}}{\chi^{\overline{p_i}(x)}}dx<+\infty\text{ for some }\chi>0\right\},$$

on $E_i$ we use the following norm

$$\|\omega\|_{E_i}:=\inf\left\{\chi>0:\varrho_{E_i}\left(\frac{\omega}{\chi}\right)\leq 1\right\},\quad i=1,2,$$

where

$$\varrho_{E_i}(\omega):=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{p_i(x,y)}}{|x-y|^{N+p_i(x,y)s(x,y)}}dxdy+\int_{\mathbb{R}^N}V(x)|\omega(x)|^{\overline{p_i}(x)}dx,$$

is a modular on $E_i$. Then $(W,\|\cdot\|_W)$ and $(E_i,\|\cdot\|_{E_i})$ are the separable reflexive Banach spaces (see [27, 29]).

**Lemma 2.4.** ([27]) Suppose that $\omega_n,\omega\in E_i$. Then the following properties hold
(i) $\chi=\|\omega\|_{E_i}$ if and only if $\varrho_{E_i}(\frac{\omega}{\chi})=1$;
(ii) $\|\omega\|_{E_i}>1\Rightarrow\|\omega\|^{p_i^-}_{E_i}\leq\varrho_{E_i}(\omega)\leq\|\omega\|^{p_i^+}_{E_i}$;
(iii) $\|\omega\|_{E_i}<1\Rightarrow\|\omega\|^{p_i^+}_{E_i}\leq\varrho_{E_i}(\omega)\leq\|\omega\|^{p_i^-}_{E_i}$;
(iv) $\|\omega\|_{E_i}<1\ (=1;>1)\Leftrightarrow\varrho_{E_i}(\omega)<1\ (=1;>1)$;
(v) $\lim_{n\to\infty}\|\omega_n-\omega\|_{E_i}=0\Leftrightarrow\lim_{n\to\infty}\varrho_{E_i}(\omega_n-\omega)=0$.

Moreover, in order to study problems $(H_\xi)$, we consider the space $E=E_1\bigcap E_2$, endowed with the norm

$$\|\omega\|=\|\omega\|_E=\|\omega\|_{E_1}+\|\omega\|_{E_2}.$$

Obviously, the Banach space $(E,\|\cdot\|_E)$ is separable and reflexive, $E^*$ is the dual space of $E$. It is not difficult to obtain the following embedding theorem according to the above norm and Theorem 2.10 in ([28, 29]).

**Theorem 2.1.** Let $\Omega\in\mathbb{R}^N$ be a smooth bounded domain, $p(x,y)$ and $s(x,y)$ satisfying (A1) and (A2), respectively, with $p(x,y)s(x,y)<N$ for any $(x,y)\in\Omega\times\Omega$. Assume that (V1) holds and $\theta(x)\in C_+(\overline{\Omega})$ satisfies

$$1<\theta^-=\min_{x\in\overline{\Omega}}\theta(x)\leq\theta(x)<p^*_{s(\cdot)},$$

for all $x\in\overline{\Omega}$. Then, the space $E$ is continuous compact embedded in $L^{\theta(x)}(\Omega)$.

**Theorem 2.2.** Suppose that (A1)-(A2) and (V1) hold with $p(x,y)s(x,y)<N$ and let $\mu\in C_+(\overline{\Omega})$ such that $p^*_{s(\cdot)}>\mu(x)\geq\overline{p}(x)$ for any $x\in\mathbb{R}^N$. Then the embedding $E(\mathbb{R}^N)\hookrightarrow L^{\mu(x)}(\mathbb{R}^N)$ is continuous.

Note that the embedding $E(\mathbb{R}^N)\hookrightarrow L^{\mu(x)}(\mathbb{R}^N)$ is no longer compact. In order to overcome this difficulty, we introduce a new space.

**2.3.** ***The space*** $L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$**.** Assume that $b(x)$ satisfying (B1) and $\mu(x) \in C_+(\mathbb{R}^N)$, we define

$$L^{\mu(x)}_{b(x)}(\mathbb{R}^N) := \left\{\omega : \omega \text{ is a measurable and } \int_{\mathbb{R}^N} b(x)|\omega(x)|^{\mu(x)}dx < \infty\right\},$$

with the norm

$$\|\omega\|_{\mu(x),b(x)} = \|\omega\|_{L^{\mu(x)}_{b(x)}(\mathbb{R}^N)} := \inf\left\{\chi > 0 : \int_{\mathbb{R}^N} b(x)\left|\frac{\omega(x)}{\chi}\right|^{\mu(x)} dx \le 1\right\}.$$

Obviously, the semimodular $\varrho_{\mu(x),b(x)}(\omega) = \int_{\mathbb{R}^N} b(x)|\omega|^{\mu(x)}dx$. Moreover, the space $(L^{\mu(x)}_{b(x)}(\mathbb{R}^N), \|\omega\|_{\mu(x),b(x)})$ is a reflexive and separable Banach space (see [24]).

**Lemma 2.5.** ([24]) Suppose that $\omega_n, \omega \in L^{\mu(x)}_{b(x)}(\Omega)$. Then the following properties hold
(i) $\chi = \|\omega\|_{\mu(x),b(x)}$ if and only if $\varrho_{\mu(x),b(x)}(\frac{\omega}{\chi}) = 1$;
(ii) $\|\omega\|_{\mu(x),b(x)} > 1 \Rightarrow \|\omega\|^{\mu^-}_{\mu(x),b(x)} \le \varrho_{\mu(x),b(x)}(\omega) \le \|\omega\|^{\mu^+}_{\mu(x),b(x)}$;
(iii) $\|\omega\|_{\mu(x),b(x)} < 1 \Rightarrow \|\omega\|^{\mu^+}_{\mu(x),b(x)} \le \varrho_{\mu(x),b(x)}(\omega) \le \|\omega\|^{\mu^-}_{\mu(x),b(x)}$;
(iv) $\|\omega\|_{\mu(x),b(x)} < 1\ (= 1; > 1) \Leftrightarrow \varrho_{\mu(x),b(x)}(\omega) < 1\ (= 1; > 1)$);
(v) $\lim_{n\to\infty} \|\omega_n\|_{\mu(x),b(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \varrho_{\mu(x),b(x)}(\omega_n) = 0$.

We present the following embedding results.

**Theorem 2.3.** Suppose that (A1)-(A3) and (V1) hold. Let $\mu(x) \in C_+(\mathbb{R}^N)$ such that $1 < \mu^- < \mu^+ < p^*_{s(\cdot)}$ for any $x \in \mathbb{R}^N$. Let (B2) hold with $\beta(x)$ satisfying

$$\overline{p}(x) \le \eta(x) = \frac{\beta(x)\mu(x)}{\beta(x) - 1} \le p^*_{s(\cdot)}, \quad x \in \mathbb{R}^N.$$

Then, the embedding $E \hookrightarrow L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$ is continuous. Moreover, if $\eta^+ < p^*_{s(\cdot)}$ for any $x \in \mathbb{R}^N$, then the embedding $E \hookrightarrow L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$ is compact.

***Proof.*** By Theorem 2.2, we have that the embedding $E \hookrightarrow L^{\mu(x)}(\mathbb{R}^N)$ is continuous. Next, analogously to the proof of Lemma 2.4 in ([26]), we prove that $E_i \hookrightarrow L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$, so $E \hookrightarrow L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$, where the embedding is continuous and compact. □

Taking especially $b(x) = |x|^{-a(x)}$, we obtain a corollary of Theorem 2.3 as follows.

**Corollary 1.** Suppose that $p, a, \mu \in C(\mathbb{R}^N)$, $0 \le a(x) < N$ for $x \in \mathbb{R}^N$. If $\mu$ satisfies the condition

$$\overline{p}(x) \le \eta(x) = \frac{N\mu(x)}{N - a(x)} \le p^*_{s(\cdot)}, \quad x \in \mathbb{R}^N,$$

then the embedding $E(\mathbb{R}^N) \hookrightarrow L^{\mu(x)}_{|x|^{-a(x)}}(\mathbb{R}^N)$ is continuous and compact.

