ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

REMARKS ON THE STANLEY DEPTH AND HILBERT DEPTH OF MONOMIAL IDEALS WITH LINEAR QUOTIENTS

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ABSTRACT. We prove that if I is a monomial ideal with linear quotients in a ring of polynomials S in n indeterminates and $\operatorname{depth}(S/I) = n-2$, then $\operatorname{sdepth}(S/I) = n-2$ and, if I is squarefree, $\operatorname{hdepth}(S/I) = n-2$. Also, we prove that $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$ for a monomial ideal I with linear quotients which satisfies certain technical conditions.

1. Introduction

Let K be a field and let $S = K[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in n variables. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum

$$\mathscr{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

as K-vector spaces, where $m_i \in M$ are homogeneous, $Z_i \subset \{x_1, \ldots, x_n\}$ such that $m_i K[Z_i]$ is a free $K[Z_i]$-module; $m_i K[Z_i]$ is called a *Stanley subspace* of M. We define $\operatorname{sdepth}(\mathscr{D}) = \min_{i=1}^{r} |Z_i|$ and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D}) \,:\, \mathscr{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M. Herzog Vlădoiu and Zheng [8] proved that this invariant can be computed in a finite number of steps, when $M = I/J$, where $J \subset I \subset S$ are monomial ideals.

We say that the multigraded module M satisfies the *Stanley inequality* if

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M).$$

Stanley conjectured in [13] that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$, for any $\mathbb{Z}^n$-graded S-module M. In fact, in this form, the conjecture was stated by Apel in [1]. The Stanley conjecture was disproved by Duval et. al [6], in the case $M = I/J$, where $(0) \neq J \subset I \subset S$ are monomial ideals, but it remains open in the case $M = I$, a monomial ideal.

A monomial ideal $I \subset S$ has *linear quotients*, if there exists $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_m$, an ordering on the minimal set of generators $G(I)$, such that, for any $2 \leq j \leq m$, the ideal $(u_1, \ldots, u_{j-1}) : u_j$ is generated by variables.

Given a monomial ideal with linear quotients $I \subset S$, Soleyman Jahan [11] noted that I satisfies the Stanley inequality, i.e.

$$\operatorname{sdepth}(I) \geq \operatorname{depth}(I).$$

However, a similar result for S/I, if true, is more difficult to prove, only some particular cases being known. For instance, Seyed Fakhari [7] proved the inequality

$$\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$$

for weakly polymatroidal ideals $I \subset S$, which are monomial ideals with linear quotients.

2020 *Mathematics Subject Classification.* 05E40; 13A15; 13C15; 13P10.
Key words and phrases. Stanley depth; Hilbert depth; depth; monomial ideals; linear quotients.

In Theorem 2.4, we prove that if $I \subset S$ is a monomial ideal with linear quotients with $\operatorname{depth}(S/I) = n-2$, then $\operatorname{sdepth}(S/I) = n-2$. In Theorem 2.6, we prove that if $I \subset S$ is a monomial ideal with linear quotients which has a Stanley decomposition which satisfies certain conditions, then $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$. Also, we conjecture that for any monomial ideal $I \subset S$ with linear quotients, there is a variable x_i such that $\operatorname{depth}(S/(I,x_i)) \geq \operatorname{depth}(S/I)$ and $\operatorname{sdepth}(S/(I,x_i)) \leq \operatorname{sdepth}(S/I)$. In Theorem 2.12 we prove that if this conjecture is true, then $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$, for any monomial ideal $I \subset S$ with linear quotients.

Given a finitely graded S-module M, its Hilbert depth is

$$\operatorname{hdepth}(M) = \max\left\{ r \ : \ \begin{array}{c} \text{There exists a f.g. graded } S\text{-module } N \\ \text{with } H_M(t)=H_N(t) \text{ and } \operatorname{depth}(N)=r \end{array} \right\}.$$

It is well known that $\operatorname{hdepth}(M) \geq \operatorname{sdepth}(M)$. See [3] for further details.

Let $0 \subset I \subsetneq J \subset S$ be two squarefree monomial ideals. For any $0 \leq j \leq n$, we let $\alpha_j(J/I)$ to be the number of squarefree monomials $u \in S$ of degree j such that $u \in J \setminus I$. (In particular, $\alpha_j(I)$ is the number of squarefree monomials of degree j which belong to I and $\alpha_j(S/I) = \binom{n}{j} - \alpha_j(I)$ is the number of squarefree monomials of degree j which do not belong to I.)

Also, for $0 \leq k \leq d \leq n$, we let

$$\beta_k^d(J/I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \alpha_j(J/I). \tag{1.1}$$

(In particular, $\beta_k^d(S/I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \alpha_j(S/I)$ and $\beta_k^d(I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \alpha_j(I)$.)

