EDGE IDEALS OF SOME EDGE-WEIGHTED GRAPHS

GUANGJUN ZHU*, SHIYA DUAN, YIJUN CUI AND JIAXIN LI

Abstract. This paper presents exact formulas for the regularity and depth of powers of the edge ideal of an edge-weighted star graph. Additionally, we provide exact formulas for the regularity of powers of the edge ideal of an edge-weighted integrally closed path, as well as lower bounds on the depth of powers of such an edge ideal.

1. Introduction

In this article, a graph means a simple graph without loops, multiple edges, and isolated vertices. Let G be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. Suppose $w : E(G) \to \mathbb{Z}_{>0}$ is an edge weight function on G. We write G_ω for the pair (G, ω) and call it an *edge-weighted* graph with the underlying graph G. For a weighted graph G_ω, its *edge-weighted ideal* (or simply edge ideal), was introduced in [28], is the ideal of the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ in n variables over a field $\mathbb{K}$ given by

$$I(G_\omega) = (x_i^{\omega(e)} x_j^{\omega(e)} \mid e := \{x_i, x_j\} \in E(G_\omega)).$$

If w is the constant function defined by $w(e) = 1$ for all $e \in E(G)$, then $I(G_\omega)$ is the classical edge ideal of the underlying graph G of G_ω, which has been extensively studied in the literature [1, 2, 15, 16, 18, 26, 27, 30, 31].

Recently, there has been a surge of interest in characterizing weights for which the edge ideal of an edge-weighted graph is Cohen-Macaulay. For example, Paulsen and Sather-Wagstaff in [28] classified Cohen-Macaulay edge-weighted graphs G_ω where the underlying graph G is a cycle, a tree, or a complete graph. Seyed Fakhari et al. in [29] continued this study, they classified Cohen-Macaulay edge-weighted graph G_ω when G is a very well-covered graph. Recently, Diem et al. in [8] gave a complete characterization of sequentially Cohen-Macaulay edge-weighted graphs. In [30], Wei classified all Cohen-Macaulay weighted chordal graphs from a purely graph-theoretic point of view. Hien in [21] classified Cohen-Macaulay edge-weighted graphs G_ω when G has girth at least 5.

Integral closure and normality of monomial ideals is also an interesting topic. In [9], we gave a complete characterization of an integrally closed edge-weighted graph G_ω and showed that if its underlying graph G is a star graph, a path, or a cycle, then G_ω is normal.

* Corresponding author.

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The study of edge ideals of edge-weighted graphs is much more recent and consequently there are fewer results in this direction. In this paper, we decide to focus on the regularity and depth of powers of the edge ideal $I(G_\omega)$, where G_ω is a special graph. Recall that the regularity and depth are two central invariants associated to a homogeneous ideal I. It is well known that $\mathrm{reg}(I^t)$ is asymptotically a linear function for $t \gg 0$, i.e., there exist constants a, b and a positive integer t_0 such that for all $t \geq t_0$, $\mathrm{reg}\,(I^t) = at + b$ (see [7, 24]). In this regard, there has been of interest to find the exact form of this linear function and to determine the stabilization index t_0 at which $\mathrm{reg}(I^t)$ becomes linear (cf. [1, 2]). It turns out that even in the case of monomial ideals it is challenging to find the linear function and t_0 (see [6]). In [3], Brodmann showed that $\mathrm{depth}\,(S/I^t)$ is a constant for $t \gg 0$, and that this constant is bounded above by $n - \ell(I)$, where $\ell(I)$ is the analytic spread of I. In this regard, there has been an interest in determining the smallest value t_0 such that $\mathrm{depth}\,(S/I^t)$ is a constant for all $t \geq t_0$ (see [16, 20, 26]).

The article is organized as follows. In Section 2, we provide a review of important definitions and terminology, which will be necessary later. In Section 3, by choosing different exact sequences and repeatedly using Lemma 2.6, we give some exact formulas for the regularity and depth of powers of the edge ideal of an edge-weighted star graph. In Section 4, using Betti splitting and polarization approaches, we give some exact formulas for the regularity of powers of the edge ideal of an edge-weighted integrally closed path. We also provide some lower bounds on the depth of powers of such an edge ideal.

2. Preliminaries

In this section, we provide the definitions and basic facts which will be used throughout this paper. We refer to [4] and [19] for detailed information.

2.1. **Notions of simple graphs.** Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. For any subset A of $V(G)$, the *induced subgraph* of G on the set A, denoted by $G[A]$, satisfies that $V(G[A]) = A$ and for any $x_i, x_j \in A$, $\{x_i, x_j\} \in E(G[A])$ if and only if $\{x_i, x_j\} \in E(G)$. At the same time, the induced subgraph of G on the set $V(G)\setminus A$ will be denoted by $G\setminus A$. In particular, if $A = \{v\}$ then we will write $G \setminus v$ instead of $G \setminus \{v\}$ for simplicity. For any vertex $v \in V(G)$, its *neighborhood* is defined as $N_G(v) := \{u \in V(G) \mid \{u, v\} \in E(G)\}$.

An edge-weighted graph is called a *non-trivially weighted* graph if there is at least one edge with a weight greater than 1. Otherwise, it is called a *trivially weighted* graph. An edge $e \in E(G_\omega)$ with *non-trivially weight* if $w(e) \geq 2$. Otherwise, we say e with *trivial weight*.

A *walk* W of length $(n-1)$ in a graph G is a sequence of vertices w_1 through w_n, where each consecutive pair of vertices $\{w_i, w_{i+1}\}$ is connected by an edge in G. A *path* is a walk where all vertices are distinct, and a *cycle* is a walk where $w_1 = w_n$ and other vertices are distinct. To simplify notation, a path of length $(n-1)$ is denoted by P_n, and a cycle of length n is denoted by C_n.

2.2. **Notions from commutative algebra.** For any homogeneous ideal I of the polynomial ring $R = \mathbb{K}[z_1, \ldots, z_n]$, there exists a *graded minimal free resolution*

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}(R/I)} \to \bigoplus_j R(-j)^{\beta_{p-1,j}(R/I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(R/I)} \to R/I \to 0,$$

where $p \leq n$ and $R(-j)$ is obtained from R by a shift of degree j. The number $\beta_{i,j}(R/I)$, the (i,j)-th graded *Betti* number of R/I, is an invariant of R/I that equals the number of minimal generators of degree j in the i-th syzygy module of R/I. Of particular interest is the following invariant which measures the "size" of the minimal graded free resolution of R/I. The regularity of R/I, denoted $\operatorname{reg}(R/I)$, is defined by

$$\operatorname{reg}(R/I) := \max\{j - i \mid \beta_{i,j}(R/I) \neq 0\}.$$

Meanwhile, the *projective dimension* of R/I, denoted by $\operatorname{pd}(R/I)$, is

$$\operatorname{pd}(R/I) := \max\{i \mid \beta_{i,j}(R/I) \neq 0\}.$$

These two invariants measure the complexity of the minimal graded free resolution of R/I.

For a monomial ideal I, let $\mathcal{G}(I)$ denote its unique minimal set of monomial generators. We now derive some formulas for $\operatorname{pd}(I)$ and $\operatorname{reg}(I)$ in some special cases by using some tools developed in [14].

Definition 2.1. *Let I be a monomial ideal. Suppose that there exist monomial ideals J and K such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. Then $I = J + K$ is a* Betti splitting *if*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i, j \geq 0,$$

where $\beta_{i-1,j}(J \cap K) = 0$ if $i = 0$.

Definition 2.1 implies the following results.

Corollary 2.2. ([14, Corollary 2.2]) *If $I = J + K$ is a Betti splitting ideal, then*

(1) $\operatorname{reg}(I) = max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\}$,
(2) $\operatorname{pd}(I) = max\{\operatorname{pd}(J), \operatorname{pd}(K), \operatorname{pd}(J \cap K) + 1\}$.

This formula was first obtained for the total Betti numbers by Eliahou and Kervaire [10, Proposition 3.1] and extended to the graded case by Fatabbi [13]. In [14], the authors describe some sufficient conditions for an ideal I to have a Betti splitting.

Lemma 2.3. ([14, Corollary 2.7]) *Suppose that $I = J + K$ where $\mathcal{G}(J)$ contains all the generators of I divisible by some variable x_i and $\mathcal{G}(K)$ is a nonempty set containing the remaining generators of I. If J has a linear resolution, then $I = J+K$ is a Betti splitting.*

The following three lemmas are often used in this article.

Lemma 2.4. [17, Lemma 1.3]) *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ and let I be a proper non-zero homogeneous ideal in S. Then*

(1) $\operatorname{pd}(I) = \operatorname{pd}(S/I) - 1$,
(2) $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$.

In particular, if u is a monomial of degree d in S and $K = (u)$, then we have $\operatorname{reg}(S/K) = d - 1$.

By Auslander-Buchsbaum formula (see [19, Corollary A.4.3]), we have

$$\operatorname{depth}(S/I) = n - \operatorname{pd}(S/I).$$

Lemma 2.5. ([22, Lemma 2.2 and Lemma 3.2]) *Let $S_1 = \mathbb{K}[x_1, \dots, x_m]$, $S_2 = \mathbb{K}[x_{m+1}, \dots, x_n]$ and $S = \mathbb{K}[x_1, \dots, x_n]$ be three polynomial rings over $\mathbb{K}$, $I \subseteq S_1$ and $J \subseteq S_2$ be two proper non-zero homogeneous ideals. Then we have*

(1) $\operatorname{reg}(S/(I + J)) = \operatorname{reg}(S_1/I) + \operatorname{reg}(S_2/J)$,
(2) $\operatorname{depth}(S/(I + J)) = \operatorname{depth}(S_1/I) + \operatorname{depth}(S_2/J)$.

Lemma 2.6. ([22, Lemmas 2.1 and 3.1]) *Let $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ be a short exact sequence of finitely generated graded S-modules. Then we have*

(1) $\operatorname{reg}(N) \leq max\{\operatorname{reg}(M), \operatorname{reg}(P)\}$, *and the equality holds if* $\operatorname{reg}(P) \neq \operatorname{reg}(M) - 1$.
(2) $\operatorname{depth}(N) \geq min\{\operatorname{depth}(M), \operatorname{depth}(P)\}$, *and the equality holds if* $\operatorname{depth}(P) \neq \operatorname{depth}(M) - 1$.

3. Star graph

In this section, we will give precise formulas for the depth and regularity of powers of the edge ideal of an edge-weighted star graph. For a positive integer n, the notation $[n]$ denotes the set $\{1, 2, \dots, n\}$.

Theorem 3.1. *Let G_ω be an edge-weighted star graph with n vertices, and let the set of monomial generators of its edge ideal be $\mathcal{G}(I(G_\omega)) = \{(x_i x_n)^{\omega_i} \mid i \in [n-1]\}$. Then*

(1) $\operatorname{depth}(S/I(G_\omega)) = 1$.
(2) $\operatorname{reg}(S/I(G_\omega)) = \omega + \sum\limits_{i=1}^{n-1} (\omega_i - 1)$, *where* $\omega = \max\{\omega_1, \dots, \omega_{n-1}\}$.

