Best proximity points of multiplication of two operators in strictly convex Banach algebras

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Abstract

In the current paper, we introduce a concept of a best proximity point for multiplication of two non-self mappings in the setting of strictly convex Banach algebras and present some sufficient conditions to ensure the existence of such points by using a projection mapping defined on a union of proximal sets. Examples are given to support our main conclusions. We also give an extension version of Schauder's fixed point problem to non-self mappings and used to find an optimum solution for a system of differential equations.

Keywords: Best proximity point, Schauder's fixed point problem, Strictly convex Banach algebra, Projection operator

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1 Introduction and Preliminaries

Let X be a normed linear space. Throughout this article $\mathcal{B}(x;r)$ ($\mathcal{N}(x;r)$) denotes the closed (open) ball at $x \in X$ with radius $r > 0$.

Brouwer's theorem (1911) states that any continuous function $T : \mathcal{B}(0;1) \subseteq \mathbb{R}^n \to \mathcal{B}(0;1)$ has at leat one fixed point. Brouwer's theorem has many applications in various fields of mathematics. For example, it is related to existence theorems in differential and integral equations and plays an important role in proving the existence of the Nash equilibrium problem in game theory. We mention that the one dimensional case of this problem is a conclusion of the intermediate value theorem.

Because of importance of Brouwer's theorem it was extended from Euclidean spaces to Banach spaces by Schauder in 1930 as follows:

Theorem 1.1. *Let U be a nonempty, compact and convex subset of a Banach space X and T be a continuous self mapping on U. Then T has a fixed point.*

It is worth noticing that Schauder's fixed point theorem is valid in locally convex spaces (see Theorem 2.3.8 in [14]).

Definition 1.2. *Let X and Y be normed linear spaces and K be a subset of X. A mapping $T : K \to Y$ is said to be a compact operator if T maps bounded sets into relatively compact sets.*

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The next theorem is a well-known extension of Schauder's fixed point theorem.

Theorem 1.3. *Let K be a nonempty, bounded, closed and convex subset of a Banach space X and $T : K \to K$ be a continuous and compact operator. Then T has a fixed point.*

Let U and V be two nonempty subsets of a metric space (M, d). Because the functional equation $Tx = x$, where $T : U \to V$ is a given non-self mapping, does not necessarily have a solution, it is desirable in this case to find an approximate solution $x^\star \in U$ such that the error $d(x^\star, Tx^\star)$ is minimized.

Definition 1.4. *Let U and V be nonempty subsets of a metric space (M, d) and $T : U \to V$ be a non-self mapping. A point $x^\star \in U$ is called a best proximity point of T if*

$$d(x^\star, Tx^\star) = \operatorname{dist}(U, V) := \inf\{d(u, v) : (u, v) \in U \times V\}.$$

Indeed, best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{u\in U} d(u, Tu), \tag{1}$$

has at least one solution. Moreover, if the underlying mapping is a self mapping, then it can be observed that a best proximity point is essentially a fixed point. We refer to [3, 4, 10, 13] for some existence, uniqueness and convergence of a best proximity point for various classes of non-self mappings. We also point out the paper [1], where the authors introduced appropriate dominating property to obtain both the existence and uniqueness of a common best proximity point for suitable mappings.

Definition 1.5. *A Banach space X is said to be strictly convex if the following implication holds $x, y, p \in X$ and $R > 0$:*

$$\begin{cases} \|x - p\| \le R, \\ \|y - p\| \le R, \\ x \ne y \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| < R.$$

It is well known that Hilbert spaces and l^p spaces $(1 < p < \infty)$ are strictly convex Banach spaces.

We will say that a pair (U, V) of subsets of a Banach space X has a property, whenever both U and V have that property. For example (U, V) is convex means that both U and V are convex subsets of X.

For a nonempty pair (U, V) of subsets of a Banach space X, its *proximal pair* is denoted by (U_0, V_0) and is defined as follows

$$U_0 = \{u \in U : \exists\ v' \in V \mid \|u - v'\| = \operatorname{dist}(U, V)\},$$
$$V_0 = \{v \in V : \exists\ u' \in U \mid \|u' - v\| = \operatorname{dist}(U, V)\}.$$

It is worth mentioning that if (U, V) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X then its proximal pair (U_0, V_0) is also nonempty, closed and convex.

For a nonempty subset U of a Banach space X a *metric projection operator* $\mathcal{P}_U : X \to 2^U$ is defined as

$$\mathcal{P}_U(x) := \{u \in U : \|x - u\| = \operatorname{dist}(\{x\}, U)\},$$

where 2^U denotes the set of all subsets of U. It is well known that if U is a nonempty, closed and convex subset of a reflexive and strictly convex Banach space X, then the metric projection $\mathcal{P}_U$ is single valued from X to U, that is, $\mathcal{P}_U : X \to U$ is a mapping with $\|x - \mathcal{P}_U(x)\| = \operatorname{dist}(\{x\}, U)$ for any $x \in X$.