***Proof.*** For any $x \in \mathbb{R}^N$, we can find $\varepsilon > 0$ small enough such that

$$a(x) < N - \varepsilon, \quad \overline{p}(x) \le \eta(x) = \frac{(N-\varepsilon)\mu(x)}{N - \varepsilon - a(x)} \le p^*_{s(\cdot)}.$$

Applying Theorem 2.3 to the case that $b(x) = |x|^{-a(x)}$ and $\beta(x) = \frac{N-\varepsilon}{a(x)}$, we obtain the corollary. □

**Remark 2.1.** The $p^*(x) = \frac{p(x)(N-a(x))}{N-p(x)}$ is called the critical Sobolev Hardy exponent. In this paper, we only deal with the case involving subcritical Sobolev Hardy exponents.

## 3. Statement of the main theorems

For the sake of the following statement, we give some definitions and corresponding variational forms related to the problem $(H_\xi)$.

**Definition 3.1.** We say that $\omega \in E$ is a weak solution of the problem $(H_\xi)$, if

$$\sum_{i=1}^{2}\langle\Psi'_{p_i}(\omega), \psi\rangle = \int_{\mathbb{R}^N} \frac{\xi|\omega|^{q(x)-2}\omega\psi}{|x|^{a(x)}}dx + \int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))h(y,\omega(y))\psi(y)}{|x-y|^{\phi(x,y)}}dxdy, \tag{3.1}$$

for all $\psi \in E$, where

$$\Psi_{p_i}(\omega) = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{p_i(x,y)}}{p_i(x,y)|x-y|^{N+p_i(x,y)s(x,y)}}dxdy + \int_{\mathbb{R}^N} \frac{V(x)|\omega|^{\overline{p_i}(x)}}{\overline{p_i}(x)}dx,$$

and

$$\langle\Psi'_{p_i}(\omega), \psi\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{p_i(x,y)-2}(\omega(x)-\omega(y))(\psi(x)-\psi(y))}{|x-y|^{N+p_i(x,y)s(x,y)}}dxdy + \int_{\mathbb{R}^N} V(x)|\omega|^{\overline{p_i}(x)-2}\omega\psi dx.$$

The functional $\Phi : E \to \mathbb{R}$ associated with equations $(H_\xi)$ is defined by

$$\Phi(\omega) := \sum_{i=1}^{2} \Psi_{p_i}(\omega) - \int_{\mathbb{R}^N} \frac{\xi|\omega|^{q(x)}}{q(x)|x|^{a(x)}} dx - \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}} dxdy, \tag{3.2}$$

for all $\omega \in E$. Under our assumptions, the functional $\Phi : E \to \mathbb{R}$ is of class $C^1(E,\mathbb{R})$ and for all $\omega, \psi \in E$

$$\langle \Phi'(\omega), \psi\rangle := \sum_{i=1}^{2} \langle \Psi'_{p_i}(\omega), \psi\rangle - \int_{\mathbb{R}^N} \frac{\xi|\omega|^{q(x)-2}\omega\psi}{|x|^{a(x)}} dx - \int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))h(y,\omega(y))\psi(y)}{|x-y|^{\phi(x,y)}} dxdy. \tag{3.3}$$

Moreover, we can observe that $\omega \in E$ is a critical point of the functional $\Phi$ if and only if $\omega \in E$ is a weak solution of problems $(H_\xi)$.

**Lemma 3.1.** ([28]) Let (A3) hold and $m_1(x,y), m_2(x,y) \in C_+(\mathbb{R}^{2N})$ satisfy

$$\frac{1}{m_1(x,y)} + \frac{\phi(x,y)}{N} + \frac{1}{m_2(x,y)} = 2, \text{ for any } (x,y) \in \mathbb{R}^{2N}.$$

If $f \in L^{m_1^+}(\mathbb{R}^N) \bigcap L^{m_1^-}(\mathbb{R}^N)$ and $g \in L^{m_2^+}(\mathbb{R}^N) \bigcap L^{m_2^-}(\mathbb{R}^N)$, then

$$\left| \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(x)}{|x-y|^{\phi(x,y)}} dxdy \right| \le C_1 \left( \|f\|_{L^{m_1^+}(\mathbb{R}^N)} \|g\|_{L^{m_2^+}(\mathbb{R}^N)} + \|f\|_{L^{m_1^-}(\mathbb{R}^N)} \|g\|_{L^{m_2^-}(\mathbb{R}^N)} \right),$$

where $C_1$ is a positive constant , independent of $f$ and $g$.

**Corollary 2.** In particular, by taking $f(x) = g(x) = |\omega(x)|^{\theta(x)}$, $\omega \in W$ and $m_1(x,y) = m_2(x,y) = m(x,y)$, one has $\frac{2}{m(x,y)} + \frac{\phi(x,y)}{N} = 2, (x,y) \in \mathbb{R}^{2N}$ with

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\omega(x)|^{\theta(x)}|\omega(y)|^{\theta(y)}}{|x-y|^{\phi(x,y)}} dxdy \le C_2 \left( \||\omega|^{\theta(\cdot)}\|^2_{L^{m^+}(\mathbb{R}^N)} + \||\omega|^{\theta(\cdot)}\|^2_{L^{m^-}(\mathbb{R}^N)} \right),$$

where $m \in C_+(\mathbb{R}^N)$ and $p(x,x) \le \theta(x)m^- \le \theta(x)m^+ < p^*_{s(\cdot)}$. Furthermore, $C_2$ is a positive constant, independent of $\omega$.

**Theorem 3.1.** Suppose that (A1)-(A3), (V1), (H1)-(H2) and (H5) hold. Then, for any $\xi \in (0, \xi^*]$, equations $(H_\xi)$ has infinitely many large energy solutions.

**Theorem 3.2.** Suppose that (A1)-(A3), (V1) and (H1)-(H3) hold. Then, equations $(H_\xi)$ has infinitely many nontrivial solutions with negative energy converging to 0.

**Theorem 3.3.** Suppose that (A1)-(A3) and (V1) hold and $h$ satisfy (H1)-(H3). Then, equations $(H_\xi)$ possess infinitely many small negative energy solutions.

## 4. Cerami condition

The main task of this section is to verify the Cerami $(Ce)$ condition. As being known, the Cerami condition is weaker than the Palais-Smale compactness condition.

**Definition 4.1.** Let $E$ be a Banach space, $\Phi \in C^1(E,\mathbb{R})$. If any $(Ce)_c$ sequence $\{\omega_n\}_{n\in\mathbb{N}} \subset E$, namely

$$\Phi(\omega_n) \to c, \ (1 + \|\omega_n\|)\Phi'(\omega_n) \to 0 \text{ in } E^*, \text{ as } n \to \infty, \tag{4.1}$$

have a convergent subsequence in $E$, then $\Phi$ satisfies the $(Ce)$ condition at the level $c \in \mathbb{R}$.

**Lemma 4.1.** If the conditions (A1)-(A3), (B1)-(B2), (V1) and (H5)-(H6) are satisfied, then the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $E$.

***Proof.*** Let $\{\omega_n\}_{n\in\mathbb{N}} \subset E$ be a Cerami sequence of $\Phi$ satisfying

$$|\Phi(\omega_n)| \le c, \tag{4.2}$$

for some constant $c > 0$, and

$$(1 + \|\omega_n\|)\Phi'(\omega_n) \to 0 \text{ in } E^* \text{ as } n \to \infty, \tag{4.3}$$

which implies that

$$\langle \Phi'(\omega_n), \omega_n \rangle \to 0 \text{ as } n \to \infty. \tag{4.4}$$

Now, we prove that $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $E$. By contradiction, assume that

$$\|\omega_n\| \to \infty, \text{ as } n \to \infty. \tag{4.5}$$

Set $\upsilon_n = \frac{\omega_n}{\|\omega_n\|}$. Then $\{\upsilon_n\}_{n\in\mathbb{N}} \subset E$ and $\|\upsilon_n\| = 1$. By Theorem 2.3, there exists a subsequence $\{\upsilon_n\}_{n\in\mathbb{N}}$ such that

$$\upsilon_n \rightharpoonup \upsilon \text{ weakly in } E, \ \upsilon_n \to \upsilon \text{ strongly in } L^{\mu(x)}_{b(x)}(\mathbb{R}^N), \ \upsilon_n \to \upsilon \text{ a.e. in } \mathbb{R}^N, \tag{4.6}$$

for $\mu(x) \in (1, p^*_{s(\cdot)})$ and $|\upsilon| \geq 0$.