Proof. Let us assume, without loss of generality, that $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_{n-1}$. We will now proceed to prove the given statements by induction on n, we first establish the base case where $n = 2$, which is trivial. For $n = 3$, the following equalities hold: $I(G_\omega) : x_3^{\omega_2} = (x_1^{\omega_1} x_3^{\omega_1 - \omega_2}, x_2^{\omega_2})$, $(I(G_\omega), x_3^{\omega_2}) = (x_3^{\omega_2})$, $I(G_\omega) : x_2^{\omega_2} = (x_3^{\omega_2})$, and $(I(G_\omega), x_2^{\omega_2}) = ((x_1 x_3)^{\omega_1}, x_2^{\omega_2})$. Therefore, $\operatorname{depth}(S/(I(G_\omega) \colon x_3^{\omega_2})) = 1$, $\operatorname{depth}(S/(I(G_\omega), x_3^{\omega_2})) = 2$, $\operatorname{reg}(S/(I(G_\omega) \colon x_2^{\omega_2})) = \omega_2 - 1$, and $\operatorname{reg}(S/(I(G_\omega), x_2^{\omega_2})) = 2\omega_1 + \omega_2 - 2$. By using Lemma 2.6 and analyzing the short exact sequences

$$0 \rightarrow \frac{S}{I(G_\omega) \colon x_3^{\omega_2}}(-\omega_2) \xrightarrow{\cdot x_3^{\omega_2}} \frac{S}{I(G_\omega)} \rightarrow \frac{S}{(I(G_\omega), x_3^{\omega_2})} \rightarrow 0,$$
$$0 \rightarrow \frac{S}{I(G_\omega) \colon x_2^{\omega_2}}(-\omega_2) \xrightarrow{\cdot x_2^{\omega_2}} \frac{S}{I(G_\omega)} \rightarrow \frac{S}{(I(G_\omega), x_2^{\omega_2})} \rightarrow 0,$$

we can obtain that $\operatorname{depth}(S/I(G_\omega)) = 1$ and $\operatorname{reg}(S/I(G_\omega)) = 2\omega_1 + \omega_2 - 2$.

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In the following, we assume that $n \geq 4$ and that the results hold for $n-1$. Since $I(G_\omega) : x_{n-1}^{\omega_{n-1}} = (x_n^{\omega_{n-1}})$ and $(I(G_\omega), x_{n-1}^{\omega_{n-1}}) = J + (x_{n-1}^{\omega_{n-1}})$, where $J = ((x_1x_n)^{\omega_1}, \ldots, (x_{n-2}x_n)^{\omega_{n-2}})$, we obtain depth $\left(\frac{S}{I(G_\omega)\colon x_{n-1}^{\omega_{n-1}}}\right) = n-1$, depth $\left(\frac{S}{(I(G_\omega), x_{n-1}^{\omega_{n-1}})}\right)$ $=$ depth$(\frac{S'}{J}) = 1$. Furthermore, reg $\left(\frac{S}{I(G_\omega)\colon x_{n-1}^{\omega_{n-1}}}\right) = \omega_{n-1}-1$ and reg $\left(\frac{S}{(I(G_\omega), x_{n-1}^{\omega_{n-1}})}\right) =$ reg$(\frac{S'}{J}) + (\omega_{n-1} - 1) = \omega_1 + \sum\limits_{i=1}^{n-1}(\omega_i - 1)$ by the inductive hypothesis, where $S' = \mathbb{K}[x_1, \ldots, x_{n-2}, x_n]$. Applying Lemma 2.6 to the following short exact sequence

$$0 \rightarrow \frac{S}{I(G_\omega):x_{n-1}^{\omega_{n-1}}}(-\omega_{n-1}) \xrightarrow{\cdot x_{n-1}^{\omega_{n-1}}} \frac{S}{I(G_\omega)} \rightarrow \frac{S}{(I(G_\omega), x_{n-1}^{\omega_{n-1}})} \rightarrow 0,$$

we get that depth$(S/I(G_\omega)) = 1$ and reg$(S/I(G_\omega)) = \omega_1 + \sum\limits_{i=1}^{n-1}(\omega_i - 1)$. □

Lemma 3.2. *Let G_ω be an edge-weighted star graph as in Theorem 3.1. Suppose $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_{n-1}$. Then, for any $t \geq 2$, we have*

(1) $(I(G_\omega)^t : (x_{n-1}x_n)^{\omega_{n-1}}) = I(G_\omega)^{t-1}$;
(2) $((I(G_\omega)^t : x_{n-1}^{\omega_{n-1}}), x_n^{\omega_{n-1}}) = (x_n^{\omega_{n-1}})$;
(3) $(I(G_\omega)^t, x_{n-1}^{\omega_{n-1}}) = I((G_\omega \setminus x_{n-1})^t, x_{n-1}^{\omega_{n-1}})$.

Proof. (1) For any monomial u in $\mathcal{G}(I(G_\omega)^t : (x_{n-1}x_n)^{\omega_{n-1}})$, we have $u(x_{n-1}x_n)^{\omega_{n-1}} \in I(G_\omega)^t$. Let $u(x_{n-1}x_n)^{\omega_{n-1}} = u_{i1} \cdots u_{it}h$, where each $u_{ij} \in \mathcal{G}(I(G_\omega))$ and h is a monomial. If there exists some $j \in [t]$ such that $x_{n-1}|u_{ij}$, then $u_{ij} = (x_{n-1}x_n)^{\omega_{n-1}}$ because $N_G(x_{n-1}) = \{x_n\}$. This implies that $u \in I(G_\omega)^{t-1}$. If $x_{n-1} \nmid u_{ij}$ for all $j \in [t]$, then $x_{n-1}|h$. Thus, it can be concluded that $u \in I(G_\omega)^{t-1}$, since $x_n^{\omega_{n-1}}|u_{ij}$ for all $j \in [t]$.

(2) For any monomial $u \in \mathcal{G}(I(G_\omega)^t : x_{n-1}^{\omega_{n-1}})$, we have $ux_{n-1}^{\omega_{n-1}} = u_{i1} \cdots u_{it}h$ for some monomial h, where each $u_{ij} \in \mathcal{G}(I(G_\omega))$. Let u_{ij} be represented as $u_{ij} = (x_{ij}x_n)^{\omega_{ij}}$ with $ij \in [n-1]$, then it can be inferred that $x_n^{\omega_{ij}}|u$. Therefore, $x_n^{\omega_{n-1}}|u$, since $\omega_{ij} \geq \omega_{n-1}$. This forces that $u \in (x_n^{\omega_{n-1}})$.

(3) It is clear that $(I(G_\omega \setminus x_{n-1})^t, x_{n-1}^{\omega_{n-1}}) \subseteq (I(G_\omega)^t, x_{n-1}^{\omega_{n-1}})$. If the monomial $u \in \mathcal{G}(I(G_\omega)^t) \setminus \mathcal{G}(I(G_\omega \setminus x_{n-1})^t)$, then $x_{n-1}|u$. It follows that $(x_{n-1}x_n)^{\omega_{n-1}}|u$, since $N_G(x_{n-1}) = \{x_n\}$. This implies that $u \in (x_{n-1}^{\omega_{n-1}})$. □

Theorem 3.3. *Let G_ω be an edge-weighted star graph as in Lemma 3.2. Then, for $t \geq 2$, we have*

(1) depth$(S/I(G_\omega)^t) = 1$.
(2) reg$(S/I(G_\omega)^t) = 2(t-1)\omega_1 + $ reg$(S/I(G_\omega))$.

Proof. Let $I = I(G_\omega)$. We will prove the assertions by induction on n and t. The case where $n = 2$ is trivial.

In the following, we assume that $n \geq 3$ and that the results hold for $n-1$ and $t-1$. Using Lemma 3.2, Lemma 2.5, Theorem 3.1 and the inductive hypothesis, we can conclude that

$$\begin{aligned}
\operatorname{depth}(S/((I^t : x_{n-1}^{\omega_{n-1}}) : x_n^{\omega_{n-1}})) &= \operatorname{depth}(S/I^{t-1}) = 1,\\
\operatorname{depth}(S/((I^t : x_{n-1}^{\omega_{n-1}}), x_n^{\omega_{n-1}})) &= \operatorname{depth}(S/(x_n^{\omega_{n-1}})) = n-1,\\
\operatorname{depth}(S/(I^t, x_{n-1}^{\omega_{n-1}})) &= \operatorname{depth}(S/I((G_\omega \setminus x_{n-1})^t, x_{n-1}^{\omega_{n-1}})) = 1,\\
\operatorname{reg}(S/((I^t : x_{n-1}^{\omega_{n-1}}) : x_n^{\omega_{n-1}})) &= \operatorname{reg}(S/I^{t-1}) = 2(t-2)\omega_1 + \operatorname{reg}(S/I(G_\omega)),\\
\operatorname{reg}(S/((I^t : x_{n-1}^{\omega_{n-1}}), x_n^{\omega_{n-1}})) &= \operatorname{reg}(S/(x_n^{\omega_{n-1}})) = \omega_{n-1} - 1,\\
\operatorname{reg}(S/(I^t, x_{n-1}^{\omega_{n-1}})) &= \operatorname{reg}(S/I((G_\omega \setminus x_{n-1})^t, x_{n-1}^{\omega_{n-1}}))\\
&= 2(t-1)\omega_1 + \operatorname{reg}(S/I(G_\omega)).
\end{aligned}$$

The desired results hold by Lemma 2.6 and the following short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \frac{S}{I^t : x_{n-1}^{\omega_{n-1}}}(-\omega_{n-1}) & \stackrel{\cdot x_{n-1}^{\omega_{n-1}}}{\longrightarrow} & \frac{S}{I^t} & \to & \frac{S}{(I^t, x_{n-1}^{\omega_{n-1}})} & \to & 0,\\
0 & \to & \frac{S}{(I^t : x_{n-1}^{\omega_{n-1}}) : x_n^{\omega_{n-1}}}(-\omega_{n-1}) & \stackrel{\cdot x_n^{\omega_{n-1}}}{\longrightarrow} & \frac{S}{I^t : x_{n-1}^{\omega_{n-1}}} & \to & \frac{S}{((I^t : x_{n-1}^{\omega_{n-1}}), x_n^{\omega_{n-1}})} & \to & 0.
\end{array}$$

□

4. PATH GRAPH

This section provides precise formulas for the regularity of powers of the edge ideal of an edge-weighted integrally closed path using Betti splitting and polarization approaches. Additionally, it offers lower bounds on the depth of powers of this edge ideal. The section begins by defining polarization.

Definition 4.1. ([11, Definition 2.1]) *Let $I \subset S$ be a monomial ideal with $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, where $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ for $i = 1, \ldots, m$. The polarization of I, denoted by $I^{\mathcal{P}}$, is a squarefree monomial ideal in the polynomial ring $S^{\mathcal{P}}$*

$$I^{\mathcal{P}} = (\mathcal{P}(u_1), \ldots, \mathcal{P}(u_m))$$

where $\mathcal{P}(u_i) = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$ is a squarefree monomial in $S^{\mathcal{P}} = \mathbb{K}[x_{j1}, \ldots, x_{ja_j} \mid j = 1, \ldots, n]$ and $a_j = \max\{a_{ij} | i = 1, \ldots, m\}$ for $1 \le j \le n$.

A monomial ideal and its polarization share numerous homological and algebraic properties. The following is a useful property of polarization.

Lemma 4.2. ([19, Corollary 1.6.3]) *Let $I \subset S$ be a monomial ideal and $I^{\mathcal{P}} \subset S^{\mathcal{P}}$ be its polarization. Then*

(1) $\beta_{ij}(I) = \beta_{ij}(I^{\mathcal{P}})$ *for all i and j,*
(2) $\operatorname{reg}(I) = \operatorname{reg}(I^{\mathcal{P}})$,
(3) $\operatorname{pd}(I) = \operatorname{pd}(I^{\mathcal{P}})$.