Proposition 1.6. ([7, 8]) *Let (U, V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X. Define a projection mapping $\mathcal{P} : U_0 \cup V_0 \to U_0 \cup V_0$ as*

$$\mathcal{P}(x) = \begin{cases} \mathcal{P}_{U_0}(x); & \text{if} \quad x \in V_0, \\ \mathcal{P}_{V_0}(x); & \text{if} \quad x \in U_0. \end{cases} \tag{2}$$

Then the following statements hold:

(i) $\|x - \mathcal{P}x\| = \operatorname{dist}(U, V)$ *for any* $x \in U_0 \cup V_0$ *and* $\mathcal{P}$ *is* cyclic *on* $U_0 \cup V_0$, *that is,* $\mathcal{P}(U_0) \subseteq V_0$ *and* $\mathcal{P}(V_0) \subseteq U_0$*;*

(ii) $\mathcal{P}|_{U_0}$ *and* $\mathcal{P}|_{V_0}$ *are isometry;*

(iii) $\mathcal{P}|_{U_0}$ *and* $\mathcal{P}|_{V_0}$ *are affine;*

(iv) $\mathcal{P}^2|_{U_0} = i_{U_0}$ *and* $\mathcal{P}^2|_{V_0} = i_{V_0}$*, where* i_A *denotes the identity mapping on a nonempty subset* A *of* X.

Definition 1.7. ([12]) *Let (U, V) be a pair of nonempty subsets of a metric space (M, d) with $U_0 \neq \emptyset$. The pair (U, V) is said to have* P*-property if and only if*

$$\begin{cases} d(x_1, y_1) = \operatorname{dist}(U, V), \\ d(x_2, y_2) = \operatorname{dist}(U, V), \end{cases} \quad \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in U_0$ and $y_1, y_2 \in V_0$.

It is remarkable to note that every nonempty, closed and convex pair in a strictly convex Banach space X has the P-property (see Example 2.3 of [9]).

This paper is organized as follows: In Section 2. we prove a best proximity point theorem for the multiplication of two non-self mappings in strictly Banach algebras. We also present examples to illustrate the main existence results. In Section 3. we give a generalization of Schauder's fixed point theorem for non-self mappings by using the concept of measure of noncompactness in strictly convex Banach spaces. Finally, in Section 4. as an application of the extension version of Schauder's fixed point problem, we investigate the existence of an optimum solution for a system of differential equations.

2 Best proximity points in Banach algebras

The next theorem is the first main result of this article.

Theorem 2.1. *Let (U, V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach algebra X. Assume that $\mathcal{T} : U \to X$ and $\mathcal{S} : V \to X$ are two operators satisfying the following conditions:*

(i) *There exists a real number $k \in [0, 1)$ such that $\|\mathcal{T}x - \mathcal{T}y\| \leq k\|x - y\|$ for any $x, y \in U_0$;*

(ii) *$\mathcal{S}$ is a continuous and compact operator on V_0;*

(iii) $\mathcal{T}(U_0)\mathcal{S}(V_0) \subseteq U_0$.

If $kM < 1$, then there is an element $v^\star \in V_0$ for which

$$\|v^\star - \mathcal{T}(\mathcal{P}v^\star)\mathcal{S}v^\star\| = \operatorname{dist}(U, V), \tag{3}$$

where $M := \sup\{\|\mathcal{S}v\| : v \in V_0\}$ and $\mathcal{P}$ is a projection mapping defined in (2). That is, $v^\star$ is a best proximity point for the multiplicative mapping $\mathcal{T}\mathcal{S}$.

Proof. Since (U, V) is a bounded, closed and convex pair in a reflexive Banach space X, its proximal (U_0, V_0) is also nonempty, closed and convex. Let $v \in V_0$ be an arbitrary element and define $\mathcal{T}_v : U_0 \to U_0$ with

$$\mathcal{T}_v(u) = (\mathcal{T}u)(\mathcal{S}v), \quad \forall u \in U_0.$$

From the assumption (iii), $\mathcal{T}_v$ is well-defined. Also, for any $u_1, u_2 \in U_0$ we have

$$\begin{aligned}
\|\mathcal{T}_v(u_1) - \mathcal{T}_v(u_2)\| &= \|(\mathcal{T}u_1)(\mathcal{S}v) - (\mathcal{T}u_2)(\mathcal{S}v)\| \\
&\le \|\mathcal{S}v\|\|\mathcal{T}u_1 - \mathcal{T}u_2\| \\
&\le \underbrace{kM}_{<1}\|u_1 - u_2\|,
\end{aligned}$$

which implies that the mapping $\mathcal{T}_v$ is a contraction map and since the set U_0 is complete, by the Banach contraction principle, $\mathcal{T}_v$ has a unique fixed point, say $g(v) \in U_0$. Thus

$$g(v) = \mathcal{T}_v\big(g(v)\big) = \big(\mathcal{T}(gv)\big)(\mathcal{S}v).$$

Therefore, $g : V_0 \to U_0$ is a mapping for which

$$g(v) = \big(\mathcal{T}(gv)\big)(\mathcal{S}v), \quad \forall v \in V_0.$$

We show that g is continuous on V_0. Let $\{v_n\}$ be a sequence in V_0 such that $v_n \to v \in V_0$. Then

$$\begin{aligned}
\|gv_n - gv\| &= \|\big(\mathcal{T}(gv_n)\big)(\mathcal{S}v_n) - \big(\mathcal{T}(gv)\big)(\mathcal{S}v)\| \\
&\le \|\big(\mathcal{T}(gv_n)\big)(\mathcal{S}v_n) - \big(\mathcal{T}(gv)\big)(\mathcal{S}v_n)\| + \|\big(\mathcal{T}(gv)\big)(\mathcal{S}v_n) - \big(\mathcal{T}(gv)\big)(\mathcal{S}v)\| \\
&\le \|\mathcal{T}(gv_n) - \mathcal{T}(gv)\|\|\mathcal{S}v_n\| + \|\mathcal{T}(gv)\|\|\mathcal{S}v_n - \mathcal{S}v\| \\
&\le kM\|gv_n - gv\| + \|\mathcal{T}(gv)\|\|\mathcal{S}v_n - \mathcal{S}v\|,
\end{aligned}$$

and so,

$$\|gv_n - gv\| \le \frac{\|\mathcal{T}(gv)\|}{1 - kM}\|\mathcal{S}v_n - \mathcal{S}v\|.$$

By the fact that $\mathcal{S}$ is continuous, if $n \to \infty$ in above relation, we obtain $gv_n \to gv$, that is, g is continuous.