Let $\Omega_0 := \{x \in \mathbb{R}^N : |\upsilon(x)| > 0\}$. Thus, we have $|\omega_n(x)| \to +\infty$ for all $x \in \Omega_0$. Therefore, by the hypothesis (H5), for any $x \in \Omega_0$ and sufficiently large $n$, we obtain

$$\lim_{n\to\infty} \frac{H(x,\omega_n)}{\|\omega_n\|^{\frac{1}{2}p^+_{max}}} = \lim_{n\to\infty} \frac{H(x,\omega_n)|\upsilon_n|^{\frac{1}{2}p^+_{max}}}{|\omega_n|^{\frac{1}{2}p^+_{max}}} = +\infty. \tag{4.7}$$

From Fatou's lemma, we get

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{H(y,\omega_n)|\upsilon_n|^{\frac{1}{2}p^+_{max}}}{|x-y|^{\phi(x,y)}|\omega_n|^{\frac{1}{2}p^+_{max}}} dy \geq \int_{\Omega_0} \liminf_{n\to\infty} \frac{H(y,\omega_n)|\upsilon_n|^{\frac{1}{2}p^+_{max}}}{|x-y|^{\phi(x,y)}|\omega_n|^{\frac{1}{2}p^+_{max}}} dy = +\infty. \tag{4.8}$$

Combing (4.7) and (4.8), we have

$$\left(\int_{\mathbb{R}^N} \frac{H(y,\omega_n)}{|x-y|^{\phi(x,y)}} dy\right) \frac{H(x,\omega_n)}{\|\omega_n\|^{p^+_{max}}} \to +\infty, \text{ as } n \to \infty. \tag{4.9}$$

Therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{H(y,\omega_n)}{|x-y|^{\phi(x,y)}} dy\right) \frac{H(x,\omega_n)}{\|\omega_n\|^{p^+_{max}}} dx = +\infty. \tag{4.10}$$

As a consequence of (4.1), we derive

$$\int_{\mathbb{R}^{2N}} \frac{H(x,\omega_n(x))H(y,\omega_n(y))}{|x-y|^{\phi(x,y)}} dxdy \leq 2\sum_{i=1}^{2} \Psi_{p_i}(\omega_n) - 2\int_{\mathbb{R}^N} \frac{\xi|\omega_n|^{q(x)}}{q(x)|x|^{\alpha(x)}} dx + C_3.$$

Without loss of generality, taking $p_1(x,\cdot) < p_2(x,\cdot)$, and we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}\|\omega_n\|^{p^+_{max}}} dxdy \\
\leq& \frac{1}{\|\omega_n\|^{p^+_{max}}} \left[\frac{2}{p_1^-}\|\omega_n\|^{p_1^+} + \frac{2}{p_2^-}\|\omega_n\|^{p_2^+}\right] - \frac{2\xi\|\upsilon_n\|^{q^-}}{q^+\|\omega_n\|^{p^+_{max}-q^-}} + \frac{C_3}{\|\omega_n\|^{p^+_{max}}} \\
\leq& \frac{2}{p_1^-} - \frac{2\xi\|\upsilon_n\|^{q^-}}{q^+\|\omega_n\|^{p^+_{max}-q^-}} + \frac{C_3}{\|\omega_n\|^{p^+_{max}}}.
\end{aligned} \tag{4.11}$$

Hence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}\|\omega_n\|^{p^+_{max}}} dxdy \leq \frac{2}{p_1^-}, \tag{4.12}$$

and this contradicts (4.10).

Therefore, we assume that $\upsilon = 0$ and again arrive at a contradiction. We have $\upsilon_n \to 0$ in $L^{\mu(x)}_{b(x)}(\mathbb{R}^N)$ and $\upsilon_n \to 0$ a.e. in $\mathbb{R}^N$. As $\Phi(t\omega_n)$ is continuous function in $t \in [0,1]$, there exists $t_n \subset [0,1]$ such that

$$\Phi(t_n\omega_n) := \max_{t\in[0,1]} \Phi(t\omega_n). \tag{4.13}$$

Let $u_n := (2\zeta)^{1/p_2^-}\upsilon_n = \frac{(2\zeta)^{1/p_2^-}\omega_n}{\|\omega_n\|}$, and $\zeta > \frac{1}{2}\left(\frac{p_1^+}{p_2^+}\right)^{\frac{p_2^-}{p_1^- - p_2^-}}$. Hence, using the continuity of $H$, we deduce $\lim_{n\to+\infty} H(x,u_n) = 0$. Therefore, as $n \to +\infty$

$$\int_{\mathbb{R}^{2N}} \frac{H(x,u_n(x))H(y,u_n(y))}{|x-y|^{\phi(x,y)}} dxdy \to 0. \tag{4.14}$$

According to $\|\omega_n\| \to \infty$ as $n \to \infty$, we have $\frac{(2\zeta)^{1/p_2^-}}{\|\omega_n\|} \in (0,1)$ for large enough $n$. Thus, from (4.14) we obtain

$$\begin{aligned}
\Phi(t_n\omega_n) \geq \Phi(u_n) &= \sum_{i=1}^{2} \Psi_{p_i}(u_n) - \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{H(x,u_n(x))H(y,u_n(y))}{|x-y|^{\phi(x,y)}} dxdy - \int_{\mathbb{R}^N} \frac{\xi|u_n|^{q(x)}}{q(x)|x|^{\alpha(x)}} dx \\
&= \sum_{i=1}^{2} \Psi_{p_i}(u_n) + o_n(1) \geq \frac{(2\zeta)^{p_1^-/p_2^-}}{p_1^+}\|\upsilon_n\|^{p_1^+}_{E_1} + \frac{2\zeta}{p_2^+}\|\upsilon_n\|^{p_2^+}_{E_2} + o_n(1) \\
&\geq \frac{\zeta}{2^{p_1^+-2}p_2^+}\left(\|\upsilon_n\|_{E_1} + \|\upsilon_n\|_{E_2}\right)^{p_1^+} + o_n(1) \\
&= \frac{\zeta}{2^{p_1^+-2}p_2^+} + o_n(1),
\end{aligned} \tag{4.15}$$

where we have used that $\|\upsilon_n\|_{X_2} \le \|\upsilon_n\|_{X_1} + \|\upsilon_n\|_{X_2} = \|\upsilon_n\| = 1$, and also that $2^{p-1}(\mathbf{a}^p+\mathbf{b}^p) \ge (\mathbf{a}+\mathbf{b})^p$ for $\mathbf{a},\mathbf{b}>0$. Due to $\zeta$ being arbitrary, we have the following conclusion

$$\Phi(t_n\omega_n) = \infty, \text{ as } n \to \infty. \tag{4.16}$$

Since $0 \le t_n\omega_n \le \omega_n$ and the hypothesis (H6) yields

$$\begin{aligned} 2\int_{\mathbb{R}^N} h(x,t_n\omega_n)t_n\omega_n dx &= \int_{\mathbb{R}^N} p^+_{max}H(x,t_n\omega_n)dx + \int_{\mathbb{R}^N} \vartheta(x,t_n\omega_n)dx \\ &\le \int_{\mathbb{R}^N} p^+_{max}H(x,t_n\omega_n)dx + \int_{\mathbb{R}^N} \lambda\vartheta(x,\omega_n)dx. \end{aligned} \tag{4.17}$$