Definition 4.3. ([19, Definition 1.4.1]) *Let I be an ideal in a ring R. An element $f \in R$ is said to be* integral *over I if there exists an equation*

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \quad \text{with} \quad c_i \in I^i.$$

The set $\overline{I}$ of elements in R which are integral over I is the integral closure *of I. If $I = \overline{I}$, then I is said to be* integrally closed. *An edge-weighted graph G_ω is said to be* integrally closed *if its edge ideal $I(G_\omega)$ is integrally closed.*

According to [19, Theorem 1.4.6], every edge-weighted graph G_ω with trivial weights is integrally closed. The following lemma offers a complete characterization of a non-trivial edge-weighted graph that is integrally closed.

Lemma 4.4. ([9, Theorem 3.6]) *If G_ω is a non-trivially edge-weighted graph, then $I(G_\omega)$ is integrally closed if and only if G_ω does not contain any of the following three graphs as induced subgraphs.*

(1) *A path P_ω^3 with three vertices, whose all edges have non-trivial weights.*
(2) *The disjoint union $P_\omega^2 \sqcup P_\omega^2$ of two paths P_ω^2 where all edges have non-trivial weights.*
(3) *A 3-cycle C_ω^3 where all edges have non-trivial weights.*

From the lemma above, we can derive

Corollary 4.5. *Let P_ω^n be a non-trivial edge-weighted integrally closed path with n vertices, then it can have at most two edges with non-trivial weights.*

The following lemma provides exact formulas for the regularity and depth of powers of the edge ideal of integrally closed paths with trivial weights.

Lemma 4.6. ([12, Lemma 2.8], [23, Corollary 7.7.34], [2, Theorem 4.7 and Remark 2.12], and [25, Theorem 3.4]) *Let P_ω^n be a trivial edge-weighted path with n vertices, then the following results hold:*

(1) $\operatorname{depth}(S/I(P_\omega^n)) = \lceil \frac{n}{3} \rceil$.
(2) $\operatorname{depth}(S/I(P_\omega^n)^t) = \max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$.
(3) $\operatorname{reg}(S/I(P_\omega^n)) = \lfloor \frac{n+1}{3} \rfloor$.
(4) $\operatorname{reg}(S/I(P_\omega^n)^t) = \lfloor \frac{n+1}{3} \rfloor + 2(t-1) = \operatorname{reg}(S/I(P_\omega^n)) + 2(t-1)$.

Thus, we will now consider the non-trivial edge-weighted integrally closed path that satisfies the following conditions

Remark 4.7. *Let $n \geq 2$ be an integer and P_ω^n be a non-trivial edge-weighted integrally closed path with the vertex set $\{x_1, \dots, x_n\}$ and the edge set $\{e_1, \dots, e_{n-1}\}$, where $e_i = \{x_i, x_{i+1}\}$ and $\omega_i = \omega(e_i)$ for all $i \in [n-1]$.*

We begin by calculating the regularity and depth of the edge ideal of a path P_ω^n with $n \leq 4$.

Theorem 4.8. *Let P_ω^n be a path as in Remark 4.7, where $n \leq 4$. Let $\omega = \max\{\omega_1, \dots, \omega_{n-1}\}$. Then*

(1) *If $n = 2$ or 3, then* $\operatorname{reg}(S/I(P_\omega^n)) = 2\omega - 1$ *and* $\operatorname{depth}(S/I(P_\omega^n)) = 1$.
(2) *If $n = 4$, then* $\operatorname{reg}(S/I(P_\omega^n)) = 2\omega - 1$ *and* $\operatorname{depth}(S/I(P_\omega^n)) = 2 - a$, *where*
$$a = \begin{cases} 0, & \text{if } \omega_2 = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let $I = I(P_\omega^n)$. We distinguish between the following two cases:

(1) The case where $n = 2$ is trivial. If $n = 3$, then we can suppose $\omega_1 \geq 2$ by symmetry. Thus the following equalities hold: $I : x_2 = (x_1^{\omega_1} x_2^{\omega_1 - 1}, x_3)$ and $(I, x_2) = (x_2)$. It follows from Lemma 2.4 that $\operatorname{depth}(S/(I : x_2)) = 1$ and $\operatorname{depth}(S/(I, x_2)) = 2$. Additionally, $\operatorname{reg}(S/(I : x_2)) = 2\omega_1 - 2$ and $\operatorname{reg}(S/(I, x_2)) = 0$. Applying Lemma 2.6 to the following short exact sequence

$$0 \rightarrow \tfrac{S}{I:x_2}(-1) \xrightarrow{\cdot x_2} \tfrac{S}{I} \rightarrow \tfrac{S}{(I,x_2)} \rightarrow 0, \qquad (1)$$

we can determine that $\operatorname{depth}(S/I) = 1$ and $\operatorname{reg}(S/I) = 2\omega_1 - 1$.

(2) If $n = 4$, then $I = ((x_1x_2)^{\omega_1}, (x_2x_3)^{\omega_2}, (x_3x_4)^{\omega_3})$ with $\omega_2 = 1$ or $\omega_1 = \omega_3 = 1$. There are two cases to consider:

(i) Suppose $\omega_2 = 1$. By symmetry, it can be assumed that $\omega_1 \geq \omega_3$. Therefore, we have $(I : x_2) = (x_1^{\omega_1} x_2^{\omega_1 - 1}, x_3)$ and $(I, x_2) = (x_2, (x_3x_4)^{\omega_3})$. As a result, $\operatorname{depth}(S/(I : x_2)) = \operatorname{depth}(S/(I, x_2)) = 2$, along with $\operatorname{reg}(S/(I : x_2)) = 2\omega_1 - 2$ and $\operatorname{reg}(S/(I, x_2)) = 2\omega_3 - 1$, are presented. Using Lemma 2.6 and the exact sequence (1), we can determine that $\operatorname{depth}(S/I) = 2$ and $\operatorname{reg}(S/I) = 2\omega_1 - 1$.

(ii) If $\omega_1 = \omega_3 = 1$, then $\omega_2 \geq 2$. In this case, we have $(I : x_2) = (x_1, x_2^{\omega_2 - 1} x_3^{\omega_2}, x_3x_4)$ and $(I, x_2) = (x_2, x_3x_4)$. Thus $\operatorname{depth}(S/(I, x_2)) = 2$ and $\operatorname{reg}(S/(I, x_2)) = 1$. If we let $I' = (I : x_2)$, then $(I' : x_3) = (x_1, (x_2x_3)^{\omega_2 - 1}, x_4)$ and $(I', x_3) = (x_1, x_3)$. Therefore, $\operatorname{depth}(S/(I' : x_3)) = 1$, $\operatorname{depth}(S/(I', x_3)) = 2$, $\operatorname{reg}(S/(I' : x_3)) = 2\omega_2 - 3$ and $\operatorname{reg}(S/(I', x_3)) = 0$. By applying Lemma 2.6 to the following two short exact sequences

$$\begin{array}{lllllllll} 0 & \rightarrow & \frac{S}{I':x_3}(-1) & \xrightarrow{\cdot x_3} & \frac{S}{I'} & \rightarrow & \frac{S}{(I',x_3)} & \rightarrow & 0, \\ 0 & \rightarrow & \frac{S}{(I:x_2)}(-1) & \xrightarrow{\cdot x_2} & \frac{S}{I} & \rightarrow & \frac{S}{(I,x_2)} & \rightarrow & 0, \end{array}$$

we can determine that $\operatorname{depth}(S/I) = 1$ and $\operatorname{reg}(S/I) = 2\omega_2 - 1$. ☐

Theorem 4.9. *Let P_ω^n be a path as in Remark 4.7, where $n \geq 5$. By symmetry and Corollary 4.5, we can assume that $\omega_i \geq \omega_{i+2}$ and $\omega_i \geq 2$ for some $i \in [n-3]$. Then*

$$\operatorname{reg}(S/I(P_\omega^n)) = \max\{2\omega_i + \lfloor\frac{i-1}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, 2\omega_{i+2} + \lfloor\frac{i-2}{3}\rfloor + \lfloor\frac{n-i}{3}\rfloor\} - 1,$$

$$\operatorname{depth}(S/I(P_\omega^n)) = \min\{\lceil\frac{i}{3}\rceil + \lceil\frac{n-i-a}{3}\rceil, \lceil\frac{i-2}{3}\rceil + \lceil\frac{n-i-2}{3}\rceil + 1\},$$

where $a = \begin{cases} 0, & \text{if } \omega_{i+2} = 1, \\ 1, & \text{otherwise.} \end{cases}$

Proof. Let $I = I(P_\omega^n)$ and $I^{\mathcal{P}}$ be its polarization, then $I^{\mathcal{P}} = J^{\mathcal{P}} + K^{\mathcal{P}}$, which is Betti splitting by Lemma 2.3, where $J = (x_ix_{i+1})^{\omega_i}$ and $\mathcal{G}(K) = \mathcal{G}(I) \setminus \mathcal{G}(J)$, since $\omega_i \geq 2$ and $\omega_{i-1} = 1$. Note that $(J \cap K)^{\mathcal{P}} = J^{\mathcal{P}} \cap K^{\mathcal{P}}$. It follows from Corollary 2.2 and Lemmas 2.4 and 4.2 that

$$\begin{aligned} \operatorname{reg}(S/I) &= \operatorname{reg}(I) - 1 = \operatorname{reg}(I^{\mathcal{P}}) - 1 \\ &= \max\{\operatorname{reg}(J^{\mathcal{P}}), \operatorname{reg}(K^{\mathcal{P}}), \operatorname{reg}(J^{\mathcal{P}} \cap K^{\mathcal{P}}) - 1\} - 1 \\ &= \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\} - 1 \qquad (2) \end{aligned}$$

and

$$\begin{aligned}\mathrm{pd}(S/I) &= \mathrm{pd}(I)+1=\mathrm{pd}(I^{\mathcal{P}})+1\\ &= \max\{\mathrm{pd}(J^{\mathcal{P}}),\mathrm{pd}(K^{\mathcal{P}}),\mathrm{pd}(J^{\mathcal{P}}\cap K^{\mathcal{P}})+1\}+1\\ &= \max\{\mathrm{pd}(J),\mathrm{pd}(K),\mathrm{pd}(J\cap K)+1\}+1\\ &= \max\{\mathrm{pd}(S/J),\mathrm{pd}(S/K),\mathrm{pd}(S/(J\cap K))+1\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{depth}(S/I) &= n-\mathrm{pd}(S/I)=n-\max\{\mathrm{pd}(S/J),\mathrm{pd}(S/K),\mathrm{pd}(S/(J\cap K))+1\}\\ &= \min\{\mathrm{depth}(S/J),\mathrm{depth}(S/K),\mathrm{depth}(S/(J\cap K))-1\}. \qquad (3)\end{aligned}$$

Now, we calculate the depth of S/I.