Next we prove that g is a compact operator. At first note that for any $u \in U_0$ we have

$$\begin{aligned}
\|\mathcal{T}u\| &\le \|\mathcal{T}u'\| + \|\mathcal{T}u - \mathcal{T}u'\| \\
&\le \|\mathcal{T}u'\| + k\|u - u'\| \\
&\le \underbrace{\|\mathcal{T}u'\| + k\operatorname{diam}(U_0)}_{:=\rho},
\end{aligned}$$

where $u' \in U_0$ is an arbitrary and fixed element. Hence, $\|\mathcal{T}u\| \le \rho$ for all $u \in U_0$. Now let $\epsilon > 0$ be given. Since $\mathcal{S}$ is a compact operator, the set $\overline{\mathcal{S}(V_0)}$ is compact and so $\mathcal{S}(V_0)$ is totally bounded. This implies that there exists a finite $(\frac{1-kM}{\rho})\epsilon$-net in V_0, say $E = \{v_1, v_2, \ldots, v_n\}$. That is,

$$\mathcal{S}(V_0) \subseteq \bigcup_{j=1}^{n} \mathcal{N}\big(\mathcal{S}v_j; (\frac{1-kM}{\rho})\epsilon\big).$$

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So, for any $v \in V_0$ there is an element v_j for some $j \in \{1, 2, \dots, n\}$ such that $\|\mathcal{S}v - \mathcal{S}v_j\| < (\frac{1-kM}{\rho})\epsilon$. Thereby,

$$\begin{aligned}
\|gv - gv_j\| &= \|\big(\mathcal{T}(gv)\big)(\mathcal{S}v) - \big(\mathcal{T}(gv_j)\big)(\mathcal{S}v_j)\| \\
&\le \|\big(\mathcal{T}(gv)\big)(\mathcal{S}v) - \big(\mathcal{T}(gv_j)\big)(\mathcal{S}v)\| + \|\big(\mathcal{T}(gv_j)\big)(\mathcal{S}v) - \big(\mathcal{T}(gv_j)\big)(\mathcal{S}v_j)\| \\
&\le \|\mathcal{T}(gv) - \mathcal{T}(gv_j)\|\|\mathcal{S}v\| + \|\mathcal{T}(gv_j)\|\|\mathcal{S}v - \mathcal{S}v_j\| \\
&< kM\|gv - gv_j\| + \rho(\frac{1-kM}{\rho})\epsilon.
\end{aligned}$$

Hence, $\|gv - gv_j\| < \epsilon$ which concludes that

$$g(V_0) \subseteq \bigcup_{j=1}^{n} \mathcal{N}(gv_j; \epsilon),$$

that is, $g(V_0)$ is totally bounded which ensures that $\overline{g(V_0)}$ is a compact set. Then g is a compact operator.

Consider the projection mapping defined in (2). Then

$$\mathcal{P}g(V_0) \subseteq \mathcal{P}(U_0) \subseteq V_0,$$

which deduces that $\mathcal{P}g$ maps V_0 into itself. Continuity of the mapping g on V_0 and the projection mapping $\mathcal{P}$ on U_0 yields that $\mathcal{P}g$ is continuous too. Since $\overline{g(V_0)}$ is a compact set and that $\mathcal{P}|_{U_0}$ is continuous, $\mathcal{P}\big(\overline{g(V_0)}\big)$ is compact. Besides,

$$\overline{(\mathcal{P}g)V_0} = \overline{\mathcal{P}\big(g(V_0)\big)} \subseteq \overline{\mathcal{P}\big(\overline{g(V_0)}\big)} = \mathcal{P}\big(\overline{g(V_0)}\big),$$

which implies that $\mathcal{P}g$ is a compact operator. It now follows from the Schauder's fixed point theorem (Theorem 1.3) that there exists an element $v^\star \in V_0$ for which $\mathcal{P}gv^\star = v^\star$. By the statement (i) of the Proposition 1.6 we have

$$\|gv^\star - v^\star\| = \|gv^\star - \mathcal{P}gv^\star\| = \mathrm{dist}(U, V) = \|\mathcal{P}v^\star - v^\star\|.$$

Because of the fact that the pair (U, V) has the P-property, we must have $gv^\star = \mathcal{P}v^\star$ and so,

$$\|\mathcal{T}(\mathcal{P}v^\star)\mathcal{S}v^\star - v^\star\| = \mathrm{dist}(U, V),$$

and this completes the proof. □

Remark 2.2. *It is worth noticing that if in Theorem 2.1 $U_0 \neq \emptyset$, then we do not need the reflexivity assumption of a Banach space X. Besides, if $U \cap V \neq \emptyset$, then $U_0 = V_0 = U \cap V$, and so the projection mapping $\mathcal{P}$ defined on $U_0 \cup V_0$ is identity. That is, the strict convexity assumption of the Banach algebra X can be dropped. In this case, we obtain a fixed point for the mapping $\mathcal{T}\mathcal{S}$, i.e., there exists a point $u^\star \in U_0$ such that $\mathcal{T}u^\star\mathcal{S}u^\star = u^\star$.*

The next corollary is a main result of [2].