From (1.1), using an inversion formula, it follows that

$$\alpha_k(J/I) = \sum_{j=0}^{k} \binom{d-j}{k-j} \beta_j^d(J/I) \text{ for all } 0 \leq k \leq d \leq n. \tag{1.2}$$

With the above notations, we proved in [2, Theorem 2.4] that

$$\operatorname{hdepth}(J/I) = \max\{d \ : \ \beta_k^d(J/I) \geq 0 \text{ for all } 0 \leq k \leq d\}.$$

If $I \subset S$ is a proper squarefree monomial ideal, we claim that

$$\operatorname{hdepth}(S/I) \leq \max\{k \ : \ \alpha_k(S/I) > 0\}. \tag{1.3}$$

Note that $\alpha_n(S/I) = 0$, since $x_1 \cdots x_n \in I$, and thus $m := \max\{k \ : \ \alpha_k(S/I) > 0\} < n$. From (1.3) it follows that

$$\alpha_{m+1}(S/I) = \sum_{j=0}^{m+1} \beta_j^{m+1}(S/I).$$

Since $I \neq S$ it follows that $1 \notin I$ and thus $\beta_0^{m+1}(S/I) = \alpha_0(S/I) = 1$. The above identity implies that there exists some $1 \leq k \leq m+1$ with $\beta_k^{m+1}(S/I) < 0$ and therefore $\operatorname{hdepth}(S/I) \leq m$, as required.

Also, we will make use of the well known fact that

$$\operatorname{hdepth}(J/I) \geq \operatorname{sdepth}(J/I). \tag{1.4}$$

In Section 3 of our paper we study the Hilbert depth of S/I, where I is a squarefree monomial ideal with linear quotients. In Proposition 3.2 we compute the numbers $\beta_k^d(I)$'s and $\beta_k^d(S/I)$'s. In Corollary 3.3, we express these numbers in combinatorial terms, thus showing the difficulty in finding explicit formulas for $\operatorname{hdepth}(I)$ and $\operatorname{hdepth}(S/I)$.

The main result of this section is Theorem 3.4, in which we show that if I is a squarefree monomial ideal with linear quotients with $\operatorname{depth}(S/I) = n-2$ then

$$\operatorname{hdepth}(S/I) = \operatorname{sdepth}(S/I) = \operatorname{depth}(S/I) = n-2.$$

2. Main results

Let $I \subset S$ be a monomial ideal and let $G(I)$ be the set of minimal monomial generators of I. We recall that I has *linear quotients*, if there exists a linear order $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_m$ on $G(I)$, such that for every $2 \leq j \leq m$, the ideal $(u_1,\ldots,u_{j-1}) : u_j$ is generated by a subset of n_j variables.

We let $I_j := (u_1,\ldots,u_j)$, for $1 \leq j \leq m$.

Let $Z_1 = \{x_1,\ldots,x_n\}$ and $Z_j = \{x_i \; : \; x_i \notin (I_{j-1} : u_j)\}$ for $2 \leq j \leq m$.

Note that, for any $2 \leq j \leq m$, we have

$$I_j/I_{j-1} = u_j(S/(I_{j-1} : u_j)) = u_jK[Z_j].$$

Hence the ideal I has the Stanley decomposition

$$I = u_1K[Z_1] \oplus u_2K[Z_2] \oplus \cdots \oplus u_mK[Z_m]. \tag{2.1}$$

According to [10, Corollary 2.7], the projective dimension of S/I is

$$\operatorname{pd}(S/I) = \max\{n_j \; : \; 2 \leq j \leq m\} + 1.$$

Hence, Ausländer-Buchsbaum formula implies that

$$\operatorname{depth}(S/I) = n - \max\{n_j : 2 \leq j \leq m\} - 1 = \min\{n - n_j : 2 \leq j \leq m\} - 1 = \min\{|Z_j| : 2 \leq j \leq m\} - 1. \tag{2.2}$$

Note that, (2.1) and (2.2) implies $\operatorname{sdepth} I \geq \operatorname{depth} I$, a fact which was proved in [11]. We recall the following results:

Proposition 2.1. *Let $I \subset S$ be a monomial ideal and $u \in S \setminus I$ a monomial. Then:*

(1) $\operatorname{depth}(S/(I : u)) \geq \operatorname{depth}(S/I)$. *(*[9, Corollary 1.3]*)*

(2) $\operatorname{sdepth}(S/(I : u)) \geq \operatorname{sdepth}(S/I)$. *(*[5, Proposition 2.7(2)]*)*

Proposition 2.2. *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded S-modules. Then* $\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}$. *(*[9, Lemma 2.2]*)*

Note that, a proper monomial ideal $I \subset S$ is principal if and only if $\operatorname{depth}(S/I) = n-1$ if and only if $\operatorname{sdepth}(S/I) = n-1$.

Lemma 2.3. *Let $I \subset S$ be a monomial ideal with linear quotients with* $\operatorname{depth}(S/I) = n-2$. *Then there exists some $i \in [n]$ and a monomial $u \in G(I)$ such that $(I,x_i) = (u,x_i)$ and $(I : x_i)$ has linear quotients.*

Proof. If $n = 2$ and $I = S$ then $u = 1 \in G(I)$ and the assertion is obvious. Hence, we may assume that I is proper.