Note that $K=(x_1x_2,\dots,x_{i-1}x_i)+K'$ where $K'=(x_{i+1}x_{i+2},x_{i+2}^{\omega_{i+2}}x_{i+3}^{\omega_{i+2}},\dots,x_{n-1}x_n)$, and $J\cap K=JL$ with $L=I(P_\omega^n\setminus\{x_{i-1},x_i,x_{i+1},x_{i+2}\})+(x_{i-1},x_{i+2})$, where $x_i=0$ if $i\le 0$. We have the following two cases:

(a) If $\omega_{i+2}=1$, then $\mathrm{depth}(S/J)=n-1$, $\mathrm{depth}(S/K)=\lceil\frac{i}{3}\rceil+\lceil\frac{n-i}{3}\rceil$ and $\mathrm{depth}(S/J\cap K)=\lceil\frac{i-2}{3}\rceil+\lceil\frac{n-i-2}{3}\rceil+2$ by Lemma 4.6 and Lemma 2.5. It follows from the equation (3) that

$$\mathrm{depth}(S/I)=\min\{\lceil\frac{i}{3}\rceil+\lceil\frac{n-i}{3}\rceil,\lceil\frac{i-2}{3}\rceil+\lceil\frac{n-i-2}{3}\rceil+1\}. \qquad (4)$$

(b) If $\omega_{i+2}>1$, then $\mathrm{depth}(S/J)=n-1$, $\mathrm{depth}(S/J\cap K)=\lceil\frac{i-2}{3}\rceil+\lceil\frac{n-i-2}{3}\rceil+2$ by Lemma 4.6 and Lemma 2.5. By substituting K' for I in the equation (4), we can see that $\mathrm{depth}(S/K)=\lceil\frac{i}{3}\rceil+\lceil\frac{n-i-1}{3}\rceil$. Thus

$$\mathrm{depth}(S/I)=\min\{\lceil\frac{i}{3}\rceil+\lceil\frac{n-i-1}{3}\rceil,\lceil\frac{i-2}{3}\rceil+\lceil\frac{n-i-2}{3}\rceil+1\}$$

by the equation (3).

Next, we calculate the regularity of S/I. There are two cases to consider:

(a) If $\omega_{i+2}=1$, then $\mathrm{reg}(S/J)=2\omega_i-1$, $\mathrm{reg}(S/K)=\lfloor\frac{i+1}{3}\rfloor+\lfloor\frac{n-i+1}{3}\rfloor$ and $\mathrm{reg}(S/J\cap K)=2\omega_i+\lfloor\frac{i-1}{3}\rfloor+\lfloor\frac{n-i-1}{3}\rfloor$ by applying Lemma 4.6 and Lemma 2.5. From the equation (2), it follows that

$$\mathrm{reg}(S/I)=2\omega_i+\lfloor\frac{i-1}{3}\rfloor+\lfloor\frac{n-(i+1)}{3}\rfloor-1. \qquad (5)$$

(b) If $\omega_{i+2}>1$, then $\mathrm{reg}(S/J)=2\omega_i-1$, $\mathrm{reg}(S/J\cap K)=2\omega_i+\lfloor\frac{i-1}{3}\rfloor+\lfloor\frac{n-i-1}{3}\rfloor$ by applying Lemma 4.6 and Lemma 2.5, and $\mathrm{reg}(S/K)=2\omega_{i+2}+\lfloor\frac{i+1}{3}\rfloor+\lfloor\frac{n-i}{3}\rfloor-2$ by substituting K' for I in equation (5). Therefore,

$$\mathrm{reg}(S/I(P_\omega^n))=\max\{2\omega_i+\lfloor\frac{i-1}{3}\rfloor+\lfloor\frac{n-(i+1)}{3}\rfloor,2\omega_{i+2}+\lfloor\frac{i-2}{3}\rfloor+\lfloor\frac{n-i}{3}\rfloor\}-1$$

by the equation (2). ☐

We will now analyze the powers of the edge ideal of an edge-weighted path P_ω^n.

Lemma 4.10. *Let P_ω^n be an edge-weighted path with the vertex set $\{x_1,\dots,x_n\}$ and the edge set $\{e_1,\dots,e_{n-1}\}$, where $e_i=\{x_i,x_{i+1}\}$ and $\omega_i=\omega(e_i)$ for all $i\in[n-1]$. If $\omega_{n-1}=1$, then for all $t\ge 2$, we have*

(1) $(I(P_\omega^n)^t : x_{n-1}x_n) = I(P_\omega^n)^{t-1}$;
(2) $((I(P_\omega^n)^t : x_n), x_{n-1}) = (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$;
(3) $(I(P_\omega^n)^t, x_n) = (I(P_\omega^n \setminus x_n)^t, x_n)$;
(4) $(I(P_\omega^n)^t, x_{n-1}) = (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$;
(5) $((I(P_\omega^n)^t : x_{n-1}), x_n) = ((I(P_\omega^n \setminus x_n)^t : x_{n-1}), x_n)$.

Proof. Let $I = I(P_\omega^n)$.

(1) For any monomial $u \in \mathcal{G}(I^t : x_{n-1}x_n)$, it follows that $ux_{n-1}x_n \in I^t$. If

$$ux_{n-1}x_n = u_{i1} \cdots u_{it}h \tag{6}$$

where each $u_{ij} \in \mathcal{G}(I)$ and h is a monomial, and $x_n | u_{ij}$ for some $j \in [t]$, then $u_{ij} = x_{n-1}x_n$, since $N_G(x_n) = \{x_{n-1}\}$. Therefore, $u \in I^{t-1}$ by the equation (6). If $x_n \nmid u_{ij}$ for any $j \in [t]$, then $x_n | h$ according to the equation (6). This implies that $u \in I^{t-1}$, since $\omega_{n-1} = 1$.

(2) It is clear that $(I(P_\omega^n \setminus x_{n-1})^t, x_{n-1}) \subseteq ((I^t : x_n), x_{n-1})$. For any monomial $u \in \mathcal{G}(((I^t : x_n), x_{n-1}))$. If $x_{n-1}|u$, then $u \in (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$. Otherwise, $ux_n \in I^t$. We can write ux_n as

$$ux_n = u_{\ell 1} \cdots u_{\ell t}v$$

where each $u_{\ell j} \in \mathcal{G}(I(P_\omega^n \setminus x_{n-1}))$ and v is a monomial. It follows that $x_n|v$, since $x_{n-1} \nmid u$. Therefore, $u \in I(P_\omega^n \setminus x_{n-1})^t$.

(3) For any monomial $u \in \mathcal{G}(I^t) \setminus \mathcal{G}(I(P_\omega^n \setminus x_n)^t)$, it follows that $x_n|u$. So $u \in (x_n)$.

(4) If u is a monomial in $\mathcal{G}(I^t) \setminus \mathcal{G}(I(P_\omega^n \setminus x_{n-1})^t)$, then $x_{n-1}|u$, indicating that $u \in (x_{n-1})$.

(5) For any monomial $u \in \mathcal{G}((I^t : x_{n-1}), x_n)$, if $x_n|u$, then $u \in ((I(P_\omega^n \setminus x_n)^t : x_{n-1}), x_n)$. Otherwise, we have $ux_{n-1} \in I^t$. Let $ux_{n-1} = u_{i1} \dots u_{it}g$, where each $u_{ij} \in \mathcal{G}(I) \setminus \{x_{n-1}x_n\}$ and g is a monomial. This implies that each $u_{ij} \in \mathcal{G}(I(P_\omega^n \setminus x_n))$. Hence $u \in I(P_\omega^n \setminus x_n)^t : x_{n-1}$. $\square$

Lemma 4.11. *Let P_ω^4 be a path as in Remark 4.7. Suppose $\omega_1 \geq \omega_3 \geq 2$ and $\omega_2 = 1$. Then, for any $t \geq 2$, we have*
(1) $(I(P_\omega^4)^t : (x_2x_3)^{t-1}) = I(P_\omega^4)$;
(2) $((I(P_\omega^4)^t : (x_2x_3)^\ell), x_2x_3) = ((x_1x_2)^{(t-\ell)\omega_1}, x_2x_3, (x_3x_4)^{(t-\ell)\omega_3})$ *for* $\ell \in [t-2]$;
(3) $(I(P_\omega^4)^t, x_2x_3) = ((x_1x_2)^{t\omega_1}, x_2x_3, (x_3x_4)^{t\omega_3})$.

Proof. Let $I = I(P_\omega^4)$.

(1) It is evident that $I \subseteq (I^t\colon (x_2x_3)^{t-1})$. For any monomial u in $\mathcal{G}(I^t\colon (x_2x_3)^{t-1})$, if $x_2x_3|u$, then $u \in I$. Otherwise, we write $u(x_2x_3)^{t-1}$ as $u(x_2x_3)^{t-1} = (x_2x_3)^{t-j}h\prod_{\ell=1}^{j} u_{i\ell}$ for some $j \in [t]$, where each $u_{i\ell} \in \mathcal{G}(I)$, $x_2x_3 \nmid h$ and $x_2x_3 \nmid u_{i\ell}$. Thus $u(x_2x_3)^{j-1} = h\prod_{\ell=1}^{j} u_{i\ell}$. Suppose that $u_{i\ell} = (x_1x_2)^{\omega_1}$ for any $\ell \in [k]$ and $u_{i\ell} = (x_3x_4)^{\omega_3}$ for $k+1 \leq \ell \leq j$, then

$$u(x_2x_3)^{j-1} = (x_1x_2)^{k\omega_1}(x_3x_4)^{(j-k)\omega_3}h \tag{7}$$

We can distinguish six cases as follows:

(i) If $j = 2k+1$, then $u(x_2x_3)^{2k} = (x_1x_2)^{k\omega_1}(x_3x_4)^{(k+1)\omega_3}h$ by the equation (7). As $\omega_3 \geq 2$, $\omega_3(k+1) \geq \omega_3 + 2k$. Therefore, $(x_3x_4)^{\omega_3}|u$.

(ii) If $j > 2k+1$ and j is odd, then $2(j-k-1) > j-1$. Hence $\omega_3(j-k) > (j-1)+\omega_3$, since $\omega_3 \geq 2$. By comparing the powers of x_3 in the equation (7), we can deduce that $(x_3x_4)^{\omega_3}|u$.

(iii) If $j < 2k+1$ and j is odd, then $j-1 \leq 2(k-1)$. Thus, we have $k\omega_1 \geq 2(k-1)+\omega_1 \geq (j-1)+\omega_1$. Therefore, we can conclude that $(x_1x_2)^{\omega_1}|u$ by comparing powers of x_2 in the equation (7).

(iv) If $j = 2k$, then $\omega_1 k, \omega_3(j-k) \geq 2k$. Hence, $x_2x_3|u$ by the equation (7).

(v) If $j > 2k$ and j is even, then $2(j-k-1) \geq j-1$. It follows that $\omega_3(j-k) \geq 2(j-k-1)+\omega_3 \geq (j-1)+\omega_3$. As a result, $(x_3x_4)^{\omega_3}|u$ according to the equation (7).

(vi) If $j < 2k$ and j is even, then $k\omega_1 = (k-1)\omega_1+\omega_1 \geq 2(k-1)+\omega_1 \geq (j-1)+\omega_1$. Hence, $(x_1x_2)^{\omega_1}|u$ by using the equation (7).

In any case, we always have $u \in I$.