Corollary 2.3. *Let U be a nonempty, bounded, closed and convex subset of a Banach algebra X and let $\mathcal{T}, \mathcal{S} : U \to X$ be two mappings such that*

(i) *There exists a real number $k \in [0, 1)$ such that $\|\mathcal{T}x - \mathcal{T}y\| \le k\|x - y\|$ for any $x, y \in U$;*

(ii) $\mathcal{S}$ *is a continuous and compact operator;*

(iii) $\mathcal{T}(U)\mathcal{S}(U) \subseteq U$.

If $kM < 1$, *then*

$$\exists\, u^\star \in U \text{ s.t. } \mathcal{T}u^\star\mathcal{S}u^\star = u^\star, \tag{4}$$

where $M = \sup\{\mathcal{S}u : u \in U\}$.

Proof. It is sufficient to consider $V = U$ in Theorem 2.1. In this case, $U_0 = V_0 = U$ and the projection mapping $\mathcal{P}$ should be identity on U and so, the result follows from Theorem 2.1. □

Let us illustrate Theorem 2.1 with the next example.

Example 2.4. *Consider the complex plane* $X = \mathbb{C}$ *with the Euclidean norm as a strictly convex Banach algebra and let*

$$U = [0, 2], \quad V = \{v + i : 0 \leq v \leq 1\}.$$

Clearly, $\mathrm{dist}(U, V) = 2$ *and that*

$$U_0 = [0, 1], \quad V_0 = V.$$

In this situation, the projection mapping $\mathcal{P} : V_0 \to U_0$ *is defined by*

$$\mathcal{P}(v + i) = v, \quad \forall v \in [0, 1].$$

Define $\mathcal{T} : U \to X$ *and* $\mathcal{S} : V \to X$ *with*

$$\mathcal{T}(u) = \frac{u+1}{2}, \quad \mathcal{S}(v + i) = \sqrt{v}, \quad \forall (u, v) \in [0, 2] \times [0, 1].$$

Then $\mathcal{T}$ *is contraction with contractive constant* $k = \frac{1}{2}$. *Also,* $\mathcal{S}$ *is a continuous and compact operator. Moreover,*

$$\mathcal{T}(u)\mathcal{S}(v + i) = (\frac{u+1}{2})(\sqrt{v}), \quad \forall (u, v) \in [0, 1] \times [0, 1],$$

which implies that $\mathcal{T}(U_0)\mathcal{S}(V_0) \subseteq U_0$. *It is worth noticing that*

$$M := \sup\{\underbrace{\mathcal{S}(v + i)}_{=\sqrt{v}} : v \in [0, 1]\} = 1,$$

and so, $kM < 1$. *Therefore, from Theorem 2.1 the equation* (3) *has a solution, that is, there exists a point* $v^\star \in V_0$ *such that*

$$\|v^\star - \mathcal{T}(\mathcal{P}v^\star)\mathcal{S}v^\star\| = \mathrm{dist}(U, V),$$

and in this case, $v^\star \in \{i, 1 + i\}$.

Example 2.5. *Let* $K = [0, 1]$ *and* $X = L^\infty(K)$ *with the pointwise multiplication* $(fg)x = f(x)g(x)$ *and with the supremum norm. Then* X *is a Banach algebra which is not strictly convex. Set*

$$U = \Big\{f \in X\ ;\ 0 \leq f \leq \frac{1}{2}\ \text{(a.e.)}\Big\}, \quad V = \Big\{g \in X\ ;\ 0 \leq g \leq i_K\ \text{(a.e.)}\Big\},$$

where i_K denotes the identity mapping on the set K. Obviously, (U,V) is a bounded, closed and convex pair with $\text{dist}(U,V)=0$ *and*

$$U_0 = V_0 = \Big\{ h \in X \ ; \ 0 \le h \le \min\{\frac{1}{2}, i_K\} \ \text{(a.e.)} \Big\}.$$

Define $\mathcal{T} : U \to X$ and $\mathcal{S} : V \to X$ with

$$(\mathcal{T}f)x = \frac{1}{4} \quad \& \quad (\mathcal{S}g)x = x + \int_0^x g(t)dt, \quad \forall x \in K.$$

It is easy to see that $\mathcal{T}$ is contraction and $\mathcal{S}$ is a continuous and compact operator. Also, for any $h_1, h_2 \in U_0$ we have

$$\begin{aligned}
\big(\mathcal{T}h_1\mathcal{S}h_2\big)(x) &= \big(\mathcal{T}h_1\big)(x)\big(\mathcal{S}h_2\big)(x) \\
&= \frac{1}{4}\Big(x + \int_0^x h_2(t)dt\Big) \\
&\le \min\big\{\frac{1}{2}, i_K\big\},
\end{aligned}$$

which shows that $\mathcal{T}h_1\mathcal{S}h_2 \in U_0$ for all $h_1, h_2 \in U_0$ and so, $\mathcal{T}(U_0)\mathcal{S}(U_0) \subseteq U_0$. Furthermore,

$$M := \sup\{\|\mathcal{S}h\| : h \in U_0\} \le 1 + \sup_{x\in K} \int_0^x \frac{1}{2}dt = \frac{3}{2},$$

and so, $kM < 1$ for $k \in (0, \frac{2}{3})$. It now follows from the Remark 2.2 that there exists an element $u^\star \in U_0$ for which $\mathcal{T}u^\star\mathcal{S}u^\star = u^\star$ or equivalently,

$$u^\star(x) = \frac{1}{4}x + \frac{1}{4}\int_0^x u^\star(t)dt.$$

It is worth noticing that the only solution of the above integral equation is $u^\star(x) = e^{\frac{1}{4}x} - 1$.