First, note that I is not principal. Since I has linear quotients, we can assume that $G(I) = \{u_1,\ldots,u_m\}$ such that $((u_1,\ldots,u_{j-1}) : u_j)$ is generated by variables, for every $2 \leq j \leq m$. We consider the decomposition (2.1), that is

$$I = u_1K[Z_1] \oplus u_2K[Z_2] \oplus \cdots \oplus u_mK[Z_m],$$

where $Z_1 = \{x_1,\ldots,x_n\}$ and Z_j is the set of variables which do not belong to $((u_1,\ldots,u_{j-1}) : u_j)$, for $2 \leq j \leq m$. From (2.2), it follows that $|Z_j| = n-1$ for $2 \leq j \leq m$. Since $((u_1,\ldots,u_{j-1}) : u_j) = (x_i \; : \; x_i \notin Z_j)$, it follows that for any $2 \leq j \leq m$ we have

$$(u_1,\ldots,u_{j-1}) \cap u_jK[Z_j] = \{0\}. \tag{2.3}$$

We assume, by contradiction, that for any $i \in [n]$ there exists $k_i > \ell_i \in [m]$ such that $x_i \nmid u_{\ell_i}$. We claim that $x_i \in Z_{k_i}$. Indeed, otherwise $Z_{k_i} = \{x_1,\ldots,x_n\} \setminus \{x_i\}$ and therefore $u_{k_i}u_{\ell_i} \in u_{\ell_i}S \cap u_{k_i}K[Z_{k_i}]$, a contradiction to (2.3) for $j = k_i$.

Without any loss of generality, we can assume $k_n = \max\{k_i : i \in [n]\}$. Since $x_n \in Z_{k_n}$, it follows that $Z_{k_n} = \{x_1,\dots,x_n\} \setminus \{x_t\}$ for some $t \leq n-1$. Since $x_t \in Z_{k_t}$ it follows that $k_t < k_n$ and, moreover, $x_t \nmid u_{k_t}$, that is $u_{k_t} \in K[Z_{k_n}]$. Therefore $u_{k_t}u_{k_n} \in u_{k_t}S \cap u_{k_n}K[Z_{k_n}]$, a contradiction to (2.3) for $j = k_n$.

Thus, that there exists $i \in [n]$ such that for any $k_i > \ell_i \in [m]$, $x_i \mid u_{\ell_i}$. It implies that $x_i \mid u_j$ for $j = 1,\dots,m-1$. It follows that $(I : x_i) = (u'_1,\dots,u'_m)$ where $u'_j = u_j/x_i$ for $j = 1,\dots,m-1$ and $u'_m = u_m$ if $x_i \nmid u_m$ and $u'_m = u_m/x_i$ if $x_i \mid u_m$. It is clear that $\{u'_1,\dots,u'_{m-1}\} \subset G(I : x_i)$ and

$$((u'_1,\dots,u'_{j-1}) : u'_j) = ((u_1,\dots,u_{j-1}) : u_j) \text{ for all } 2 \leq j \leq m-1. \tag{2.4}$$

We have $u'_m \in (u_m : x_i) \subseteq (I : x_i)$. If $u'_m \notin G(I : x_i)$ then $G(I : x_i) = \{u'_1,\dots,u'_{m-1}\}$ and, from (2.4), it follows that $(I : x_i)$ has linear quotients. On the other hand, assume $u'_m \in G(I : x_i)$ then we claim that $x_i \mid u_m$, hence $((u'_1,\dots,u'_{m-1}) : u'_m) = ((u_1,\dots,u_{m-1}) : u_m)$ and, again, from (2.4), it follows that $(I : x_i)$ has linear quotients. Indeed, otherwise, $u'_m = u_m \in G(I : x_i)$ and $G(I : x_i) = \{u'_1,\dots,u'_{m-1},u_m\}$, then for $1 \leq j \leq m-1$ there exists $\ell_j \in [m] \setminus [i]$ such that $x_{\ell_j} \mid u'_j$ and $x_{\ell_j} \nmid u_m$, and thus $x_{\ell_j}, x_i \in \operatorname{supp}(u_i) \setminus \operatorname{supp}(u_m)$, which contradicts that $((u_1,\dots,u_{m-1}) : u_m)$ is generated by variables. □

Theorem 2.4. *Let $I \subset S$ be a monomial ideal with linear quotients. If $\operatorname{depth}(S/I) = n-2$, then $\operatorname{sdepth}(S/I) = n-2$.*

Proof. If $n = 2$ then $S/I = 0$ and there is nothing to prove, so we may assume $n \geq 3$ and I is proper with $G(I) = \{u_1,\dots,u_m\}$ for some $m \geq 2$. We use induction on m and $d = \sum_{j=1}^m \deg(u_i)$. If $m = 2$, then from [4, Proposition 1.6] it follows that $\operatorname{sdepth}(S/I) = n-2$. If $d = 2$, then I is generated by two variables and there is nothing to prove.