By passing to a new subsequence, if necessary, we can assume that $0 < t_n < 1$ for $n$ sufficiently large. Indeed, the fact that $\Phi(0) = 0$ implies that $t_n \neq 0$ and (4.16) combined with (4.2) implies that $t_n \neq 1$. Thus,

$$\begin{aligned} 0 =& t_n\frac{d}{dt}\Phi(t\omega_n)|_{t=t_n} = \langle \Phi'(t_n\omega_n), t_n\omega_n\rangle \\ =& \sum_{i=1}^{2}\left[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|t_n\omega_n(x)-t_n\omega_n(y)|^{p_i(x,y)}}{|x-y|^{N+p_i(x,y)s(x,y)}}dxdy + \int_{\mathbb{R}^N} V(x)|t_n\omega_n|^{\overline{p_i}(x)}dx\right] \\ &- \int_{\mathbb{R}^{2N}} \frac{H(x,t_n\omega_n(x))h(y,t_n\omega_n(y))t_n\omega_n(y)}{|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{\xi|t_n\omega_n|^{q(x)}}{|x|^{a(x)}}dx. \end{aligned} \tag{4.18}$$

Therefore, for each sufficiently large $n$, combining (4.2), (4.4), (4.17) and (4.18), we have

$$\begin{aligned} &\frac{1}{\lambda}\Phi(t_n\omega_n) + o_n(1) = \frac{1}{\lambda}\left[\Phi(t_n\omega_n) - \frac{1}{p^+_{max}}\langle\Phi'(t_n\omega_n), t_n\omega_n\rangle\right] \\ =& \frac{1}{\lambda}\sum_{i=1}^{2}\Psi_{p_i}(t_n\omega_n) - \frac{1}{\lambda}\int_{\mathbb{R}^N}\frac{\xi|t_n\omega_n|^{q(x)}}{q(x)|x|^{a(x)}} - \frac{1}{\lambda p^+_{max}}\sum_{i=1}^{2}\langle\Psi'_{p_i}(t_n\eta_n), t_n\eta_n\rangle + \frac{1}{\lambda p^+_{max}}\int_{\mathbb{R}^N}\frac{\xi|t_n\omega_n|^{q(x)}}{|x|^{a(x)}} \\ &+ \frac{1}{2\lambda p^+_{max}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{H(y,t_n\omega_n(y))}{|x-y|^{\phi(x,y)}}dy\right)(2h(x,t_n\omega_n(x))t_n\omega_n(x) - p^+_{max}H(x,t_n\omega_n(x)))dx \\ \le& \sum_{i=1}^{2}\Psi_{p_i}(t_n\omega_n) - \int_{\mathbb{R}^N}\frac{\xi|t_n\omega_n|^{q(x)}}{q(x)|x|^{a(x)}} - \frac{1}{p^+_{max}}\sum_{i=1}^{2}\langle\Psi'_{p_i}(t_n\eta_n), t_n\eta_n\rangle + \frac{1}{p^+_{max}}\int_{\mathbb{R}^N}\frac{\xi|t_n\omega_n|^{q(x)}}{|x|^{a(x)}} \\ &+ \frac{1}{2p^+_{max}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(y,t_n\omega_n(y))}{|x-y|^{\phi(x,y)}}\vartheta(x,\omega_n(x))dxdy \\ \le& \sum_{i=1}^{2}\Psi_{p_i}(\omega_n) - \int_{\mathbb{R}^N}\frac{\xi|\omega_n|^{q(x)}}{q(x)|x|^{a(x)}} - \frac{1}{p^+_{max}}\sum_{i=1}^{2}\langle\Psi'_{p_i}(\eta_n), t_n\eta_n\rangle + \frac{1}{p^+_{max}}\int_{\mathbb{R}^N}\frac{\xi|\omega_n|^{q(x)}}{|x|^{a(x)}} \\ &+ \frac{1}{2p^+_{max}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(y,\omega_n(y))}{|x-y|^{\phi(x,y)}}\vartheta(x,\omega_n(x))dxdy \\ =& \Phi(\omega_n) - \frac{1}{p^+_{max}}\langle\Phi'(\omega_n), \omega_n\rangle = c + o_n(1), \end{aligned} \tag{4.19}$$

as $n \to \infty$, which contradicts (4.16). Hence, we have that the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $E$. □

**Lemma 4.2.** If conditions (A1)-(A3), (B1)-(B2), (V1), and (H2) are satisfied, then the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ has a strong convergent subsequence.

***Proof.*** By Lemma 4.1, $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $E$. Thus, there exists $\omega \in E$, and we can extract a subsequence, denoted by $\{\omega_n\}_{n\in\mathbb{N}}$ again, satisfies

$$\omega_n \rightharpoonup \omega \text{ weakly in } E,\ \omega_n \to \omega \text{ strongly in } L^{\mu(x)}_{b(x)}(\mathbb{R}^N),\ \omega_n \to \omega\ a.e. \text{ in } \mathbb{R}^N. \tag{4.20}$$

Furthermore, we have

$$|\langle\Phi'(\omega_n), \omega_n - \omega\rangle| \le \|\Phi'(\omega_n)\|(\|\omega_n\|_E + \|\omega\|_E) \to 0, \text{ as } n \to \infty.$$

Since $\omega_n$ is bounded in $E$ and $\Phi'(\omega_n) \to 0$, we derive that

$$\langle\Phi'(\omega_n), \omega_n - \omega\rangle \to 0, \text{ as } n \to \infty,$$

and it follows that

$$
\begin{aligned}
o_n(1) =&\langle \Phi'(\omega_n), \omega_n - \omega\rangle \\
=&\langle \Psi'_{p_1}(\omega_n), \omega_n - \omega\rangle + \langle \Psi'_{p_2}(\omega_n), \omega_n - \omega\rangle \\
&- \int_{\mathbb{R}^{2N}} \frac{H(x,\omega_n(x))h(y,\omega_n(y))(\omega_n-\omega)(y)}{|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{\xi|\omega_n|^{q(x)-2}\omega_n(\omega_n-\omega)(x)}{|x|^{a(x)}}dx. \quad (4.21)
\end{aligned}
$$

From (H2), Lemma 2.5 and Theorem 2.3, we obtain

$$
\begin{aligned}
\|H(\cdot,\omega_n)\|_{m^+} \le& C_4\left(\int_\Omega b(x)|\omega_n|^{\theta(x)m^+}dx\right)^{\frac{1}{m^+}} \le C_4 \max\left\{\|\omega_n\|^{\theta^-}_{\theta(x)m^+,b(x)}, \|\omega_n\|^{\theta^+}_{\theta(x)m^+,b(x)}\right\} \quad (4.22)\\
\le& C_4 \max\left\{C^{\theta^-}_{\theta(x)m^+}\|\omega_n\|^{\theta^-}, C^{\theta^+}_{\theta(x)m^+}\|\omega_n\|^{\theta^+}\right\},
\end{aligned}
$$

that is $H(\cdot,\omega_n) \in L^{m^+}(\mathbb{R}^N)$. Similarly, we have

$$
\|H(\cdot,\omega_n)\|_{m^-} \le C_5 \max\left\{C^{\theta^-}_{\theta(x)m^-}\|\omega_n\|^{\theta^-}, C^{\theta^+}_{\theta(x)m^-}\|\omega_n\|^{\theta^+}\right\}. \quad (4.23)
$$

Thus, combined with (4.22)-(4.23) and Lemma 3.1, we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,\omega_n(x))h(y,\omega_n(y))(\omega_n-\omega)(y)}{|x-y|^{\phi(x,y)}}dxdy\right| \\
\le& C_6\left(\|H(x,\omega_n(x))\|_{m^+}\|h(y,\omega_n(y))(\omega_n-\omega)(y)\|_{m^+} + \|H(x,\omega_n(x))\|_{m^-}\|h(y,\omega_n(y))(\omega_n-\omega)(y)\|_{m^-}\right) \\
\le& C_7 \max\left\{C^{\theta^-}_{\theta(x)m^+}\|\omega_n\|^{\theta^-}, C^{\theta^+}_{\theta(x)m^+}\|\omega_n\|^{\theta^+}\right\}\|h(y,\omega_n(y))(\omega_n-\omega)(y)\|_{m^+} \\
&+ C_7 \max\left\{C^{\theta^-}_{\theta(x)m^-}\|\omega_n\|^{\theta^-}, C^{\theta^+}_{\theta(x)m^-}\|\omega_n\|^{\theta^+}\right\}\|h(y,\omega_n(y))(\omega_n-\omega)(y)\|_{m^-}. \quad (4.24)
\end{aligned}
$$