3 An extension of Schauder's fixed point theorem

In this section we present a generalization of Schauder's fixed point in order to study the existence of a best proximity point. To this end we fix the following notation.

Notation. Let (U,V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X and $\mathcal{T} : U \to V$ be a non-self mapping such that $\mathcal{T}(U_0) \subseteq V_0$. Put $\mathcal{U}_0 := U_0$ and $\mathcal{V}_0 := V_0$ and for all $n \in \mathbb{N} \cup \{0\}$ define

$$\mathcal{V}_{n+1} = \overline{\text{con}}\big(\mathcal{T}\mathcal{U}_n\big), \quad \mathcal{U}_{n+1} = \mathcal{P}(\mathcal{V}_{n+1}),$$

where $\mathcal{P}$ is a projection mapping defined in Proposition 1.6.

Proposition 3.1. *Let (U,V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X and $\mathcal{T} : U \to V$ be a non-self mapping such that $\mathcal{T}(U_0) \subseteq V_0$. Then $\{(\mathcal{U}_n, \mathcal{V}_n)\}_{n\in\mathbb{N}\cup\{0\}}$ is a descending sequence of nonempty, closed and convex pairs in $U_0 \times V_0$. Also,*

$$\text{dist}(\mathcal{U}_n, \mathcal{V}_n) = \text{dist}(U,V), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Moreover, $\mathcal{T}(\mathcal{U}_n) \subseteq \mathcal{V}_n$ for all $n \in \mathbb{N}$.

Proof. Clearly, $\mathcal{V}_n$ is nonempty, closed and convex for any $n \in \mathbb{N} \cup \{0\}$. So, each $\mathcal{U}_n$ is also nonempty. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{U}_n$ such that $u_j \to u \in U_0$. Then for any $j \in \mathbb{N}$ there exists an element $v_j \in \mathcal{V}_n$ for which $u_j = \mathcal{P}v_j$. By this reality that $\mathcal{P}^2|_{V_0} = i_{V_0}$ (statement (iv) of Proposition 1.6), we obtain

$$\mathcal{P}u_j = \mathcal{P}^2(v_j) = v_j, \quad \forall j \in \mathbb{N}.$$

Continuity of $\mathcal{P}$ on U_0 deduces that $\mathcal{P}u_j \to \mathcal{P}u$ and so, $v_j \to \mathcal{P}u$ which concludes that $\mathcal{P}u \in \mathcal{V}_n$. Thus

$$u = \mathcal{P}(\mathcal{P}u) \in \mathcal{P}(\mathcal{V}_n) = \mathcal{U}_n,$$

that is, $\mathcal{U}_n$ is closed. Now assume that $u, u' \in \mathcal{U}_n$. Then there are points $v, v' \in \mathcal{V}_n$ for which $u = \mathcal{P}v$ and $u' = \mathcal{P}v'$. Affinity of the projection mapping $\mathcal{P}$ on V_0 implies that for any $\lambda \in [0,1]$ we have

$$\begin{aligned}\lambda u + (1-\lambda)u' &= \lambda \mathcal{P}v + (1-\lambda)\mathcal{P}v' \\ &= \mathcal{P}\big(\lambda v + (1-\lambda)v'\big) \in \mathcal{P}(\mathcal{V}_n) = \mathcal{U}_n,\end{aligned}$$

which ensures that the set $\mathcal{U}_n$ is convex. Besides,

$$\begin{aligned}&\mathcal{V}_1 = \overline{\text{con}}\big(\mathcal{T}U_0\big) \subseteq \mathcal{V}_0, \\ &\mathcal{U}_1 = \mathcal{P}(\mathcal{V}_1) \subseteq \mathcal{P}(\mathcal{V}_0) \subseteq \mathcal{U}_0,\end{aligned}$$

which concludes that

$$\begin{aligned}&\mathcal{V}_2 = \overline{\text{con}}\big(\mathcal{T}\mathcal{U}_1\big) = \overline{\text{con}}\big(\mathcal{T}(\mathcal{P}\mathcal{V}_1)\big) \subseteq \overline{\text{con}}\big(\mathcal{T}(\mathcal{U}_0)\big) = \mathcal{V}_1, \\ &\mathcal{U}_2 = \mathcal{P}(\mathcal{V}_2) \subseteq \mathcal{P}(\mathcal{V}_1) = \mathcal{U}_1.\end{aligned}$$

Continuing this process and by induction we obtain

$$\begin{aligned}&\mathcal{V}_n = \overline{\text{con}}\big(\mathcal{T}\mathcal{U}_{n-1}\big) = \overline{\text{con}}\big(\mathcal{T}(\mathcal{P}\mathcal{V}_{n-1})\big) \subseteq \overline{\text{con}}\big(\mathcal{T}(\mathcal{U}_{n-2})\big) = \mathcal{V}_{n-1}, \\ &\mathcal{U}_n = \mathcal{P}(\mathcal{V}_n) \subseteq \mathcal{P}(\mathcal{V}_{n-1}) = \mathcal{U}_{n-1} \quad \forall n \in \mathbb{N}.\end{aligned}$$