Assume $m > 2$ and $d > 2$. According to Lemma 2.3, there exist $i \in [n]$ such that $(I, x_i) = (u_m, x_i)$. Since $(I, x_i) = (u_m, x_i)$, from [4, Proposition 1.2] it follows that $\operatorname{sdepth}(S/(I, x_i)) \geq n-2$. If $(I : x_i)$ is principal, then $\operatorname{sdepth}(S/(I : x_i)) = \operatorname{depth}(S/(I : x_i)) = n-1$.

Assume that $(I : x_i)$ is not principal. We have that $\operatorname{depth}(S/(I : x_i)) \leq n-2$. On the other hand, by Proposition 2.1(1) we have $\operatorname{depth}(S/(I : x_i)) \geq \operatorname{depth}(S/I) = n-2$ and thus $\operatorname{depth}(S/(I : x_i)) = n-2$. From the proof of Lemma 2.3, we have $G(I : x_i) \subset \{u_1/x_i,\dots,u_{m-1}/x_i,u_m\}$. It follows that

$$d' := \sum_{u \in G(I:x_i)} \deg(u) < d,$$

thus, by induction hypothesis, we have $\operatorname{sdepth}(S/(I : x_i)) = n-2$. In both cases,

$$\operatorname{sdepth}(S/(I : x_i)) \geq n-2.$$

From Proposition 2.2 and the short exact sequence

$$0 \to S/(I : x_i) \to S/I \to S/(I, x_i) \to 0,$$

it follows that $\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/(I : x_i)), \operatorname{sdepth}(S/(I, x_i))\} \geq n-2$. Since I is not principal, it follows that $\operatorname{sdepth}(S/I) = n-2$, as required. □

Lemma 2.5. *Let $I \subset S$ be a monomial ideal and $u \in S$ a monomial with $(I : u) = (x_1,\dots,x_m)$. Assume that S/I has a Stanley decomposition*

$$\mathscr{D} : S/I = \bigoplus_{i=1}^r v_i K[Z_i], \tag{2.5}$$

such that there exists i_0 with $Z_{i_0} = \{x_{m+1},\dots,x_n\}$ and $v_{i_0} \mid u$. Then:

$$\operatorname{sdepth}(S/(I, u)) \geq \min\{\operatorname{sdepth}(\mathscr{D}), n-m-1\}.$$

Proof. If $\operatorname{sdepth}(S/I)=0$ or $m=n-1$, then there is nothing to prove. We assume that $\operatorname{sdepth}(S/I)\geq 1$ and $m\leq n-2$. Since $S/(I:u)=S/(x_1,\dots,x_m)\cong K[x_{m+1},\dots,x_n]$, from the short exact sequence

$$0\longrightarrow S/(I:u)\xrightarrow{\cdot u} S/I\longrightarrow S/(I,u)\longrightarrow 0,$$

it follows that we have the K-vector spaces isomorphism

$$S/I\cong S/(I,u)\oplus uK[x_{m+1},\dots,x_n]. \tag{2.6}$$

From our assumption, $uK[x_{m+1},\dots,x_n]=uK[Z_{i_0}]\subset v_{i_0}K[Z_{i_0}]$. Hence, from (2.5) and (2.6) it follows that

$$S/(I,u)\cong\left(\bigoplus_{i\neq i_0}v_iK[Z_i]\right)\oplus\frac{v_{i_0}K[Z_{i_0}]}{uK[Z_{i_0}]}\cong\left(\bigoplus_{i\neq i_0}v_iK[Z_i]\right)\oplus\frac{K[x_{m+1},\dots,x_n]}{w_0K[x_{m+1},\dots,x_n]}, \tag{2.7}$$

where $w_0=\frac{u}{v_{i_0}}$. On the other hand, $\operatorname{sdepth}\left(\frac{K[x_{m+1},\dots,x_n]}{w_0K[x_{m+1},\dots,x_n]}\right)=n-m-1$. Hence (2.7) and Proposition 2.2 yields the required conclusion. □

Theorem 2.6. *Let $I\subset S$ be a monomial ideal with linear quotients, $G(I)=\{u_1,\dots,u_m\}$. Let $I_j=(u_1,\dots,u_j)$ for $1\leq j\leq m$, such that $(I_{j-1}:u_j)=(\{x_1,\dots,x_n\}\setminus Z_j)$, where $Z_j\subset\{x_1,\dots,x_n\}$, for all $2\leq j\leq m$.*

We assume that for any $2\leq j\leq m$, there exists a Stanley decomposition $\mathscr{D}_{j-1}$ of S/I_{j-1} such that $\operatorname{sdepth}(\mathscr{D}_{j-1})=\operatorname{sdepth}(S/I_{j-1})$ and there exists a Stanley subspace $w_{j-1}K[W_{j-1}]$ of $\mathscr{D}_{j-1}$ with $w_{j-1}\mid u_j$ and $W_{j-1}=Z_j$.

Then $\operatorname{sdepth}(S/I)\geq\operatorname{depth}(S/I)$.