Next, using (H2), we get

$$
\begin{aligned}
&\|h(y,\omega_n)(\omega_n-\omega)\|^{m^+}_{m^+} \\
\le& \int_{\mathbb{R}^N} b(y)|\omega_n|^{(\theta(y)-1)m^+}(\omega_n-\omega)^{m^+}dy \\
\le& 2^{(\theta^+-1)m^+}\left(\int_{\mathbb{R}^N} b(y)|\omega_n-\omega|^{\theta(y)m^+}dy + \int_{\mathbb{R}^N} b(y)|\omega|^{(\theta(y)-1)m^+}(\omega_n-\omega)^{m^+}dy\right) \\
\to& 0. \quad (4.25)
\end{aligned}
$$

It follows from (4.20) that $\int_{\mathbb{R}^N} b(y)|\omega|^{(\theta(y)-1)m^+}(\omega_n-\omega)^{m^+}dy \to 0$ as $n\to\infty$. According to Lemma 2.5 and strong convergence of sequences, we obtain $\int_{\mathbb{R}^N} b(y)|\omega_n-\omega|^{\theta(y)m^+}dy \to 0$ as $n\to\infty$.

Similarly, we have

$$
\|h(y,\omega_n)(\omega_n-\omega)\|_{m^-} = o_n(1), \text{ as } n\to\infty. \quad (4.26)
$$

Hence, combining with (4.24)-(4.26), we derive

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,\omega_n(x))h(y,\omega_n(y))(\omega_n-\omega)(y)}{|x-y|^{\phi(x,y)}}dxdy = 0. \quad (4.27)
$$

Analogously to the proof (4.25), we infer

$$
\int_{\mathbb{R}^N} \frac{\xi|\omega_n|^{q(x)-2}\omega_n(\omega_n-\omega)}{|x|^{a(x)}}dx \le 2^{q^+-1}\left(\int_{\mathbb{R}^N}\frac{\xi|\omega_n-\omega|^{q(x)}}{|x|^{a(x)}}dx + \int_{\mathbb{R}^N}\frac{\xi|\omega|^{q(x)-1}(\omega_n-\omega)}{|x|^{a(x)}}dx\right) \to 0, \quad (4.28)
$$

as $n\to\infty$. Therefore, from (4.27) and (4.28), we conclude that

$$
\lim_{n\to\infty}\left[\langle \Psi'_{p_1}(\omega_n), \omega_n-\omega\rangle + \langle \Psi'_{p_2}(\omega_n), \omega_n-\omega\rangle\right] = 0. \quad (4.29)
$$

From (4.20) and the Fatou lemma, it follows that

$$
\liminf_{n\to\infty}\langle \Psi'_{p_i}(\omega_n), \omega_n\rangle \ge \langle \Psi'_{p_i}(\omega), \omega\rangle. \quad (4.30)
$$

By (4.29), we have, as $n\to\infty$,

$$
o(1) = \langle \Psi'_{p_1}(\omega_n), \omega_n-\omega\rangle + \langle \Psi'_{p_2}(\omega_n), \omega_n-\omega\rangle \ge \langle \Psi'_{p_i}(\omega_n), \omega_n-\omega\rangle. \quad (4.31)
$$

Fixed $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, according to the Young inequality, we get

$$\begin{aligned} &|\omega_n(x)-\omega_n(y)|^{p_i(x,y)-1}|\omega(x)-\omega(y)| \\ \leq &\frac{1}{p_i'(x,y)}|\omega_n(x)-\omega_n(y)|^{p_i(x,y)} + \frac{1}{p_i(x,y)}|\omega(x)-\omega(y)|^{p_i(x,y)} \\ \leq &\frac{1}{(p_i')^-}|\omega_n(x)-\omega_n(y)|^{p_i(x,y)} + \frac{1}{p_i^-}|\omega(x)-\omega(y)|^{p_i(x,y)}, \end{aligned} \tag{4.32}$$

and

$$|\omega_n(x)|^{\overline{p}_i(x)-1}|\omega(x)| \leq \frac{1}{(\overline{p}_i')^-}|\omega_n(x)|^{\overline{p}_i(x)} + \frac{1}{(\overline{p}_i)^-}|\omega(x)|^{\overline{p}_i(x)}, \tag{4.33}$$

so that

$$\begin{aligned} \langle \Psi'_{p_i}(\omega_n), \omega_n-\omega\rangle \geq &\langle \Psi'_{p_i}(\omega_n), \omega_n\rangle - \langle \Psi'_{p_i}(\omega_n), \omega\rangle \\ \geq &C_{p_i}\left(\langle \Psi'_{p_i}(\omega_n), \omega_n\rangle - \langle \Psi'_{p_i}(\omega), \omega\rangle\right), \end{aligned} \tag{4.34}$$

which combined with (4.31) and (4.34) yield

$$\lim_{n\to\infty} \langle \Psi'_{p_i}(\omega_n), \omega_n\rangle = \langle \Psi'_{p_i}(\omega), \omega\rangle. \tag{4.35}$$

However, using (4.20) and the Brézis-Lieb type lemma for variable exponent in [30], we obtain

$$o_n(1) + \langle \Psi'_{p_i}(\omega_n-\omega), \omega_n-\omega\rangle = \langle \Psi'_{p_i}(\omega_n), \omega_n\rangle - \langle \Psi'_{p_i}(\omega), \omega\rangle, \tag{4.36}$$

which joint with (4.35), we have

$$\lim_{n\to\infty} \varrho_{E_i}(\omega_n-\omega) = 0,$$

according to Lemma 2.4, we finally achieve that $\omega_n \to \omega$ in $E$ as $n\to\infty$.

□

## 5. Proofs of Theorem 3.1

Let $E$ be a separable and reflexive real Banach space, then there exists $\{e_j\} \in E$ and $\{e_j^*\} \in E^*$ such that $E = \overline{\text{span}\{e_j : j = 1,2,...\}}$, $E^* = \overline{\text{span}\{e_j^* : j = 1,2,...\}}$ and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, i = j\,; \\ 0, i \neq j\,. \end{cases}$$

Set $E_i = \text{span}\{e_i : i = 1,2,...\}$, and denote $X_k = \bigoplus_{i=1}^k E_i, Y_k = \overline{\bigoplus_{i=k}^\infty E_i}$. We state the symmetric mountain pass theorem, i.e. Theorem 5.1 below.

**Theorem 5.1.** ([9]). Let $E$ be a real infinite dimensional Banach space, $E = X_k \bigoplus Y_k$ and $\dim X_k < \infty$. $\Phi \in C^1(E,R)$ be even with $\Phi(0) = 0$. Suppose $\Phi$ satisfying $(PS)$ condition and
(i) there are constants $\alpha, \gamma > 0$ such that $\inf_{\omega\in Y_k, \|\omega\|=\alpha} \Phi(\omega) \geq \gamma$;
(ii) for every finite dimensional subspaces $E' \subset E$ there exists $M = M(E') > 0$ such that $\max_{\omega\in E', \|\omega\|\geq M} \Phi(\omega) \leq 0$.
Then $\Phi$ possesses an unbounded sequence of critical values.