(2) Let $u \in \mathcal{G}(I^t : (x_2x_3)^\ell)$, and $x_2x_3|u$, then $u \in ((x_1x_2)^{(t-\ell)\omega_1}, x_2x_3, (x_3x_4)^{(t-\ell)\omega_3})$. Otherwise, we write $u(x_2x_3)^\ell$ as $u(x_2x_3)^\ell = (x_1x_2)^{i\omega_1}(x_2x_3)^j(x_3x_4)^{k\omega_3}h$, where h is a monomial such that $x_2x_3 \nmid h$ and $i+j+k=t$ with $j \leq \ell$, then

$$u(x_2x_3)^{\ell-j} = (x_1x_2)^{i\omega_1}(x_3x_4)^{k\omega_3}h. \tag{8}$$

We distinguish between the following two cases:

(i) When $i \leq k$, we can rewrite the equation (8) as

$$u(x_2x_3)^{\ell-j} = (x_2x_3)^{i\omega_3}(x_3x_4)^{(k-i)\omega_3}h' \tag{9}$$

in which $h' = x_1^{i\omega_1}x_2^{i(\omega_1-\omega_3)}x_4^{i\omega_3}h$, based on the power of x_3. Since $x_2x_3 \nmid h$, we have $\ell-j \geq i\omega_3$ by comparing powers of x_2x_3 in equation (9). Therefore, $u(x_2x_3)^{\ell-(j+i\omega_3)} = (x_3x_4)^{(k-i)\omega_3}h'$. Note that $\omega_1 \geq \omega_3 \geq 2$ and $i+j+k=t$ with $\ell-j \geq i\omega_3$, we have

$$\begin{aligned} & (k-i)\omega_3 + i\omega_3 - (\ell-j) = k\omega_3 + j - \ell = (t-i-j)\omega_3 + j - \ell + \ell\omega_3 - \ell\omega_3 \\ = \ & (t-\ell)\omega_3 - i\omega_3 + (\ell-j)(\omega_3-1) \geq (t-\ell)\omega_3 - i\omega_3 + (\ell-j) \\ \geq \ & (t-\ell)\omega_3 - i\omega_3 + i\omega_3 = (t-\ell)\omega_3 \end{aligned}$$

and

$$k-i = k+\ell-i-(i+j+k)+(t-\ell) = (\ell-j-2i)+(t-\ell) \geq (i\omega_3-2i)+(t-\ell) \geq (t-\ell).$$

By comparing the powers of x_3x_4 in the equation (9), we obtain $(x_3x_4)^{(t-\ell)\omega_3}|u$, which implies $u \in ((x_1x_2)^{(t-\ell)\omega_1}, x_2x_3, (x_3x_4)^{(t-\ell)\omega_3})$.

(ii) When $i > k$, the equation (8) can be rewritten as

$$u(x_2x_3)^{\ell-j} = (x_2x_3)^{k\omega_3}(x_1x_2)^{(i-k)\omega_1}v \tag{10}$$

with $v = x_1^{k\omega_1}x_2^{k(\omega_1-\omega_3)}x_4^{k\omega_3}h$. Since $(x_2x_3) \nmid u$, one has $\ell-j \geq k\omega_3$ by comparing powers of x_2x_3 in the equation (10). Thus, $u(x_2x_3)^{\ell-j-\omega_3k} = (x_1x_2)^{(i-k)\omega_1}v$. Note that $\omega_1 \geq \omega_3 \geq 2$ and $i+j+k=t$ with $\ell-j \geq k\omega_3$, we have

$$\begin{aligned} & (i-k)\omega_1 + k\omega_3 - (\ell-j) = (t-j-k)\omega_1 + k(\omega_3-\omega_1) + \ell\omega_1 - \ell\omega_1 - (\ell-j) \\ = \ & (t-\ell)\omega_1 + k(\omega_3-2\omega_1) + (\ell-j)(\omega_1-1) \\ \geq \ & (t-\ell)\omega_1 + k(\omega_3-2\omega_1) + k\omega_3(\omega_1-1) = (t-\ell)\omega_1 + k\omega_1(\omega_3-2) \\ \geq \ & (t-\ell)\omega_1 \end{aligned}$$

and

$$i-k=(t-j-k)-k+\ell-\ell=(t-\ell)-2k+(\ell-j)\geq(t-\ell)-2k+k\omega_3\geq(t-\ell).$$

By comparing powers of x_1x_2 in the equation (10), we obtain $(x_1x_2)^{(t-\ell)\omega_1}|u$, which implies that $u\in((x_1x_2)^{(t-\ell)\omega_1},x_2x_3,(x_3x_4)^{(t-\ell)\omega_3})$.

(3) Let $u\in\mathcal{G}(I^t)$, and $x_2x_3|u$, then $u\in((x_1x_2)^{t\omega_1},x_2x_3,(x_3x_4)^{t\omega_3})$. Otherwise, let $u=(x_1x_2)^{i\omega_1}(x_3x_4)^{j\omega_3}$, where $i+j=t$ and $j\leq\ell$. If $i,j\neq 0$, then $x_2x_3|u$. If $i=0$, then $j=t$, which means that $(x_3x_4)^{t\omega_3}|u$. If $j=0$, then $i=t$, which results in $(x_1x_2)^{t\omega_3}|u$. In any case, we always have $u\in((x_1x_2)^{t\omega_1},x_2x_3,(x_3x_4)^{t\omega_3})$. □

Using the previous preparatory knowledge, we can calculate the regularity and depth of the edge ideal of powers of a path P_ω^n, starting with $n\leq 4$.

Theorem 4.12. *Let P_ω^n be a path as in Remark 4.7, where $n\leq 4$. Let $\omega=\max\{\omega_1,\ldots,\omega_{n-1}\}$. Then, for all $t\geq 1$, we have the following results:*

(1) *If $n=2$ or 3, then $\operatorname{reg}(S/I(P_\omega^n)^t)=2t\omega-1$ and $\operatorname{depth}(S/I(P_\omega^n)^t)=1$.*

(2) *If $n=4$, then $\operatorname{reg}(S/I(P_\omega^n)^t)=2t\omega-1$ and*

$$\operatorname{depth}(S/I(P_\omega^n)^t)=\begin{cases}1, & \text{if } \omega_1=\omega_3=1 \text{ and } \omega_2>1,\\ 2, & \text{if } \omega_1,\omega_3>1 \text{ and } \omega_2=1.\end{cases}$$

Proof. Let $I=I(P_\omega^n)$. We will distinguish between the following two cases:

(1) The case where $n=2$ is straightforward. If $n=3$, then $I=((x_1x_2)^{\omega_1},x_2x_3)$ with $\omega_1\geq 2$, as proven in Lemma 4.4. We will prove these assertions by induction on t. The case where $t=1$ follows from Theorem 4.8. In the following, we assume that $t\geq 2$. Consider the following short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \frac{S}{I^t:x_3}(-1) & \xrightarrow{\cdot x_3} & \frac{S}{I^t} & \to & \frac{S}{(I^t,x_3)} & \to & 0,\\ 0 & \to & \frac{S}{(I^t:x_3):x_2}(-1) & \xrightarrow{\cdot x_2} & \frac{S}{I^t:x_3} & \to & \frac{S}{((I^t:x_3),x_2)} & \to & 0.\end{array}\tag{11}$$

Using Lemma 2.5, Lemma 4.10 and the inductive hypothesis, we can deduce that $\operatorname{reg}(S/((I^t:x_3):x_2))=\operatorname{reg}(S/I^{t-1})=2(t-1)\omega_1-1$, $\operatorname{reg}(S/((I^t:x_3),x_2)=\operatorname{reg}(S/(x_2))=0$ and $\operatorname{reg}(S/(I^t,x_3))=\operatorname{reg}(S/((x_1x_2)^{t\omega_1},x_3))=2t\omega_1-1$. Additionally, we can see that $\operatorname{depth}(S/((I^t:x_3):x_2))=\operatorname{depth}(S/I^{t-1})=1$, $\operatorname{depth}(S/((I^t:x_3),x_2)=\operatorname{depth}(S/(x_2))=2$ and $\operatorname{depth}(S/(I^t,x_3))=\operatorname{depth}(S/((x_1x_2)^{t\omega_1},x_3))=1$. Using Lemma 2.6 and the short exact sequences (11), we can determine that $\operatorname{reg}(S/I^t)=2t\omega_1-1$ and $\operatorname{depth}(S/I^t)=1$.

(2) If $n=4$, then $I=((x_1x_2)^{\omega_1},(x_2x_3)^{\omega_2},(x_3x_4)^{\omega_3})$ with $\omega_2=1$ or $\omega_1=\omega_3=1$ by Lemma 4.4. We will prove these assertions by induction on t with the case $t=1$ verified in Theorem 4.8. Now we assume that $t\geq 2$ and distinguish into the following three subcases:

(1) If $\omega_2=\omega_3=1$, then $(I^t,x_3x_4)=(J^t,x_3x_4)$ where $J=((x_1x_2)^{\omega_1},x_2x_3)$. Consider the following short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \frac{S}{I^t:x_3x_4}(-2) & \xrightarrow{\cdot x_3x_4} & \frac{S}{I^t} & \to & \frac{S}{(I^t,x_3x_4)} & \to & 0,\\ 0 & \to & \frac{S}{(J^t,x_3x_4):x_3}(-1) & \xrightarrow{\cdot x_3} & \frac{S}{(J^t,x_3x_4)} & \to & \frac{S}{(J^t,x_3)} & \to & 0,\\ 0 & \to & \frac{S}{((J^t,x_3x_4):x_3):x_2}(-1) & \xrightarrow{\cdot x_2} & \frac{S}{(J^t,x_3x_4):x_3)} & \to & \frac{S}{(((J^t,x_3x_4):x_3),x_2)} & \to & 0.\end{array}\tag{12}$$

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Lemma 4.10 implies that $(I^t : x_3x_4) = I^{t-1}$, $(J^t, x_3) = ((x_1x_2)^{t\omega_1}, x_3)$, $((J^t, x_3x_4) : x_3) : x_2 = (J^{t-1}, x_4)$, $(((J^t, x_3x_4) : x_3), x_2) = (x_2, x_4)$. Therefore, $\operatorname{reg}(S/(J^{t-1}, x_4)) = 2(t-1)\omega_1 - 1$, $\operatorname{reg}(S/I^{t-1}) = 2(t-1)\omega_1 - 1$, $\operatorname{reg}(S/((x_1x_2)^{t\omega_1}, x_3)) = 2t\omega_1 - 1$ and $\operatorname{reg}(S/(x_2, x_4)) = 0$ by Lemma 2.5 and the inductive hypothesis. From Lemma 2.6 and the above short exact sequences (12), we can conclude $\operatorname{reg}(S/I^t) = 2t\omega_1 - 1$.

(ii) If $\omega_2 = 1$ and $\omega_3 > 1$, then $I = ((x_1x_2)^{\omega_1}, x_2x_3, (x_3x_4)^{\omega_3})$. Consider the exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & \frac{S}{I^t:x_2x_3}(-2) & \xrightarrow{\cdot x_2x_3} & \frac{S}{I^t} & \to & \frac{S}{(I^t,x_2x_3)} & \to & 0, \\
0 & \to & \frac{S}{I^t:(x_2x_3)^2}(-2) & \xrightarrow{\cdot x_2x_3} & \frac{S}{I^t:\, x_2x_3} & \to & \frac{S}{((I^t:x_2x_3),x_2x_3)} & \to & 0, \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \frac{S}{I^t:(x_2x_3)^{t-2}}(-2) & \xrightarrow{\cdot x_2x_3} & \frac{S}{I^t:(x_2x_3)^{t-3}} & \to & \frac{S}{((I^t:(x_2x_3)^{t-3}),x_2x_3)} & \to & 0, \\
0 & \to & \frac{S}{I^t:(x_2x_3)^{t-1}}(-2) & \xrightarrow{\cdot x_2x_3} & \frac{S}{I^t:(x_2x_3)^{t-2}} & \to & \frac{S}{((I^t:(x_2x_3)^{t-2}),x_2x_3)} & \to & 0.
\end{array}$$

Since $I^t : (x_2x_3)^{t-1} = I$, $((I^t : (x_2x_3)^i), x_2x_3) = ((x_1x_2)^{(t-i)\omega_1}, x_2x_3, (x_3x_4)^{(t-i)\omega_3})$ for any $i \in [t-2]$, and $(I^t, x_2x_3) = ((x_1x_2)^{t\omega_1}, x_2x_3, (x_3x_4)^{t\omega_3})$ by Lemma 4.11. By Theorem 4.8, we can deduce that $\operatorname{reg}(S/((I^t : (x_2x_3)^i), x_2x_3)) = 2(t-i)\omega_1 - 1$ and $\operatorname{depth}(S/((I^t : (x_2x_3)^i), x_2x_3)) = 2$ for any $i = 0, \ldots, t-1$. From Lemma 2.6 and the above exact sequences, we can conclude that $\operatorname{reg}(S/I^t) = 2t\omega_1 - 1$ and $\operatorname{depth}(S/I^t) = 2$.