Proof. From the hypothesis and Lemma 2.5, we have that

$$\begin{aligned}\operatorname{sdepth}(S/I_j)&=\operatorname{sdepth}(S/(I_{j-1},u_j))\geq\min\{\operatorname{sdepth}(\mathscr{D}_{j-1}),n-n_i-1\}\\ &=\min\{\operatorname{sdepth}(S/I_{j-1}),n-n_j-1\}\},\text{ for all }2\leq j\leq m,\end{aligned} \tag{2.8}$$

where $n_j=n-|Z_j|$, $1\leq j\leq m$. On the other hand, according to (2.2),

$$\operatorname{depth}(S/I)=\min_{j=2}^{m}\{n-n_j-1\}. \tag{2.9}$$

Since $\operatorname{sdepth}(S/I_1)=\operatorname{depth}(S/I_1)=n-1$, by applying repeatedly (2.8) we deduce that

$$\operatorname{sdepth}(S/I)=\operatorname{sdepth}(S/I_m)\geq\min_{j=2}^{m}\{n-n_j-1\}.$$

Hence, from (2.9) we get the required conclusion. □

Example 2.7. Let $I=(x_1^2,x_1x_2^2,x_1x_2x_3^2)\subset S=K[x_1,x_2,x_3,x_4]$. Let $u_1=x_1^2$, $u_2=x_1x_2^2$ and $u_3=x_1x_2x_3^2$. Since $((u_1):u_2)=(x_1)$ and $((u_1,u_2):u_3)=(x_1,x_2)$, it follows that I has linear quotients with repect to the order $u_1\leqslant u_2\leqslant u_3$. Moreover,

$$I=u_1K[Z_1]\oplus u_2K[Z_2]\oplus u_3K[Z_3]=x_1^2K[x_1,x_2,x_3,x_4]\oplus x_1x_2^2K[x_2,x_3,x_4]\oplus x_1x_2x_3^2K[x_3,x_4].$$

Let $I_1=(u_1)$ and $I_2=(u_1,u_2)$. We consider the Stanley decomposition

$$\mathscr{D}_1:\ S/I_1=K[x_2,x_3,x_4]\oplus x_1K[x_2,x_3,x_4],$$

of S/I_1 with $\operatorname{sdepth}(\mathscr{D}_1)=\operatorname{sdepth}(S/I_1)=3$. Let $w_1=x_1$ and $W_1=\{x_2,x_3,x_4\}$. Clearly, $W_1=Z_2$ and $w_1\mid u_2$. As in the proof of Lemma 2.5, we obtain the Stanley decomposition

$$\mathscr{D}_2:\ S/I_2=K[x_2,x_3,x_4]\oplus\frac{x_1K[x_2,x_3,x_4]}{x_1x_2^2K[x_2,x_3,x_4]}=K[x_2,x_3,x_4]\oplus x_1K[x_3,x_4]\oplus x_1x_2K[x_3,x_4]$$

of S/I_2 with $\operatorname{sdepth}(\mathscr{D}_2) = \operatorname{sdepth}(S/I_2) = 2$.

Let $w_2 = x_1x_2$ and $W_2 = \{x_3, x_4\}$. Clearly, $W_2 = Z_3$ and $w_2 \mid u_3$. Hence, according to Theorem 2.6, $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I) = 1$. Note that

$$\mathscr{D} \,:\, S/I = K[x_2,x_3,x_4] \oplus x_1K[x_3,x_4] \oplus x_1x_2K[x_4] \oplus x_1x_2x_3K[x_4],$$

is a Stanley decomposition of S/I with $\operatorname{sdepth}(\mathscr{D}) = 1$ and thus $\operatorname{sdepth}(S/I) \geq 1$.

On the other hand, since (x_1,x_2,x_3) is an associated prime to S/I, it follows that $\operatorname{sdepth}(S/I) \leq 1$ and thus $\operatorname{sdepth}(S/I) = 1$. Finally, note that

$$\operatorname{depth}(S/I_2) = 2,\ (I_2,x_1) = (x_1) \text{ and } (I_2 : x_1) = (x_1, x_2^2).$$

In particular, we have $\operatorname{sdepth}(S/(I_2 : x_1)) = \operatorname{depth}(S/(I_2 : x_1)) = 2$, while $\operatorname{sdepth}(S/(I_2,x_1)) = \operatorname{depth}(S/(I_2,x_1)) = 3$.

We propose the following conjecture:

Conjecture 2.8. *If $I \subset S$ is a proper monomial ideal with linear quotients, then there exists $i \in [n]$ such that* $\operatorname{depth}(S/(I,x_i)) \geq \operatorname{depth}(S/I)$.

The following result is well know in literature. However, in order of completeness, we give a proof.