***Proof of Theorem 3.1.*** From (4.22) and (4.23), one has

$$\begin{aligned} \left|\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}}dxdy\right| \leq &C_8\left(\|H(\cdot,\omega(\cdot))\|^2_{m^+} + \|H(\cdot,\omega(\cdot))\|^2_{m^-}\right) \\ \leq &C_9 \max\left\{\|\omega\|^{2\theta^-}, \|\omega\|^{2\theta^+}\right\}. \end{aligned}$$

Let $\omega \in Y_k$ such that $\|\omega\| = \alpha \in (0,1)$. Thus, using the Lemma 2.4 and Theorem 2.3, we get

$$\begin{aligned}
\Phi(\omega) :=& \sum_{i=1}^{2} \Psi_{p_i}(\omega) - \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{\xi|\omega|^{q(x)}}{q(x)|x|^{a(x)}}dx \\
\geq& \frac{1}{p_1^+}\|\omega\|_{E_1}^{p_1^+} + \frac{1}{p_2^+}\|\omega\|_{E_2}^{p_2^+} - \frac{C_{12}}{2}|\omega|^{2\theta^-} - \frac{\xi}{q^-}\|\omega\|_{q(x),|x|^{-a(x)}}^{q^-} \\
\geq& \frac{1}{2^{p_{min}^+-1}p_{max}^+}\|\omega\|^{p_{min}^+} - \frac{C_{12}}{2}\|\omega\|^{2\theta^-} - \frac{\xi C_{q^-}}{q^+}\|\omega\|^{q^-} \\
=& \alpha^{p_{min}^+}\left(\frac{1}{2^{p_{min}^+-1}p_{max}^+} - \frac{C_{12}}{2}\alpha^{2\theta^- - p_{min}^+}\right) - \frac{\xi C_{q^-}}{q^-}\alpha^{q^-}. \qquad (5.1)
\end{aligned}$$

Choosing $\alpha \in (0, \min\{1, [1/2^{p_{min}^+-1}p_{max}^+C_{12}]^{1/(2\theta^- - p_{min}^+)}\})$, we deduce

$$\Phi(\omega) \geq \frac{1}{2^{p_{min}^+}p_{max}^+}\alpha^{p_{min}^+} - \frac{\xi C_{q^-}}{q^+}\alpha^{q^-}.$$

Taking $\xi^* = q^+\beta^{p_{min}^+ - q^-}/2^{p_{min}^+ + 1}p_{max}^+C_{q^-}$. Then for any $\xi \in (0,\xi^*]$, we obtain

$$\Phi(\omega) \geq \frac{1}{2^{p_{min}^+ + 1}p_{max}^+}\alpha^{p_{min}^+} = \gamma > 0.$$

Thus, condition $(i)$ holds.

By (H5), for any $C_{10} > 0$, there exists a positive constant $C_{11}$ such that

$$|H(x,\omega)| \geq C_{10}|\omega|^{\frac{p_{max}^+}{2}}, \text{ for each } x \in \mathbb{R}^N \text{ and } |\omega| > C_{11}.$$

Obviously, there exists $C_{E'} > 0$ that satisfies $\|\omega\|_{q(x),|x|^{-a(x)}} \geq C_{E'}\|\omega\|$, since all norms are equivalent on the finite dimensional Banach space $E'$. For $t > 1$, we get

$$\begin{aligned}
\Phi(t\omega) :=& \sum_{i=1}^{2} \Psi_{p_i}(t\omega) - \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{H(x,t\omega(x))H(y,t\omega(y))}{|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{|t\omega|^{q(x)}}{q(x)|x|^{a(x)}}dx \\
\leq& \frac{t^{p_1^+}}{p_1^-}\|\omega\|_{E_1}^{p_1^+} + \frac{t^{p_2^+}}{p_2^-}\|\omega\|_{E_2}^{p_2^+} - \frac{C_{10}^2 t^{p_{max}^+}}{2}\int_{\mathbb{R}^{2N}} \frac{|\omega(x)|^{\frac{p_{max}^+}{2}}|\omega(y)|^{\frac{p_{max}^+}{2}}}{|x-y|^{\phi(x,y)}}dxdy - \frac{t^{q^-}}{q^+}\|\omega\|_{q(x),|x|^{-a(x)}}^{q^+} \\
=& \frac{t^{p_{max}^+}}{p_{min}^-}\|\omega\|^{p_{max}^+} - \frac{C_{10}^2 t^{p_{max}^+}}{2}\int_{\mathbb{R}^{2N}} \frac{|\omega(x)|^{\frac{p_{max}^+}{2}}|\omega(y)|^{\frac{p_{max}^+}{2}}}{|x-y|^{\phi(x,y)}}dxdy - \frac{t^{q^-}}{q^+}C_{E'}\|\omega\|^{q^+}. \qquad (5.2)
\end{aligned}$$

If $C_{10}$ is big enough to satisfy

$$\frac{1}{p_{min}^-}\|\omega\|^{p_{max}^+} < \frac{C_{10}^2}{2}\int_{\mathbb{R}^{2N}} \frac{|\omega(x)|^{\frac{p_{max}^+}{2}}|\omega(y)|^{\frac{p_{max}^+}{2}}}{|x-y|^{\phi(x,y)}}dxdy.$$

So, it follows from (5.2) that

$$\Phi(t\omega) \to -\infty,$$

as $t \to \infty$, by $q^+ < p_{max}^+$. Therefore, there exists $M_0 > 0$ large enough such that $\Phi(\omega) < 0$ for all $\omega \in E'$ with $\|\omega\| = M > 1$ and $M \geq M_0$. This completes the proof.

□

## 6. Proofs of Theorem 3.2

In order to prove Theorem 3.2, we will use the Dual Fountain Theorem.

**Theorem 6.1.** ([12]). Suppose that $\Phi \in C^1(E,\mathbb{R})$ satisfies the $(Ce)_c^*$ condition for every $c \in [d_{k_0}, 0]$. If for any $k \geq k_0$, there exists $\varsigma_k > \rho_k > 0$ satisfies the following properties
(i) $\Phi(-\omega) = \Phi(\omega)$;
(ii) $y_k = \inf\{\Phi(\omega) : \omega \in Y_k, \|\omega\| = \alpha_k\} \geq 0$;
(iii) $x_k = \sup\{\Phi(\omega) : \omega \in X_k, \|\omega\| = \rho_k\} < 0$;
(iv) $z_k = \inf\{\Phi(\omega) : \omega \in Y_k, \|\omega\| \leq \alpha_k\} \to 0 \ as \ k \to \infty$,
then $J$ has a sequence of negative critical points $\omega_k$ such that $J(\omega_k) \to 0$.

**Definition 6.1.** If any $(Ce)_c^*$ sequence $\{\omega_k\}_{k\in\mathbb{N}}$ in $E$ with $\omega_k \in X_k$, namely

$$\Phi(\omega_k) \to c, \ (1+\|\omega_k\|)(\Phi|_{X_k})'(\omega_k) \to 0 \text{ in } E^*, \text{ as } n\to\infty, \tag{6.1}$$

have a convergent subsequence in $E$, then $\Phi$ satisfies the $(Ce)_c^*$ condition at the level $c\in\mathbb{R}$.

**Lemma 6.1.** Assume that the hypotheses in Theorem 3.2 hold. Then $\Phi$ satisfies the $(Ce)_c^*$ condition.

***Proof*.** Let $c\in\mathbb{R}$ and the sequence $\{\omega_j\}_{j\in\mathbb{N}}\subset E$ such that $\{\omega_j\}\in X_j$, $\Phi(\omega_j)\to c$ and $(1+\|\omega_j\|)(\Phi|_{X_j})'(\omega_j)\to 0$ as $j\to+\infty$, which implies that

$$\langle\Phi'(\omega_j),\omega_j\rangle = \langle(\Phi|_{X_j})'(\omega_j),\omega_j\rangle \to 0.$$

Similar to the proof of Lemma 4.1, we can prove that $\{\omega_j\}$ is bounded. So, there exists a subsequence, denoted for $\{\omega_j\}$, and $\omega_0\in E$ such that $\omega_j \rightharpoonup \omega_0$ weakly in $E$. As $E=\overline{\bigcup_j X_j}=\overline{span\{e_j : j\geq 1\}}$, we choose $\nu_j\in X_j$ such that $\nu_j\to\omega_0$ strongly in $E$. Hence, using the facts $\Phi'|_{X_j}(\omega_j)\to 0$ and $\omega_j-\nu_j\rightharpoonup 0$ in $X_j$, we obtain

$$\langle\Phi'(\omega_j),\omega_j-\omega_0\rangle = \langle\Phi'(\omega_j),\omega_j-\nu_j\rangle+\langle\Phi'(\omega_j),\nu_j-\omega_0\rangle\to 0.$$

Again recalling the proof of Lemma 4.2, we deduce $\omega_j\to\omega_0$ strongly in $E$. Then, we conclude that $\Phi$ satisfies the $(Ce)_c^*$ condition. Furthermore, we obtain that $\Phi'(\omega_j)\to\Phi'(\omega_0)$ as $j\to+\infty$.