Hence $\{(\mathcal{U}_n, \mathcal{V}_n)\}_{n\in\mathbb{N}\cup\{0\}}$ is a descending sequence.
Also, we note that $\text{dist}(\mathcal{U}_0, \mathcal{V}_0) = \text{dist}(U, V)$. Let $u_0 \in \mathcal{U}_0$. Since $\mathcal{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and X is strictly convex, there exists a unique element $u_1 \in \mathcal{U}_0$ such that $\|u_1 - \mathcal{T}u_0\| = \text{dist}(U, V)$. In this case, $\mathcal{T}u_0 \in \mathcal{V}_1$ and $\mathcal{P}(\mathcal{T}u_0) = u_1 \in \mathcal{U}_1$ which implies that $\text{dist}(\mathcal{U}_1, \mathcal{V}_1) = \text{dist}(U, V)$. Again by the fact that $\mathcal{T}(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and X is strictly convex, there exists a unique element $u_2 \in \mathcal{U}_0$ for which $\|u_2 - \mathcal{T}u_1\| = \text{dist}(U, V)$. We now have $\mathcal{T}u_1 \in \mathcal{V}_2$ and $\mathcal{P}(\mathcal{T}u_1) = u_2 \in \mathcal{U}_2$ which concludes that $\text{dist}(\mathcal{U}_2, \mathcal{V}_2) = \text{dist}(U, V)$. By a similar argument we can see that $\text{dist}(\mathcal{U}_n, \mathcal{V}_n) = \text{dist}(U, V)$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, for any $n \in \mathbb{N}$ we have

$$\mathcal{T}(\mathcal{U}_n) \subseteq \overline{\text{con}}\big(\mathcal{T}\mathcal{U}_n\big) = \mathcal{V}_{n+1} \subseteq \mathcal{V}_n,$$

and this completes the proof.

□

Let $\sum$ be a family of all nonempty and bounded subsets of a Banach space X. We recall that a function $\mu : \sum \to [0, \infty)$ is called a measure of noncompactness (MNC) if it satisfies the following conditions:
(i) $\mu(A) = 0$ iff A is relatively compact,
(ii) $\mu(A) = \mu(\overline{A})$ for all $A \in \sum$,
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \sum$.

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We mention that just recently, Keyvanloo et al. ([11]) introduced a notion of weighted sequence spaces and constructed a Hausdorff measure of noncompactness in these spaces. They used this notion to obtain the existence of solutions of certain infinite systems of third-order three-point nonhomogeneous boundary value problems.

Definition 3.2. *Let (U,V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X and $\mathcal{T}: U \to V$ be a non-self mapping such that $\mathcal{T}(U_0) \subseteq V_0$. We will say that $\mathcal{T}$ is a generalized condensing non-self mapping whenever*

$$\exists\ n_0 \in \mathbb{N}\ :\ \mu(\mathcal{V}_{n_0}) = 0,$$

where μ is an MNC *on $\sum$.*

We are now in position to state another main result of this article.

Theorem 3.3. *Let (U,V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X and $\mathcal{T}: U \to V$ be a non-self mapping such that $\mathcal{T}(U_0) \subseteq V_0$. If $\mathcal{T}$ is a continuous and generalized condensing non-self mapping, then $\mathcal{T}$ has a best proximity point, that is, there exists a point $u^\star \in U$ such that $\|u^\star - \mathcal{T}u^\star\| = \operatorname{dist}(U,V)$.*

Proof. It follows from Proposition 3.1 that $\{(\mathcal{U}_n, \mathcal{V}_n)\}_{n\in\mathbb{N}\cup\{0\}}$ is a descending sequence of nonempty, closed and convex pairs in X. Since $\mathcal{T}$ is generalized condensing,

$$\exists\ n_0 \in \mathbb{N}\ :\ \mu(\mathcal{V}_{n_0}) = 0,$$

which concludes that $\mathcal{V}_{n_0}$ is relatively compact by the definition of MNC. On the other hand, from Proposition 3.1, $\mathcal{T}(\mathcal{U}_{n_0}) \subseteq \mathcal{V}_{n_0}$ and so $\mathcal{T}$ is a compact and continuous non-self mapping from $\mathcal{U}_{n_0}$ to $\mathcal{V}_{n_0}$. We now have

$$\mathcal{P}\big(\mathcal{T}(\mathcal{U}_{n_0})\big) \subseteq \mathcal{P}(\mathcal{V}_{n_0}) = \mathcal{U}_{n_0},$$

which ensures that $\mathcal{P}\mathcal{T}$ maps $\mathcal{U}_{n_0}$ into itself and that $\mathcal{P}\mathcal{T}$ is a continuous and compact operator. Using Schauder's fixed point theorem, $\mathcal{P}\mathcal{T}$ has a fixed point in $\mathcal{U}_{n_0}$, say $u^\star$. Thus

$$\|u^\star - \mathcal{T}u^\star\| = \|\mathcal{P}\mathcal{T}u^\star - \mathcal{T}u^\star\| = \operatorname{dist}(U,V),$$

and we are finished.