Lemma 2.9. *Let $I \subset S$ be a monomial ideal with linear quotients and x_i a variable. Then (x_i,I) has linear quotients. Moreover, if $S' = K[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]$, then $(x_i,I) = (x_i,J)$, where $J \subset S'$ is a monomial ideal with linear quotients.*

Proof. We consider the order $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_m$ on $G(I)$, such that, for every $2 \leq j \leq m$, the ideal $(I_{j-1} : u_j)$ is generated by a nonempty subset $\bar{Z}_j$ of variables. We assume that $u_{j_1} \leqslant u_{j_2} \leqslant \cdots \leqslant u_{j_p}$ are the minimal monomial generators of I which are not multiple of x_i. We have that $((x_i) : u_{j_1}) = (x_i)$. Also, for $2 \leq k \leq p$, we claim that

$$((x_i,u_{j_1},\ldots,u_{j_{k-1}}) : u_{j_k}) = (x_i, \bar{Z}_{j_k}). \tag{2.10}$$

Indeed, since $((u_1,\ldots,u_{j_k-1}) : u_{j_k}) = (\bar{Z}_{j_k})$ and $x_iu_{j_k} \in (x_i,u_{j_1},\ldots,u_{j_{k-1}})$ it follows that $(x_i,\bar{Z}_{j_k}) \subset ((x_i,u_{j_1},\ldots,u_{j_{k-1}}) : u_{j_k})$. Conversely, assume that $v \in S$ is a monomial with $vu_{j_k} \in (x_i,u_{j_1},\ldots,u_{j_{k-1}}) = (x_i,u_1,\ldots,u_{j_k-1})$. If $x_i \nmid v$, then $vu_{j_k} \in (u_1,\ldots,u_{j_k-1})$, hence $v \in (\bar{Z}_{j_k})$. If $x_i \mid v$, then $v \in (x_i,\bar{Z}_{j_k})$. Hence the claim (2.10) is true and therefore (x_i,I) has linear quotients. Now, let $J = (u_{j_1},\ldots,u_{j_p})$. For any $2 \leq k \leq p$, we have that

$$((u_{j_1},\ldots,u_{j_{k-1}}) : u_{j_k}) \subset ((u_1,\ldots,u_{j_k-1}) : u_{j_k}) = (\bar{Z}_{j_k}). \tag{2.11}$$

From (2.10) and (2.11), one can easily deduce that $((u_{j_1},\ldots,u_{j_{k-1}}) : u_{j_k}) = (\bar{Z}_{j_k} \setminus \{x_i\})$. Hence, J has linear quotients. □

Remark 2.10. Let $I \subset S$ be a monomial ideal with linear quotients, $G(I) = \{u_1,\ldots,u_m\}$, $I_j = (u_1,\ldots,u_j)$ for $1 \leq j \leq m$, such that $(I_{j-1} : u_j) = (\{x_1,\ldots,x_n\} \setminus Z_j)$, where $Z_j \subset \{x_1,\ldots,x_n\}$, for all $2 \leq j \leq m$. I has the Stanley decomposition:

$$I = u_1K[Z_1] \oplus u_2K[Z_2] \oplus \cdots \oplus u_mK[Z_m],$$

where $Z_1 = \{x_1,\ldots,x_n\}$. We have that

$$\operatorname{depth}(S/I) = n-s-1, \text{ where } n-s = \min\{|Z_j| \,:\, 1 \leq j \leq m\}.$$

We claim that Conjecture 2.8 is equivalent to the fact that there exists $i \in [n]$ such that there is no $1 \leq j \leq m$ with $x_i \nmid u_j$, $x_i \in Z_j$ and $|Z_j| = n-s$. Indeed, with the notations of Lemma 2.9, if there is some u_{j_k} with $x_i \nmid u_{j_k}$ and $x_i \in Z_{j_k}$ then $u_{j_k}K[Z_{j_k} \setminus \{x_i\}]$ is a subspace in the decomposition of the ideal with linear quotients $J \subset S' = K[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]$ and thus

$$\operatorname{depth}(S/(I,x_i)) = \operatorname{depth}(S'/J) \leq (n-1)-s-1 = n-s-2 < \operatorname{depth}(S/I).$$

The converse is similar.

We propose a stronger form of Conjecture 2.8.

Conjecture 2.11. *If $I \subset S$ is a proper monomial ideal with linear quotients, then there exists $i \in [n]$ such that:*

i) $\operatorname{depth}(S/(I,x_i)) \geq \operatorname{depth}(S/I)$ *and*
ii) $\operatorname{sdepth}(S/(I,x_i)) \leq \operatorname{sdepth}(S/I)$.

Note that, if x_i is a minimal generator of I, then conditions i) and ii) from Conjecture 2.11 are trivial.

Theorem 2.12. *If Conjecture 2.11 is true and $I \subset S$ is a proper monomial ideal with linear quotients, then* $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$.