Next, we prove that $\Phi'(\omega_0)=0$. Indeed, taking $\omega_l\in X_l$, for $j\geq l$, we get

$$\begin{aligned}\langle\Phi'(\omega_0),\omega_l\rangle =&\langle\Phi'(\omega_0)-\Phi'(\omega_j),\omega_l\rangle+\langle\Phi'(\omega_j),\omega_l\rangle\\ =&\langle\Phi'(\omega_0)-\Phi'(\omega_j),\omega_l\rangle+\langle\Phi'|_{X_j}(\omega_j),\omega_l\rangle\to 0,\end{aligned}$$

as $j\to+\infty$. Thus, $\Phi'(\omega_0)=0$ in $E^*$, this show that $\Phi$ satisfies the $(Ce)_c^*$ condition for each $c\in\mathbb{R}$. The proof is over. □

**Lemma 6.2.** Let $\mu(x)\in C_+(\mathbb{R}^N)$, and $\mu(x)<p_{s(\cdot)}^*$ for any $x\in\mathbb{R}^N$. For each $k\in\mathbb{N}$, define

$$\vartheta_k = \sup_{\omega\in Y_k,\|\omega\|_E=1}\|\omega\|_{L_{b(x)}^{\mu(x)}(\mathbb{R}^N)}.$$

Then, $\lim_{k\to\infty}\vartheta_k=0$.

***Proof*.** It is clear that $0<\vartheta_{k+1}\leq\vartheta_k<\infty$, and so that $\vartheta_k\to\vartheta\geq 0$ as $k\to\infty$. For each $k\geq 0$, there exists $\omega_k\in Y_k$ satisfies $\|\omega_k\|_E=1$ and $\|\omega_k\|_{L_{b(x)}^{\mu(x)}(\mathbb{R}^N)}\geq\frac{\vartheta_k}{2}$. By definition of $Y_k$, $\omega_k\rightharpoonup 0$ in $E$. Theorem 2.3 implies that $\omega_k\to 0$ in $L_{b(x)}^{\mu(x)}(\mathbb{R}^N)$, and as result $\vartheta=0$. The proof is over. □

***Proof of Theorem 3.2*.** From (H1) and Lemma 6.1, we have that $\Phi(\omega)$ is even and satisfies $(Ce)_c^*$ condition for each $c\in\mathbb{R}$. Next, we prove conditions $(ii)$-$(iv)$ are true for $\Phi(\omega)$. Firstly, for every $\omega\in Y_k$ with $\|\omega\|<1$, we derive

$$\begin{aligned}\Phi(\omega) :=& \sum_{i=1}^{2}\Psi_{p_i}(\omega)-\frac{1}{2}\int_{\mathbb{R}^{2N}}\frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}}dxdy-\int_{\mathbb{R}^N}\frac{|\omega|^{q(x)}}{q(x)|x|^{a(x)}}dx\\ \geq&\frac{1}{p_1^+}\|\omega\|_{E_1}^{p_1^+}+\frac{1}{p_2^+}\|\omega\|_{E_2}^{p_2^+}-\frac{C_9}{2}\|\omega\|^{2\theta^-}-\frac{1}{q^-}\|\omega\|_{q(x),|x|^{-a(x)}}^{q^-}\\ \geq&\frac{1}{2^{p_{min}^+-1}p_{max}^+}\|\omega\|^{p_{min}^+}-C_{12}\|\omega\|^{2\theta^-}-C_{13}\vartheta_k^{q^-}\|\omega\|^{q^-},\end{aligned}\tag{6.2}$$

we may choose $M\in(0,1)$ small such that

$$\frac{1}{2^{p_{min}^+}p_{max}^+}\|\omega\|^{p_{min}^+}\geq C_{12}\|\omega\|^{2\theta^-},$$

holds for any $\omega\in E$ with $\|\omega\|<M$. Then, we get

$$\Phi(\omega)\geq\frac{1}{2^{p_{min}^+}p_{max}^+}\|\omega\|^{p_{min}^+}-C_{13}\vartheta_k^{q^-}\|\omega\|^{q^-}.$$

We choose

$$\varsigma_k=(C_{13}2^{p_{min}^+}p_{max}^+\vartheta_k^{q^-})^{\frac{1}{p_{min}^+-q^-}},$$

since $p_{min}^+>q^-$, it follows that

$$\varsigma_k\to 0,\ k\to+\infty.$$

Thus, there exists $k_0$ such that $\varsigma_k\leq M$ as $k>k_0$. Hence, we get

$$y_k=\inf_{\omega\in Y_k,\|\omega\|=\varsigma_k}\Phi(\omega)\geq 0,$$

as $k\to+\infty$. So, the condition $(ii)$ is fulfilled.

Secondly, for any $\omega \in X_k$, $\|\omega\| = \rho_k$ with $\varsigma_k > \rho_k > 0$, by (H3) and all norms are equivalent on the finite dimensional Banach space, we have

$$\begin{aligned}\Phi(\omega) :=& \sum_{i=1}^{2} \Psi_{p_i}(\omega) - \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{|\omega|^{q(x)}}{q(x)|x|^{a(x)}}dx \\ \leq& \frac{1}{p_1^-}\|\omega\|_{E_1}^{p_1^-} + \frac{1}{p_2^-}\|\omega\|_{E_2}^{p_2^-} - \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{b'(x)|\omega|^{\theta'(x)}b'(y)|\omega|^{\theta'(y)}}{\theta'(x)\theta'(y)|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{|\omega|^{q(x)}}{q(x)|x|^{a(x)}}dx \\ \leq& \frac{1}{p_{min}^-}\|\omega\|^{p_{min}^-} - C_{X_k}\|\omega\|^{q^+} - \frac{1}{2}d \\ <& 0, \end{aligned} \tag{6.3}$$

as $p_{min}^- > q^+$, $d = \int_{\mathbb{R}^{2N}} \frac{b'(x)|\omega|^{\theta'(x)}b'(y)|\omega|^{\theta'(y)}}{\theta'(x)\theta'(y)|x-y|^{\phi(x,y)}}dxdy$ and $\rho_k$ small enough. Thus, the condition $(iii)$ also holds.

Finally, from verification of $(i)$, one has that for $k \geq k_0$ and $\omega \in Y_k$ with $\|\omega\| \leq \varsigma_k$,

$$\Phi(\omega) \geq -C_{13}\vartheta_k^{q^-}\|\omega\|^{q^-} \geq -C_{13}\vartheta_k^{q^-}\varsigma_k^{q^-} \to 0,$$

by $\vartheta_k \to 0$ and $\varsigma_k \to 0$ as $k \to \infty$. Moreover, $X_k \bigcap Y_k \neq \emptyset$, we obtain $z_k < y_k < 0$, so $\lim_{k\to\infty} z_k = 0$. Therefore, all conditions of Theorem 6.1 are satisfied. The proof is completed.

□

## 7. Proofs of Theorem 3.3

In order to prove Theorem 3.3, we recall some related knowledge of Krasnoselskii's genus.