□

Corollary 3.4. *Let (U,V) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X and $\mathcal{T}: U \to V$ be a non-self mapping such that $\mathcal{T}(U_0) \subseteq V_0$. If $\mathcal{T}$ is compact and continuous, then $\mathcal{T}$ has a best proximity point.*

Proof. It is sufficient note that since $\mathcal{T}$ is compact, $\mathcal{T}$ is a generalized condensing non-self mapping with $n_0 = 1$ and the result follows from Theorem 3.3, directly. □

Remark 3.5. *It is worth mentioning that the reflexivity assumption of a Banach space X in Proposition 3.1, Theorem 3.3 and Corollary 3.4 ensures that the proximal pair (U_0, V_0) is nonempty. So, this condition can be removed if we suppose that the set U_0 is nonempty.*

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4 Application to a system of differential equations

In the latest section of this article, we apply the existence of a best proximity point of Corollary 3.4 to find an optimum solution for a system of differential equations. In this order, we fix the following notations:

For $(u_0, v_0) \in \mathbb{R}^2$ let

$$E = \{(u,v) \in \mathbb{R}^2 : |u-u_0| \le 1,\ |v-v_0| \le 1\}.$$

Suppose that $(u,v_1),(u,v_2) \in E$ are such that $v_1 < v_2$ and let f_1 and f_2 be two different real-valued functions on the set E. Consider the following system of differential equations:

$$\begin{cases} \frac{dv}{du} = f_1(u,v)\ ; \quad v(u_0) = v_2, \\ \frac{dv}{du} = f_2(u,v)\ ; \quad v(u_0) = v_1. \end{cases} \tag{5}$$

It is worth noticing that this system does not have a solution. Let X be a Banach space $\mathcal{C}([0,1])$ consists of all continuous real functions defined on $[0,1]$ which is renormed according to

$$\|v\| = \|v\|_2 + \|v\|_\infty, \quad \forall v \in X.$$

Due to presence of $\|.\|_2$, the Banach space $(X, \|.\|)$ is strictly convex. Moreover, it is easy to check that

$$\|v\|_\infty \le \|v\| \le 2\|v\|_\infty, \quad \forall v \in X. \tag{6}$$

Set

$$\begin{aligned} U &= \{\nu \in X : |\nu(u) - v_0| \le 1,\ \nu(u_0) = v_2,\ \nu(u) \ge v_2\}, \\ V &= \{v \in X : |v(u) - v_0| \le 1,\ v(u_0) = v_1,\ v(u) \le v_1\}. \end{aligned}$$

Clearly, (U,V) is a closed and convex pair in X. Let $(\nu, v) \in U \times V$. Then we have $|\nu(u) - v(u)| \ge |v_2 - v_1|$ and so,

$$\begin{aligned} \|\nu - v\|_2 &= \Big[\int_0^1 |\nu(u) - v(u)|^2 du\Big]^{\frac{1}{2}} \\ &\ge \Big[\int_0^1 |v_2 - v_1|^2 du\Big]^{\frac{1}{2}} \\ &= |v_2 - v_1|. \end{aligned}$$

Therefore,

$$\|\nu - v\| = \|\nu - v\|_2 + \|\nu - v\|_\infty \ge 2|v_2 - v_1|.$$

Since $(v_2, v_1) \in U \times V$, we must have

$$\operatorname{dist}(U,V) = 2|v_2 - v_1|,$$

and so, the proximal pair (U_0, V_0) is nonempty. Clearly (U_0, V_0) is a convex pair and by the fact that X is strictly convex, the pair (U,V) has the P-property and hence by Lemma 3.1 of [6], the proximal pair (U_0, V_0) is closed. Define a mapping $\mathcal{T} : U \to X$ by

$$(\mathcal{T}\nu)u = v_1 - \int_{u_0}^{u} \big|f_2\big(x, \nu(x)\big)\big| dx\ ; \quad \nu \in U.$$

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Then $\big(\mathcal{T}\nu\big)(u_0) = v_1$ and $\big(\mathcal{T}\nu\big)(u) \le v_1$. Suppose that $L > 0$ is a common bound of the functions f_1 and f_2 and let

$$\delta < \min\Big\{1,\ \frac{1-|v_1 - v_0|}{L},\ \frac{1-|v_2 - v_0|}{L},\ \frac{|v_2 - v_1|}{4L}\Big\}.$$

Now by considering the strictly convex Banach space $X = \big(\mathcal{C}([0,\delta]), \|.\|\big)$ we obtain

$$\begin{aligned}\big|\big(\mathcal{T}\nu\big)(u) - v_0\big| &= \Big|v_1 - \int_{u_0}^{u} |f_2\big(x, \nu(x)\big)|dx - v_0\Big| \\ &\le |v_1 - v_0| + \int_{u_0}^{u} |f_2\big(x, \nu(x)\big)|dx \\ &\le |v_1 - v_0| + L\underbrace{|u - u_0|}_{<\delta} \le 1.\end{aligned}$$

Thereby, $\mathcal{T}$ maps U to the set V.

Definition 4.1. *We say that $\nu^\star \in U$ is an optimum solution for the system* (5) *whenever*

$$\|\nu^\star - \mathcal{T}\nu^\star\| = \operatorname{dist}(U, V),$$

that is, $\nu^\star$ is a best proximity point for the non-self mapping $\mathcal{T}$.

The next theorem guarantees the existence of an optimum solution of the system (5).