(iii) If $\omega_1 = \omega_3 = 1$, then $\omega_2 \geq 2$, since P^4_ω has non-trivial weight. We consider the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{S}{I^t:x_4}(-1) & \xrightarrow{\cdot x_4} & \frac{S}{I^t} & \longrightarrow & \frac{S}{(I^t,x_4)} & \longrightarrow & 0, \\
0 & \longrightarrow & \frac{S}{(I^t:x_4):x_3}(-1) & \xrightarrow{\cdot x_3} & \frac{S}{I^t:x_4} & \longrightarrow & \frac{S}{((I^t:x_4),x_3)} & \longrightarrow & 0.
\end{array} \tag{13}$$

Let $J = (x_1x_2, (x_2x_3)^{\omega_2})$, then, using Lemma 4.10, Lemma 2.5 and the inductive hypothesis, we can determine that $\operatorname{reg}(S/((I^t : x_4) : x_3)) = \operatorname{reg}(S/I^{t-1}) = 2(t-1)\omega_2 - 1$, $\operatorname{reg}(S/((I^t : x_4), x_3)) = \operatorname{reg}(S/((x_1x_2)^t, x_3)) = 2t - 1$ and $\operatorname{reg}(S/(I^t, x_4)) = \operatorname{reg}(S/(J^t, x_4)) = 2t\omega_2 - 1$. Additionally, we can also see that $\operatorname{depth}(S/((I^t : x_4) : x_3)) = \operatorname{depth}(S/I^{t-1}) = 1$, $\operatorname{depth}(S/((I^t : x_4), x_3) = \operatorname{depth}(S/((x_1x_2)^t, x_3)) = 2$ and $\operatorname{depth}(S/(I^t, x_4)) = \operatorname{depth}(S/(J^t, x_4)) = 1$. Applying Lemma 2.6 to the exact sequences (13), we obtain that $\operatorname{reg}(S/I^t) = 2t\omega_2 - 1$ and $\operatorname{depth}(S/I^t) = 1$. This concludes the proof. □

The following proposition can be shown using similar arguments as the proof of Theorem 4.12, provided that $\omega_2 \geq 2$ and $\omega_1 = \omega_3 = 1$, we omit its proof.

Proposition 4.13. *Let P^4_ω be a path as in Remark 4.7. If $\omega_1 > 0$ and $\omega_2 = \omega_3 = 1$, then* $\operatorname{depth}(S/I(P^4_\omega)^t) \geq 1$ *for any $t \geq 2$.*

In the following, we compute the regularity of powers of the edge ideal of a path P^n_ω with $n \geq 5$, as described in Remark 4.7.

Theorem 4.14. *Let P^n_ω be a path as in Remark 4.7, where $n \geq 5$. If $\omega_1 = \max\{\omega_i \mid i \in [n-1]\}$, then* $\operatorname{reg}(S/I(P^n_\omega)^t) \leq 2(t-1)\omega_1 + \operatorname{reg}(S/I(P^n_\omega))$.

Proof. Let $I = I(P_\omega^n)$. We will prove the assertions by induction on n and t. The case where $t = 1$ is trivial. In the following, we assume that $t \geq 2$. We have two cases:

(a) If $n = 5$, then, by Theorems 4.8, 4.9, Lemma 4.10 and the inductive hypothesis, we can derive that $\operatorname{reg}(S/((I^t : x_5) : x_4) = \operatorname{reg}(S/I^{t-1}) \leq 2\omega_1 + 2(t-2)\omega_1$, $\operatorname{reg}(S/((I^t : x_5), x_4) = \operatorname{reg}(S/(I(P_\omega^5 \setminus x_4)^t, x_4) \leq (2\omega_1 - 1) + 2(t-1)\omega_1$ and $\operatorname{reg}(S/(I^t, x_5)) = \operatorname{reg}(S/(I(P_\omega^5 \setminus x_5)^t, x_5)) \leq (2\omega_1 - 1) + 2(t-1)\omega_1$. The desired result follows from Theorem 4.9 and the following exact sequences

$$\begin{array}{llllllll} 0 & \rightarrow & \frac{S}{I^t:x_5}(-1) & \xrightarrow{\cdot x_5} & \frac{S}{I^t} & \rightarrow & \frac{S}{(I^t,x_5)} & \rightarrow 0, \\ 0 & \rightarrow & \frac{S}{(I^t:x_5):x_4}(-1) & \xrightarrow{\cdot x_4} & \frac{S}{I^t:x_5} & \rightarrow & \frac{S}{((I^t:x_5),x_4)} & \rightarrow 0. \end{array}$$

(b) Suppose that $n \geq 6$. In this case, let $J = I(P_\omega^n \setminus x_n)$, then, by Lemma 4.10, we have $(J^t : x_{n-1}) : x_{n-2} = J^{t-1}$, $((J^t : x_{n-1}), x_{n-2}) = (I(P_\omega^n \setminus \{x_n, x_{n-2}\})^t, x_{n-2})$, $(I^t : x_{n-1}) : x_n = I^{t-1}$, $((I^t : x_{n-1}), x_n) = ((I(P_\omega^n \setminus x_n)^t : x_{n-1}), x_n)$ and $(I^t, x_{n-1}) = (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$. It follows from Theorem 4.9 and the inductive hypothesis that

$$\operatorname{reg}(S/((J^t : x_{n-1}) : x_{n-2})) \leq (2\omega_1 - 1) + \lfloor \frac{n-3}{3} \rfloor + 2(t-2)\omega_1,$$
$$\operatorname{reg}(S/((J^t : x_{n-1}), x_{n-2})) \leq (2\omega_1 - 1) + \lfloor \frac{n-5}{3} \rfloor + 2(t-1)\omega_1,$$
$$\operatorname{reg}(S/((I^t : x_{n-1}) : x_n)) \leq (2\omega_1 - 1) + \lfloor \frac{n-2}{3} \rfloor + 2(t-2)\omega_1,$$
$$\operatorname{reg}(S/(I^t, x_{n-1})) \leq (2\omega_1 - 1) + \lfloor \frac{n-4}{3} \rfloor + 2(t-1)\omega_1.$$

Applying Lemma 2.6 again to the following exact sequences

$$\begin{array}{llllllll} 0 & \rightarrow & \frac{S}{I^t:x_{n-1}}(-1) & \xrightarrow{\cdot x_{n-1}} & \frac{S}{I^t} & \rightarrow & \frac{S}{(I^t,x_{n-1})} & \rightarrow 0, \\ 0 & \rightarrow & \frac{S}{(I^t:x_{n-1}):x_n}(-1) & \xrightarrow{\cdot x_n} & \frac{S}{I^t:x_{n-1}} & \rightarrow & \frac{S}{((I^t:x_{n-1}),x_n)} & \rightarrow 0, \\ 0 & \rightarrow & \frac{S}{(J^t:x_{n-1}):x_{n-2}}(-1) & \xrightarrow{\cdot x_{n-2}} & \frac{S}{J^t:x_{n-1}} & \rightarrow & \frac{S}{((J^t:x_{n-1}),x_{n-2})} & \rightarrow 0, \end{array}$$

one has $\operatorname{reg}(S/I^t) \leq 2\omega_1 - 1 + \lfloor \frac{n-2}{3} \rfloor + 2(t-1)\omega_1 = \operatorname{reg}(S/I) + 2(t-1)\omega_1$. □

Theorem 4.15. *Let P_ω^n be a path as in Remark 4.7, where $n \geq 5$. Then, for any $t \geq 1$, we have*

$$\operatorname{reg}(S/I(P_\omega^n)^t) \leq \operatorname{reg}(S/I(P_\omega^n)) + 2(t-1)\omega$$

where $\omega = \max\{\omega_i \mid i \in [n-1]\}$.

Proof. Let $I = I(P_\omega^n)$. Since P_ω^n has non-trivial weight, $\omega \geq 2$. If $\omega = \omega_i$, and $i = 1$, then the desired results follow from Theorem 4.14. Assume that $i \geq 2$ and we can prove the statements by induction on n and t. The case where $t = 1$ is trivial. Now, assuming that $t \geq 2$.

First, since P_ω^n is integrally closed, it has at most two edges with non-trivial weights by Corollary 4.5. Therefore, for ω_i and ω_{i+2}, by symmetry, there exist two cases: (a) $\omega_i > \omega_{i+2} \geq 1$, or (b) $\omega_i = \omega_{i+2} \geq 2$, while for other edges ω_j where $j \neq i, i+2$, we have $\omega_j = 1$. We distinguish into two cases:

20 Mar 2024 05:51:26 PDT
240103-ZhuGuangjun Version 2 - Submitted to Rocky Mountain J. Math.

(1) If $i=2$, then $(I^t:x_1):x_2=I^{t-1}$, $((I^t:x_1),x_2)=(I(P_\omega^n\setminus x_2)^t,x_2)$ and $(I^t,x_1)=(I(P_\omega^n\setminus x_1)^t,x_1)$ by Lemma 4.10. We have two subcases:

(a) If $\omega_i>\omega_{i+2}\geq 1$, then by the inductive hypothesis and Theorem 4.9, we can conclude that

$$\operatorname{reg}(S/((I^t:x_1):x_2)\leq(2\omega-1)+\lfloor\frac{n-3}{3}\rfloor+2(t-2)\omega,$$
$$\operatorname{reg}(S/(I^t,x_1))\leq(2\omega-1)+\lfloor\frac{n-3}{3}\rfloor+2(t-1)\omega. \qquad (14)$$

We will now prove by induction on t that $\operatorname{reg}(S/((I^t:x_1),x_2))\leq(2\omega_4-1)+\lfloor\frac{n-5}{3}\rfloor+2(t-1)\omega_4$ holds for all $t\geq 1$. The case $t=1$ has been verified in Theorem 4.9. Now assume that $t\geq 2$. Let $L=I(P_\omega^n\setminus x_2)$, $M=(x_5x_6,\dots,x_{n-1}x_n)$, then by Lemma 4.10, we obtain that $L=M+(x_3x_4,(x_4x_5)^{\omega_4})$, $(L^t:x_3):x_4=L^{t-1}$, $((L^t:x_3),x_4)=(M^t,x_4)$ and $(L^t,x_3)=(N^t,x_3)$, where $N=M+((x_4x_5)^{\omega_4})$. Therefore, we can deduce from Theorem 4.14 and the inductive hypothesis that

$$\begin{aligned}\operatorname{reg}(S/((L^t:x_3):x_4) &\leq (2\omega_4-1)+\lfloor\frac{n-5}{3}\rfloor+2(t-2)\omega_4,\\ \operatorname{reg}(S/((L^t:x_3),x_4)) &\leq \lfloor\frac{n-3}{3}\rfloor+2(t-1),\\ \operatorname{reg}(S/(L^t,x_3)) &\leq (2\omega_4-1)+\lfloor\frac{n-5}{3}\rfloor+2(t-1)\omega_4.\end{aligned}$$

Applying Lemma 2.6 to the following exact sequences

$$\begin{array}{ccccccccc} 0 & \rightarrow & \frac{S}{L^t:x_3}(-1) & \xrightarrow{\cdot x_3} & \frac{S}{L^t} & \rightarrow & \frac{S}{(L^t,x_3)} & \rightarrow & 0,\\ 0 & \rightarrow & \frac{S}{(L^t:x_3):x_4}(-1) & \xrightarrow{\cdot x_4} & \frac{S}{L^t:x_3} & \rightarrow & \frac{S}{((L^t:x_3),x_4)} & \rightarrow & 0,\end{array}$$

we obtain $\operatorname{reg}(S/((I^t:x_1),x_2))\leq(2\omega_4-1)+\lfloor\frac{n-5}{3}\rfloor+2(t-1)\omega_4$.