**Definition 7.1.** Let E be a real Banach space and set

$$\Lambda = \{B \in E\backslash\{0\} : \ B = -B \text{ and } B \text{ is compact }\}.$$

For $B \in \Lambda$. The genus $\gamma(B)$ of $B$ is defined as

$$\gamma(B) = \inf\{k \in N : \ \exists\varpi \in C(B, \mathbb{R}^k\backslash\{0\}),\ \varpi(-x) = -\varpi(x)\}.$$

If such a $k$ does not exist, we set $\gamma(B) = \infty$. Moreover, set $\gamma(\emptyset) = 0$.

**Lemma 7.1.** If $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric, and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$, then $\gamma(\partial\Omega) = N$. Furthermore, if $S^{k-1}$ be a $(k-1)$-dimensional sphere in $\mathbb{R}^k$, then $\gamma(S^{k-1}) = k$.

**Lemma 7.2.** ([35]) Let $\Phi \in C^1(E_k, \mathbb{R})$ be an even and bounded from below functional on infinite dimensional Banach space $E_K$ which satisfies the Palais-Smale condition. If there exists

$$\Lambda_k = \{D \in \Lambda : \gamma(D) \geq k\} \text{ such that } \sup_{\omega\in\Lambda_k} \Phi(\omega) < 0, \text{ for any } k \in N,$$

then $\Phi$ admits a sequence of critical point $\{\omega_k\}$ satisfies $\Phi(\omega_k) \leq 0$, $\omega_k \neq 0$.

***Proof of Theorem 3.3.*** Assume that $g \in C^\infty([0, +\infty), \mathbb{R})$ satisfies $0 \leq g(t) \leq 1$, $t \in [0, +\infty)$ and for every $\epsilon > 0$

$$g(t) = \begin{cases} 0, \text{ if } t \geq \epsilon, \\ 1, \text{ if } t \in [0, \frac{\epsilon}{2}]. \end{cases}$$

For $G(\omega) = g(\|\omega\|)$, we consider the functional

$$\mathcal{I}(\omega) := \sum_{i=1}^{2} \Psi_{p_i}(\omega) - \frac{1}{2}G(\omega)\int_{\mathbb{R}^{2N}} \frac{H(x,\omega(x))H(y,\omega(y))}{|x-y|^{\phi(x,y)}}dxdy - \int_{\mathbb{R}^N} \frac{\xi|\omega|^{q(x)}}{q(x)|x|^{a(x)}}dx. \tag{7.1}$$

It is clear that $\mathcal{I} \in C^1(E, \mathbb{R})$. Next, we prove that $\mathcal{I}$ has a sequence of nontrivial critical points $\{\omega_n\}$ with $\omega_n \to 0$ as $n \to \infty$ in $E$, then Theorem 3.3 is proved. In fact, for any $\epsilon > 0$, there exists $N > 0$ such that $\|\omega_n\| \leq \frac{\epsilon}{2}$ for all $n > N$, thus, $\mathcal{I}(\omega_n) = \Phi(\omega_n)$, this means that $\{\omega_n\}$ are also the critical points of $\Phi$.

For $\|\omega\| \geq 1$, by (7.1), we have

$$\begin{aligned}\mathcal{I}(\omega) \geq& \frac{1}{p_1^+}\|\omega\|_{E_1}^{p_1^+} + \frac{1}{p_2^+}\|\omega\|_{E_2}^{p_2^+} - \frac{\xi}{q^-}\|\omega\|_{q(x),|x|^{-a(x)}}^{q^-} \\ \geq& \frac{1}{2^{p_{min}^- -1}p_{max}^+}\|\omega\|^{p_{min}^-} - \frac{\xi C_{q^+}}{q^-}\|\omega\|^{q^+} \to \infty, \end{aligned} \tag{7.2}$$

as $\|\omega\| \to \infty$, $q^+ < p_{min}^-$, so $\mathcal{I}(\omega)$ is coercive. Then $\mathcal{I}(\omega)$ is bounded from below and satisfies the $(Ce)$ condition analogously to the proof of Lemma 4.1-4.2. From (H1), we obtain $\mathcal{I}(-\omega) = \mathcal{I}(\omega)$ and $\mathcal{I}(0) = 0$.

For any $k \in \mathbb{N}$, we choose a $k$-dimensional linear subspace $E_k$ of $E$. As all norms are equivalent on $E_k$, there exists $\sigma_k \leq \min\{1, \frac{\epsilon}{2}\}$ such that $\omega \in E_k$ with $\|\omega\| \leq \sigma_k$. Set

$$S_{\sigma_k} = \{\omega \in E_k : \|\omega\| = \sigma_k\}.$$

For $\|\omega\| \in S_{\sigma_k}$ and $t \in (0,1)$, from (6.3), we get

$$\begin{aligned}\mathcal{I}(t\omega) &:= \sum_{i=1}^{2} \Psi_{p_i}(t\omega) - \frac{1}{2} G(t\omega) \int_{\mathbb{R}^{2N}} \frac{H(x,t\omega(x))H(y,t\omega(y))}{|x-y|^{\phi(x,y)}} dxdy - \int_{\mathbb{R}^N} \frac{|t\omega|^{q(x)}}{q(x)|x|^{a(x)}} dx \\ &\leq \frac{t^{p_1^-}}{p_1^-} \|\omega\|_{E_1}^{p_1^-} + \frac{t^{p_2^-}}{p_2^-} \|\omega\|_{E_2}^{p_2^-} - \frac{t^{2\theta'^+}}{2} \int_{\mathbb{R}^{2N}} \frac{b'(x)|\omega|^{\theta'(x)} b'(y)|\omega|^{\theta'(y)}}{\theta'(x)\theta'(y)|x-y|^{\phi(x,y)}} dxdy - \frac{t^{q^+}}{q^+} \int_{\mathbb{R}^N} \frac{|\omega|^{q(x)}}{|x|^{a(x)}} dx \\ &= \frac{t^{p_{min}^-}}{p_{min}^-} \|\omega\|^{p_{min}^-} - \frac{t^{2\theta'^+}}{2} d - \frac{t^{q^+}}{q^+} C_{E_k} \|\omega\|^{q^+}. \end{aligned} \tag{7.3}$$

As $p_{min}^- > 2\theta'^+ > q^+$, we can find $t_k \in (0,1)$ such that

$$\mathcal{I}(t_k\omega) < 0, \text{ for all } \omega \in S_{\sigma_k},$$

that is

$$\mathcal{I}(\omega) < 0, \text{ for all } \omega \in S_{t_k\sigma_k}.$$

Therefore

$$S_{t_k\sigma_k} \subset \Lambda_k = \{\omega \in E : \mathcal{I}(\omega) < 0\}.$$

Furthermore, since $S_{t_k\sigma_k}$ is a sphere in $E_k$, we deduce that $S_{t_k\sigma_k}$ is a $k$-dimensional subspace of $E_k$. By Lemma 7.1, we have

$$\gamma(S_{t_k\sigma_k}) = k+1.$$

So

$$\gamma(D) \geq \gamma(S_{t_k\sigma_k}) = k+1.$$

Thus, there exists $\Lambda_k$ such that

$$\sup_{\omega\in\Lambda_k} \mathcal{I}(\omega) < 0.$$

Hence, by Lemma 7.2, the proof is completed.

□

## 8. Conclusions

In this article, we study a class of variable-order fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-Choquard equation. Based on the three different critical point theorems, the existence of infinitely many solutions are derived. The main innovation of this paper is the use of weighted Lebesgue spaces to overcome the difficulty of the compact embedding result in $\mathbb{R}^N$ and the double Laplace operator we consider is more complex. Moreover, the equation including Hardy nonlinearity and the function $h(x,\omega)$ does not satisfy the Ambrosetti-Rabinowitz condition. In addition, our work is inspiring for future research as regards the existence of solutions for Schrödinger double phase problems with variable exponents.

## Ethical Approval

Not applicable.

## Competing interests

The authors declare no conflict of interest.

## Authors' contributions

Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

## Funding

This work was supported by the Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX23-0669, KYCX24-0822) and the Doctoral Foundation of Fuyang Normal University under Grant (2023KYQD0044).

## Availability of data and materials

Not applicable.

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