Proof. We use induction on $n \geq 1$. If $n=1$ then there is nothing to prove. Assume $n \geq 2$. Let $I \subset S$ be a monomial ideal with linear quotients and let $i \in [n]$ such that $\operatorname{depth}(S/(I,x_i)) \geq \operatorname{depth}(S/I)$ and $\operatorname{sdepth}(S/(x_i,I)) \leq \operatorname{sdepth}(S/I)$. We consider the short exact sequence

$$0 \to \frac{S}{(I:x_i)} \to \frac{S}{I} \to \frac{S}{(I,x_i)} \to 0. \tag{2.12}$$

Let $S' := K[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]$. According to Lemma 2.9, $(x_i,I) = (x_i,J)$ where $J \subset S'$ is a monomial ideal with linear quotients. Note that:

$$\operatorname{sdepth}(S/(x_i,I)) = \operatorname{sdepth}(S/(x_i,J)) = \operatorname{sdepth}(S'/J) \text{ and } \operatorname{depth}(S/(I,x_i)) = \operatorname{depth}(S'/J).$$

From the induction hypothesis, we have $\operatorname{sdepth}(S'/J) \geq \operatorname{depth}(S'/J)$. It follows that:

$$\begin{aligned}\operatorname{sdepth}(S/I) &\geq \operatorname{sdepth}(S/(I,x_i)) = \operatorname{sdepth}(S'/J)\\ &\geq \operatorname{depth}(S'/J) = \operatorname{depth}(S/(I,x_i)) \geq \operatorname{depth}(S/I),\end{aligned}$$

as required. □

Remark 2.13. Note that, if $I \subset S$ has linear quotients, then $(I:x_i)$ has not necessarily the same property. For example, the ideal $I = (x_1x_2, x_2x_3x_4, x_3x_4x_5) \subset K[x_1,\ldots,x_5]$ has linear quotients, but $(I:x_5) = (x_1x_2, x_3x_4)$ has not. Henceforth, in the proof of Theorem 2.6, we cannot argue, inductively, that $\operatorname{sdepth}(S/(I:x_i)) \geq \operatorname{depth}(S/(I:x_i))$.

3. Remarks on the Hilbert depth

Let $I = (u_1,\ldots,u_m) \subset S$ be a proper squarefree monomial with linear quotients, where $(u_1,\ldots,u_{i-1}) : u_i$ is generated by variables for any $2 \leq i \leq m$. As we seen in the previous section, I has a decomposition

$$I = u_1K[Z_1] \oplus u_2K[Z_2] \oplus \cdots \oplus u_mK[Z_m]. \tag{3.1}$$

Moreover, since I is squarefree, $Z_1 = \{x_1,\ldots,x_n\}$ and, for $2 \leq i \leq m$, Z_i consists in the variables which are not in $(u_1,\ldots,u_{i-1}) : u_i$, it follows that $\operatorname{supp}(u_i) \subset Z_i$ for all $1 \leq i \leq m$. Therefore, if we denote $d_i = \deg(u_i)$ and $n_i = |Z_i|$, then $d_i \leq n_i$, for all $1 \leq i \leq m$.

We use the convention $\binom{r}{s} = 0$ for $s < 0$.

Lemma 3.1. *With the above notations, we have that:*

(1) $\alpha_j(I) = \sum\limits_{i=1}^{m} \binom{n_i-d_i}{j-d_i}$ *for all* $0 \leq j \leq n$.

(2) $\alpha_j(S/I) = \binom{n}{j} - \sum\limits_{i=1}^{m} \binom{n_i-d_i}{j-d_i}$ *for all* $0 \leq j \leq n$.

Proof. (1) For convenience, we assume that $u_1 = x_1x_2\cdots x_p$ for some $p \leq n$. For $j \geq p$, a squarefree monomial of degree j in $u_1K[Z_1] = u_1K[x_1,\dots,x_n]$ is of the form $v = u_1w$, where $w \in K[x_{p+1},\cdots,x_n]$ is squarefree of degree $j-p$. Hence, there are $\binom{n-p}{j-p} = \binom{n_1-d_1}{j-d_1}$ such monomials. Similarly, there are $\binom{n_i-d_i}{j-d_i}$ squarefree monomials of degree j in $u_iK[Z_i]$ for all $2 \leq i \leq m$. Hence, we get the required conclusion from (3.1).

(2) It follows immediately from (1). □

We recall the following combinatorial identity, which can be easily derived from the Chu-Vandermonde identity

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n}{j} = \binom{n-d+k-1}{k}. \tag{3.2}$$

Now, we state the following result, which follows immediately from Lemma 3.1 and (3.2):

Proposition 3.2. *With the above notations, we have that:*

(1) $\beta_k^d(I) = \sum\limits_{i=1}^{m}\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n_i-d_i}{j-d_i}$ *for all* $0 \leq k \leq d \leq n$.

(2) $\beta_k^d(S/I) = \binom{n-d+k-1}{k} - \sum\limits_{i=1}^{m}\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n_i-d_i}{j-d_i}$ *for all* $0 \leq k \leq d \leq n$.

If $k \geq D$ then, using (3.2) and taking $\ell = j - D$ we get

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{N-D}{j-D} = \sum_{\ell=0}^{k-D}(-1)^{k-D-\ell}\binom{d-D-\ell}{k-D-\ell}\binom{N-D}{\ell} = \binom{N-d+k-D-1}{k-D}. \tag{3.3}$$

Note that (3.3) is trivially satisfied for $k < D$ also.