(b) If $\omega_i=\omega_{i+2}\geq 2$, then, by Theorem 4.9 and the inductive hypothesis, we obtain

$$\begin{aligned}\operatorname{reg}(S/((I^t:x_1):x_2)) &\leq 2\omega+\lfloor\frac{n-5}{3}\rfloor+2(t-2)\omega,\\ \operatorname{reg}(S/(I^t,x_1)) &\leq (2\omega-1)+\lfloor\frac{n-3}{3}\rfloor+2(t-1)\omega. \qquad (15)\end{aligned}$$

Using similar arguments as the proof of regularity of $S/((I^t:x_1),x_2)$ in part (a), We can also obtain

$$\operatorname{reg}(S/((I^t:x_1),x_2)) \leq (2\omega-1)+\lfloor\frac{n-5}{3}\rfloor+2(t-1)\omega. \qquad (16)$$

In the two cases mentioned above, we can obtain $\operatorname{reg}(S/I(P_\omega^n)^t)\leq\operatorname{reg}(S/I(P_\omega^n))+2(t-1)\omega$ by applying formulas (14)∼(16) and Lemma 2.6 to the following exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \frac{S}{I^t:x_1}(-1) & \xrightarrow{\cdot x_1} & \frac{S}{I^t} & \to & \frac{S}{(I^t,x_1)} & \to & 0, \\ 0 & \to & \frac{S}{(I^t:x_1):x_2}(-1) & \xrightarrow{\cdot x_2} & \frac{S}{I^t:x_1} & \to & \frac{S}{((I^t:x_1),x_2)} & \to & 0. \end{array}$$

(2) If $i \geq 3$, then $n \geq 6$. By Lemma 4.10, we can see that $(I^t, x_2) = (I(P^n_\omega \setminus x_2)^t, x_2)$, $(I^t : x_2) : x_1 = I^{t-1}$, $((I^t : x_2), x_1) = ((K^t : x_2), x_1)$, $(K^t : x_2) : x_3 = K^{t-1}$ and $((K^t : x_2), x_3) = (I(P^n_\omega \setminus \{x_1, x_3\})^t, x_3)$, where $K = I(P^n_\omega \setminus x_1)$. There are also two subcases.

(a) If $\omega_i > \omega_{i+2} \geq 1$, then by Theorem 4.9 and the inductive hypothesis, we can conclude that

$$\begin{array}{rcl} \operatorname{reg}(S/(I^t, x_2)) & \leq & A + \lfloor\frac{i-3}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \\ \operatorname{reg}(S/(I^t : x_2) : x_1) & \leq & (A - 2\omega) + \lfloor\frac{i-1}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \\ \operatorname{reg}(S/(K^t : x_2) : x_3) & \leq & (A - 2\omega) + \lfloor\frac{i-2}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \\ \operatorname{reg}(S/((K^t : x_2), x_3)) & \leq & \begin{cases} A + \lfloor\frac{n-7}{3}\rfloor, & \text{if } i = 3 \\ A + \lfloor\frac{i-4}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, & \text{if } i < 3 \end{cases} \end{array} \qquad (17)$$

where $A = (2\omega - 1) + 2\omega(t-1)$.

(b) If $\omega_i = \omega_{i+2} \geq 2$, then by Theorem 4.9 and the inductive hypothesis, we have

$$\begin{array}{rcl} \operatorname{reg}\left(\frac{S}{(I^t, x_2)}\right) & \leq & A + \max\{\lfloor\frac{i-3}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \lfloor\frac{i-4}{3}\rfloor + \lfloor\frac{n-i}{3}\rfloor\} \\ & \leq & \operatorname{reg}(S/I) + 2(t-1)\omega \\ \operatorname{reg}\left(\frac{S}{(I^t : x_2) : x_1}\right) & \leq & (A - 2\omega) + \max\{\lfloor\frac{i-1}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \lfloor\frac{i-2}{3}\rfloor + \lfloor\frac{n-i}{3}\rfloor\} \\ & \leq & \operatorname{reg}(S/I) + 2(t-1)\omega - 2\omega \\ \operatorname{reg}\left(\frac{S}{(K^t : x_2) : x_3}\right) & \leq & (A - 2\omega) + \max\{\lfloor\frac{i-2}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \lfloor\frac{i-3}{3}\rfloor + \lfloor\frac{n-i}{3}\rfloor\} \\ & \leq & \operatorname{reg}(S/I) + 2(t-1)\omega - 2\omega \\ \operatorname{reg}\left(\frac{S}{(K^t : x_2), x_3}\right) & \leq & \begin{cases} A + \lfloor\frac{n-6}{3}\rfloor, & \text{if } i = 3 \\ A + \max\{\lfloor\frac{i-4}{3}\rfloor + \lfloor\frac{n-(i+1)}{3}\rfloor, \lfloor\frac{i-5}{3}\rfloor + \lfloor\frac{n-i}{3}\rfloor\}, & \text{if } i < 3 \end{cases} \\ & \leq & \operatorname{reg}(S/I) + 2(t-1)\omega - 1. \end{array} \qquad (18)$$

In the two cases mentioned above, by applying formulas (17) and (18) as well as Lemma 2.6 to the following exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \frac{S}{I^t:x_2}(-1) & \xrightarrow{\cdot x_2} & \frac{S}{I^t} & \to & \frac{S}{(I^t,x_2)} & \to & 0, \\ 0 & \to & \frac{S}{(I^t:x_2):x_1}(-1) & \xrightarrow{\cdot x_1} & \frac{S}{I^t:x_2} & \to & \frac{S}{((I^t:x_2),x_1)} & \to & 0, \\ 0 & \to & \frac{S}{(K^t:x_2):x_3}(-1) & \xrightarrow{\cdot x_3} & \frac{S}{K^t:x_2} & \to & \frac{S}{((K^t:x_2),x_3)} & \to & 0, \end{array}$$

It is always possible to ensure that $\operatorname{reg}(S/I^t) \leq \operatorname{reg}(S/I) + 2(t-1)\omega$. □

Theorem 4.16. *Let P_ω^n be a path as in Remark 4.7. Then, for any $t \geq 1$, we have*

$$\operatorname{reg}(S/I(P_\omega^n)^t) = \operatorname{reg}(S/I(P_\omega^n)) + 2(t-1)\omega$$

where $\omega = \max\{\omega_i \mid i \in [n-1]\}$.

Proof. Let $I = I(P_\omega^n)$ and $\omega = \omega_i$, then $\omega_i \geq 2$. By applying symmetry, we can assume that (a) $\omega_i > \omega_{i+2} \geq 1$, or (b) $\omega_i = \omega_{i+2} \geq 2$, and $\omega_j = 1$ with $j \neq i, i+2$. We distinguish between the following two cases:

(a) Suppose $\omega_i > \omega_{i+2} \geq 1$ and $\omega_j = 1$ for all $j \neq i, i+2$. In this case, let $(I^t)^{\mathcal{P}}$ be the polarization of I^t, then by Lemma 2.3, $(I^t)^{\mathcal{P}} = J^{\mathcal{P}} + K^{\mathcal{P}}$, which is Betti splitting, and $J^{\mathcal{P}} \cap K^{\mathcal{P}} = J^{\mathcal{P}} L^{\mathcal{P}}$, where $\mathcal{G}(J) = \{(x_i x_{i+1})^{t\omega_i}\}$, $\mathcal{G}(K) = \mathcal{G}(I^t) \setminus \mathcal{G}(J)$ and $L = (x_1x_2, \ldots, x_{i-3}x_{i-2}) + (x_{i+3}x_{i+4}, \ldots, x_{n-1}x_n) + (x_{i-1}, x_{i+2})$ with $x_i = 0$ if $i \leq 0$. By Lemma 2.5, Lemma 4.2 and Theorem 4.9, it follows that $\operatorname{reg}(J^{\mathcal{P}}) = \operatorname{reg}(J) = 2t\omega$,

$$\begin{aligned}
\operatorname{reg}(J^{\mathcal{P}} \cap K^{\mathcal{P}}) &= \operatorname{reg}(J^{\mathcal{P}} L^{\mathcal{P}}) = \operatorname{reg}(J) + \operatorname{reg}(L) \\
&= 2t\omega + (\lfloor \frac{i-1}{3} \rfloor + \lfloor \frac{n-(i+1)}{3} \rfloor + 1) \\
&= \operatorname{reg}(I) + 2(t-1)\omega + 1.
\end{aligned}$$

Let H and H' be hypergraphs associated with $\mathcal{G}((I^t)^{\mathcal{P}})$ and $\mathcal{G}(K^{\mathcal{P}})$, respectively. Then H' is an induced subhypergraph of H. Therefore,

$$\operatorname{reg}(K^{\mathcal{P}}) \leq \operatorname{reg}((I^t)^{\mathcal{P}}) \leq \operatorname{reg}(I) + 2(t-1)\omega$$

by Theorem 4.15. Based on Corollary 2.2 and Lemmas 2.4 and 4.2, we can conclude that

$$\begin{aligned}
\operatorname{reg}(S/I^t) &= \operatorname{reg}((I^t)^{\mathcal{P}}) - 1 \\
&= \max\{\operatorname{reg}(J^{\mathcal{P}}), \operatorname{reg}(K^{\mathcal{P}}), \operatorname{reg}(J^{\mathcal{P}} \cap K^{\mathcal{P}}) - 1\} - 1 \\
&= 2(t-1)\omega + \operatorname{reg}(S/I).
\end{aligned}$$

(b) If $\omega_i = \omega_{i+2} \geq 2$, then $\operatorname{reg}(S/I) = \max\{2\omega + \lfloor \frac{i-1}{3} \rfloor + \lfloor \frac{n-(i+1)}{3} \rfloor, 2\omega + \lfloor \frac{i-2}{3} \rfloor + \lfloor \frac{n-i}{3} \rfloor\} - 1$ by Theorem 4.9. We have two subcases:

(i) If $\operatorname{reg}(S/I) = (2\omega - 1) + \lfloor \frac{i-1}{3} \rfloor + \lfloor \frac{n-(i+1)}{3} \rfloor$, then the statements can be proved by arguments similar to part (a), we omit the details.

(ii) If $\operatorname{reg}(S/I) = (2\omega - 1) + \lfloor \frac{i-2}{3} \rfloor + \lfloor \frac{n-i}{3} \rfloor$, then let $(I^t)^{\mathcal{P}}$ be the polarization of I^t. In this case, we can deduce that $(I^t)^{\mathcal{P}} = J^{\mathcal{P}} + K^{\mathcal{P}}$, which is Betti splitting, where $\mathcal{G}(J) = \{(x_{i+2}x_{i+3})^{t\omega_{i+2}}\}$ and $\mathcal{G}(K) = \mathcal{G}(I^t) \setminus \mathcal{G}(J)$. By arguments similar to part (a), we can conclude that

$$\begin{aligned}
\operatorname{reg}(J^{\mathcal{P}}) &= \operatorname{reg}(J) = 2t\omega, \\
\operatorname{reg}(K^{\mathcal{P}}) &\leq \operatorname{reg}((I^t)^{\mathcal{P}}) \leq \operatorname{reg}(I) + 2(t-1)\omega, \\
\operatorname{reg}(J^{\mathcal{P}} \cap K^{\mathcal{P}}) &= \operatorname{reg}(J^{\mathcal{P}} L^{\mathcal{P}}) = \operatorname{reg}(J) + \operatorname{reg}(L) \\
&= 2t\omega + (\lfloor \frac{i-2}{3} \rfloor + \lfloor \frac{n-i}{3} \rfloor + 1) \\
&= \operatorname{reg}(I) + 2(t-1)\omega + 1.
\end{aligned}$$

It follows from Corollary 2.2 and Lemmas 2.4 and 4.2 that

$$\begin{aligned}\operatorname{reg}(S/I^t) &= reg((I^t)^{\mathcal{P}}) - 1 = \max\{\operatorname{reg}(J^{\mathcal{P}}), \operatorname{reg}(K^{\mathcal{P}}), \operatorname{reg}(J^{\mathcal{P}} \cap K^{\mathcal{P}}) - 1\} - 1 \\ &= 2(t-1)\omega + \operatorname{reg}(S/I).\end{aligned}$$

□

We conclude this paper by presenting the lower bound on the depth of powers of the edge ideal of a path P_ω^n with $n \geq 5$, as described in Remark 4.7.