Theorem 4.2. *Under the aforesaid hypothesis and notations of this section, if moreover,*

$$|f_1\big(x, v(x)\big)| + |f_2\big(x, \nu(x)\big)| \le \frac{1}{2}|\nu(x) - v(x)| - |v_2 - v_1|,$$

whenever $\frac{1}{2}|\nu(x) - v(x)| > |v_2 - v_1|$, then the system (5) *has an optimum solution.*

Proof. We first assert that $\mathcal{T}(U_0) \subseteq V_0$. Define a mapping $\mathcal{S} : V \to X$ with

$$\big(\mathcal{S}v\big)u = v_2 + \int_{u_0}^{u} \big|f_1\big(x, v(x)\big)\big|dx\ ; \quad v \in V.$$

In this situation, $(\mathcal{S}v)u_0 = v_2$ and $(\mathcal{S}v)u \ge v_2$. Moreover,

$$\begin{aligned}\big|\big(\mathcal{S}v\big)(u) - v_0\big| &= \Big|v_2 + \int_{u_0}^{u} |f_1\big(x, v(x)\big)|dx - v_0\Big| \\ &\le |v_2 - v_0| + \int_{u_0}^{u} |f_1\big(x, v(x)\big)|dx \\ &\le |v_2 - v_0| + L\underbrace{|u - u_0|}_{<\delta} \le 1,\end{aligned}$$

and so, $\mathcal{S}v \in U$ for any $v \in V$ which concludes that $\mathcal{S}(V) \subseteq U$. Now for any $(\nu, v) \in U \times V$ set

$$\begin{aligned}\Gamma_1 &= \Big\{x \in [u_0, u_0 + \delta]\ ;\ \frac{1}{2}|\nu(x) - v(x)| > |v_2 - v_1|\Big\}, \\ \Gamma_2 &= \Big\{x \in [u_0, u_0 + \delta]\ ;\ \frac{1}{2}|\nu(x) - v(x)| \le |v_2 - v_1|\Big\}.\end{aligned}$$

Then we have

$$
\begin{aligned}
&|(\mathcal{T}\nu)u-(\mathcal{S}v)u|\\
&=\Big|(v_1-\int_{u_0}^{u}\big|f_2\big(x,\nu(x)\big)\big|dx)-(v_2+\int_{u_0}^{u}\big|f_1\big(x,v(x)\big)\big|dx)\Big|\\
&\leq|v_2-v_1|+\int_{u_0}^{u}(\big|f_2\big(x,\nu(x)\big)\big|+\big|f_1\big(x,v(x)\big)\big|)dx\\
&=|v_2-v_1|+\int_{\Gamma_1}(\big|f_2\big(x,\nu(x)\big)\big|+\big|f_1\big(x,v(x)\big)\big|)dx+\int_{\Gamma_2}(\big|f_2\big(x,\nu(x)\big)\big|+\big|f_1\big(x,v(x)\big)\big|)dx\\
&=|v_2-v_1|+\frac{1}{2}\|\nu-v\|_\infty-|v_2-v_1|+\underbrace{2L\delta}_{<\frac{1}{2}|v_2-v_1|}\\
&\leq\frac{1}{2}\|\nu-v\|_\infty+\frac{1}{2}|v_2-v_1|\\
&\leq\frac{1}{2}\|\nu-v\|_\infty+\frac{1}{2}\|\nu-v\|_2\quad(\text{since }|v_2-v_1|\leq\|\nu-v\|_2)\\
&=\frac{1}{2}\|\nu-v\|.
\end{aligned}
$$

This yields that

$$
\|\mathcal{T}\nu-\mathcal{S}v\|_\infty\leq\frac{1}{2}\|\nu-v\|. \tag{7}
$$

Hence by using the relations (6) and (7) we obtain

$$
\|\mathcal{T}\nu-\mathcal{S}v\|\leq 2\|\mathcal{T}\nu-\mathcal{S}v\|_\infty\leq\|\nu-v\|,\quad\forall(\nu,v)\in U\times V. \tag{8}
$$

Now if $\nu_1\in U_0$, then there exists an element $v_1\in V$ such that $\|\nu_1-v_1\|=\operatorname{dist}(U,V)$. Thus from (8) we have

$$
\|\mathcal{T}\nu_1-\mathcal{S}v_1\|\leq\|\nu_1-v_1\|=\operatorname{dist}(U,V),
$$

which implies that $\mathcal{T}\nu_1\in V_0$. Thereby $\mathcal{T}(U_0)\subseteq V_0$.
It is obvious that the mapping $\mathcal{T}$ is continuous. Also, if $\nu\in U$, then

$$
\begin{aligned}
|(\mathcal{T}\nu)u|&=\Big|v_1-\int_{u_0}^{u}\big|f_2\big(x,\nu(x)\big)\big|dx\Big|\\
&\leq|v_1|+\int_{u_0}^{u}\big|f_2\big(x,\nu(x)\big)\big|dx\\
&\leq|v_1|+L\delta,
\end{aligned}
$$

and so the family of $\{\mathcal{T}\nu\}_{\nu\in U}$ is bounded. On the other hand, if $\nu\in U$ and $u_1,u_2\in[u_0,u_0+\delta]$ with $u_1<u_2$ then we have

$$
\begin{aligned}
&|(\mathcal{T}\nu)u_1-(\mathcal{T}\nu)u_2|\\
&=\Big|v_1-\int_{u_0}^{u_1}\big|f_2\big(x,\nu(x)\big)\big|dx-v_1+\int_{u_0}^{u_2}\big|f_2\big(x,\nu(x)\big)\big|dx\Big|\\
&=\int_{u_1}^{u_2}\big|f_2\big(x,\nu(x)\big)\big|dx\\
&\leq L|u_2-u_1|.
\end{aligned}
$$

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Thus $\{\mathcal{T}\nu\}_{\nu\in U}$ is a family of equicontinuous functions. It now follows from the Arzela-Ascoli's Theorem that $\overline{\mathcal{T}(U)}$ is compact which means that $\mathcal{T}$ is a compact operator. Therefore by Corollary 3.4, $\mathcal{T}$ has a best proximity point and this point is an optimum solution of the system (5).

□

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Conflicts of interests:
The author declares that he has no competing interests.

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