From Proposition 3.2 and (3.3) we get the following:

Corollary 3.3. *With the above notations, we have that:*

(1) $\beta_k^d(I) = \sum\limits_{i=1}^{m}\binom{n_i-d+k-d_i-1}{k-d_i}$ *for all* $0 \leq k \leq d \leq n$.

(2) $\beta_k^d(S/I) = \binom{n-d+k-1}{k} - \sum\limits_{i=1}^{m}\binom{n_i-d+k-d_i-1}{k-d_i}$ *for all* $0 \leq k \leq d \leq n$.

The problem of computing $\operatorname{hdepth}(I)$ and $\operatorname{hdepth}(S/I)$ using directly the formulas given in Corollary 3.3 seems hopeless. However, we can tackle the following particular case:

Theorem 3.4. *Let* $I \subset S$ *be a proper squarefree monomial ideal with linear quotients with* $\operatorname{depth}(S/I) = n-2$. *Then*

$$\operatorname{hdepth}(S/I) = \operatorname{sdepth}(S/I) = n-2.$$

Proof. From Theorem 2.4 and (1.4) it follows that

$$\operatorname{hdepth}(S/I) \geq \operatorname{sdepth}(S/I) = n-2.$$

Hence, in order to complete the proof it is enough to show that $\operatorname{hdepth}(S/I) \leq n-2$. If $\alpha_{n-1}(S/I) = 0$ then, according to (1.3), there is nothing to prove.

Suppose that $\alpha_{n-1}(S/I) = s > 0$. From [12, Lemma 2.1] we can assume that $\deg(u_1) \leq \deg(u_2) \leq \cdots \leq \deg(u_m)$, where $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_m$ is the linear order on $G(I)$. If $m = 1$ then $I = (u_1)$ is principal, a contradiction with the hypothesis $\operatorname{depth}(S/I) = n-2$.

Note that, if $x_1x_2\cdots x_n \in G(I)$ then, since I has linear quotients, it follows that $u_1 = x_1x_2\cdots x_n$ and $I = (u_1)$, a contradiction. Therefore

$$\deg(u_1) \leq \deg(u_2) \leq \cdots \leq \deg(u_m) \leq n-1.$$

We claim that $\deg(u_1) \geq s$. Assume by contradiction that $\deg(u_1) = \ell < s$ and let's say that $u_1 = x_1\cdots x_\ell$. Then $v_k = x_1\cdots x_n/x_k \in I$ for all $\ell < k \leq n$ and thus $\alpha_{n-1}(S/I) \leq \ell$, a contradiction. In particular, we have $\alpha_j(S/I) = \binom{n}{j}$ for all $j \leq s-1$ and thus, from (1.1) and (3.2), it follows that

$$\beta_k^{n-1}(S/I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n-1-j}{k-j}\binom{n}{j} = 1 \text{ for all } k \leq s-1. \tag{3.4}$$

We assume by contradiction that $\operatorname{hdepth}(S/I) = n-1$. From (1.2), it follows that

$$s = \alpha_{n-1}(S/I) = \sum_{j=0}^{n-1} \beta_j^{n-1}(S/I) \text{ with } \beta_j^{n-1}(S/I) \geq 0. \tag{3.5}$$

Therefore, from (3.4) we get

$$\beta_j^{n-1}(S/I) = 0 \text{ for all } s \leq j \leq n-1. \tag{3.6}$$

From (1.2), (3.4) and (3.6) it follows that

$$\alpha_k(S/I) = \sum_{j=0}^{k} \beta_j^{n-1}(S/I)\binom{n-1-j}{k-j} = \binom{n}{k} - \binom{n-s}{k-s} \text{ for all } 0 \leq k \leq n. \tag{3.7}$$

From Lemma 3.1(2) and (3.7) it follows

$$\sum_{i=1}^{m} \binom{n_i - d_i}{s - d_i} = 1. \tag{3.8}$$

Since $s \leq d_1 \leq d_2 \leq \cdots \leq d_m$ and $d_i \leq n_i$ for all $1 \leq i \leq m$, from (3.8) it follows that $d_1 = s$ and $d_i > s$ for $2 \leq i \leq m$. Since $n_1 = n$, from Lemma 3.1(2) it follows that

$$\alpha_{d_2}(S/I) = \binom{n}{d_2} - \sum_{i=1}^{m}\binom{n_i-d_i}{d_2-d_i} \leq \binom{n}{d_2} - \binom{n-d_1}{d_2-d_1} - \binom{n_2-d_2}{0} = \binom{n}{d_2} - \binom{n-s}{d_2-s} - 1,$$

which contradicts (3.7). □

Acknowledgements. We would like to express our gratitude to the anonymous referee who help us to correct, improve and clarify our manuscript.

The second author was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P1-1.1-TE-2021-1633, within PNCDI III.

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