Theorem 4.17. *Let P_ω^n be a path as in Remark 4.7, where $n \geq 5$. If $\omega_1 > \omega_3$ and $\omega_i = 1$ for any $i \neq 1, 3$. Then for any $t \geq 1$, the following holds:*

(1) *if* $\omega_3 = 1$, *then* $\operatorname{depth}(S/I(P_\omega^n)^t) \geq \max\left\{\lceil\frac{n-t+1}{3}\rceil, 1\right\}$.

(2) *if* $\omega_3 > 1$, *then* $\operatorname{depth}(S/I(P_\omega^n)^t) \geq \max\left\{\lceil\frac{n-t+1}{3}\rceil, 2\right\}$.

Proof. Let $I = I(P_\omega^n)$ and $J = I(P_\omega^n \setminus x_n)$. Then J is an ideal in S_1, where $S_1 = \mathbb{K}[x_1, \dots, x_{n-1}]$. Furthermore, by Lemma 4.10, we can see that $(J^t : x_{n-1}) : x_{n-2} = J^{t-1}$, $((J^t : x_{n-1}), x_{n-2}) = (I(P_\omega^n \setminus \{x_n, x_{n-2}\})^t, x_{n-2})$, $(I^t : x_{n-1}) : x_n = I^{t-1}$, $((I^t : x_{n-1}), x_n) = ((I(P_\omega^n \setminus x_n)^t : x_{n-1}), x_n)$ and $(I^t, x_{n-1}) = (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$.

We will prove the statements by induction on n and t. There are two cases to consider.

(1) If $\omega_3 = 1$, then the base case where $t = 1$ follows from Theorem 4.9. Now, we assume that $t \geq 2$. By using Lemma 2.5, Theorem 4.12, Proposition 4.13, and the inductive hypothesis, we can deduce that

$$\begin{aligned}\operatorname{depth}(S_1/(J^t : x_{n-1}) : x_{n-2}) &\geq \max\left\{\lceil\frac{n-t+1}{3}\rceil, 1\right\}, \\ \operatorname{depth}(S_1/((J^t : x_{n-1}), x_{n-2})) &\geq \max\left\{\lceil\frac{n-t+1}{3}\rceil, 1\right\}, \\ \operatorname{depth}(S/((I^t : x_{n-1}) : x_n)) &\geq \max\left\{\lceil\frac{n-t+2}{3}\rceil, 1\right\}, \\ \operatorname{depth}(S/(I^t, x_{n-1})) &\geq \max\left\{\lceil\frac{n-t+2}{3}\rceil, 1\right\}.\end{aligned}$$

Applying Lemma 2.6 to the following exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \frac{S}{I^t:x_{n-1}}(-1) & \xrightarrow{\cdot x_{n-1}} & \frac{S}{I^t} & \to & \frac{S}{(I^t,x_{n-1})} & \to & 0, \\ 0 & \to & \frac{S}{(I^t:x_{n-1}):x_n}(-1) & \xrightarrow{\cdot x_n} & \frac{S}{I^t:x_{n-1}} & \to & \frac{S}{((I^t:x_{n-1}),x_n)} & \to & 0, \\ 0 & \to & \frac{S_1}{(J^t:x_{n-1}):x_{n-2}}(-1) & \xrightarrow{\cdot x_{n-2}} & \frac{S_1}{J^t:x_{n-1}} & \to & \frac{S_1}{((J^t:x_{n-1}),x_{n-2})} & \to & 0, \end{array} \tag{19}$$

it follows that $\operatorname{depth}(S/I^t) \geq \max\{\lceil\frac{n-t+1}{3}\rceil, 1\}$.

(2) Suppose $\omega_3 > 1$. There are the following two subcases to consider.

(i) If $n = 5$ or 6, then $((I^t : x_n), x_{n-1}) = (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$ and $(I^t, x_n) = (I(P_\omega^n \setminus x_n)^t, x_n)$. By applying Theorem 4.9, Lemma 2.5, and the inductive hypothesis, we can conclude that

$$\begin{aligned}\operatorname{depth}(S/((I^t : x_n) : x_{n-1})) \geq 2, \quad \operatorname{depth}(S/I) = 2, \\ \operatorname{depth}(S/((I^t : x_n), x_{n-1})) \geq 2, \quad \operatorname{depth}(S/(I^t, x_n)) \geq 2.\end{aligned}$$

Thus we obtain $\operatorname{depth}(S/I^t) \geq 2$ by applying Lemma 2.6 to the following exact sequences

$$\begin{array}{llllll} 0 & \rightarrow & \frac{S}{I^t:x_n}(-1) & \xrightarrow{\cdot x_n} & \frac{S}{I^t} & \rightarrow & \frac{S}{(I^t,x_n)} & \rightarrow 0, \\ 0 & \rightarrow & \frac{S}{(I^t:x_n):x_{n-1}}(-1) & \xrightarrow{\cdot x_{n-1}} & \frac{S}{I^t:x_n} & \rightarrow & \frac{S}{((I^t:x_n),x_{n-1})} & \rightarrow 0. \end{array} \quad (20)$$

(ii) Suppose that $n \geq 7$, we can use Lemma 2.5, Theorem 4.9, and the inductive hypothesis to conclude that

$$\begin{aligned} \operatorname{depth}(S_1/(J^t : x_{n-1}) : x_{n-2}) &\geq \max\Big\{\lceil\frac{n-t+1}{3}\rceil, 2\Big\}, \\ \operatorname{depth}(S_1/((J^t : x_{n-1}), x_{n-2})) &\geq \max\Big\{\lceil\frac{n-t+1}{3}\rceil, 2\Big\}, \\ \operatorname{depth}(S/((I^t : x_{n-1}) : x_n)) &\geq \max\Big\{\lceil\frac{n-t+2}{3}\rceil, 2\Big\}, \\ \operatorname{depth}(S/(I^t, x_{n-1})) &\geq \max\Big\{\lceil\frac{n-t+2}{3}\rceil, 2\Big\}. \end{aligned}$$

The desired formulas can be obtained by applying Lemma 2.6 to the exact sequences (19) mentioned above. □

Theorem 4.18. *Let P_ω^n be a path as in Remark 4.7, where $n \geq 5$. If $\omega_i > \omega_{i+2}$ for some $i \geq 2$, and $\omega_j = 1$ for all $j \neq i, i+2$. Then, for any $t \geq 2$, we have*

(1) *if $\omega_{i+2} = 1$, then* $\operatorname{depth}\left(\frac{S}{I(P_\omega^n)^t}\right) \geq \begin{cases} \lceil\frac{n-1}{3}\rceil, & \text{if } t = 2,\ i \equiv 1 \pmod 3 \text{ and } n \equiv 2 \pmod 3, \\ \max\Big\{\lceil\frac{n-t}{3}\rceil, 1\Big\}, & \text{otherwise.} \end{cases}$

(2) *if $\omega_{i+2} > 1$, then* $\operatorname{depth}\left(\frac{S}{I(P_\omega^n)^t}\right) \geq \max\Big\{\lceil\frac{n-t}{3}\rceil, 2\Big\}$.

Proof. Using the notations from the proof of Theorem 4.17, we can observe from Lemma 4.10 that $(J^t : x_{n-1}) : x_{n-2} = J^{t-1}$, $(I^t : x_{n-1}) : x_n = I^{t-1}$, $((J^t : x_{n-1}), x_{n-2}) = (I(P_\omega^n \backslash \{x_n, x_{n-2}\})^t, x_{n-2})$, $((I^t : x_{n-1}), x_n) = ((I(P_\omega^n \backslash x_n)^t : x_{n-1}), x_n)$ and $(I^t, x_{n-1}) = (I(P_\omega^n \setminus x_{n-1})^t, x_{n-1})$.

We will prove the statements by induction on n and t. The base case, where $t = 1$, follows from Theorem 4.9. Now, we assume that $t \geq 2$. There are two cases:

(1) If $\omega_{i+2} = 1$, then we can assume that $2 \leq i \leq \lfloor\frac{n}{2}\rfloor$ due to symmetry. Since $\operatorname{depth}(S/I) = \min\{\lceil\frac{i}{3}\rceil + \lceil\frac{n-i}{3}\rceil, \lceil\frac{i-2}{3}\rceil + \lceil\frac{n-i-2}{3}\rceil + 1\} \geq \lceil\frac{n-1}{3}\rceil$ by Theorem 4.9. Applying Lemma 2.5, Theorem 4.17, and the inductive hypothesis, we can conclude that

(i) If $t = 2$, $i \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, then

$$\begin{aligned} \operatorname{depth}(S_1/(J^t : x_{n-1}x_{n-2})) \geq \lceil\frac{n-1}{3}\rceil, \quad \operatorname{depth}(S/(I^t : x_{n-1}x_n)) \geq \lceil\frac{n-1}{3}\rceil, \\ \operatorname{depth}(S_1/((J^t : x_{n-1}), x_{n-2})) \geq \lceil\frac{n-1}{3}\rceil, \quad \operatorname{depth}(S/(I^t, x_{n-1})) \geq \lceil\frac{n-1}{3}\rceil. \end{aligned}$$

(ii) Otherwise, we have

$$\begin{aligned}\operatorname{depth}(S_1/(J^t:x_{n-1}x_{n-2}))&\geq\max\big\{\lceil\frac{n-t}{3}\rceil,1\big\},\\ \operatorname{depth}(S_1/((J^t:x_{n-1}),x_{n-2}))&\geq\max\big\{\lceil\frac{n-t}{3}\rceil,1\big\},\\ \operatorname{depth}(S/(I^t:x_{n-1}x_n))&\geq\max\big\{\lceil\frac{n-t+1}{3}\rceil,1\big\},\\ \operatorname{depth}(S/(I^t,x_{n-1}))&\geq\max\big\{\lceil\frac{n-t+1}{3}\rceil,1\big\}.\end{aligned}$$

By Lemma 2.6 and the exact sequences (19) mentioned above, we obtain

$$\operatorname{depth}\big(\frac{S}{I(P_\omega^n)^t}\big)\geq\begin{cases}\lceil\frac{n-1}{3}\rceil, & \text{if } t=2,\ i\equiv 1 \pmod 3 \text{ and } n\equiv 2 \pmod 3,\\ \max\Big\{\lceil\frac{n-t}{3}\rceil,1\Big\}, & \text{otherwise.}\end{cases}$$

(2) If $\omega_{i+2}>1$, then we can assume that $2\leq i\leq\lfloor\frac{n}{2}\rfloor-1$ due to symmetry. There are the following two cases:

(i) If $n=6$ or 7, then $((I^t:x_n),x_{n-1})=(I(P_\omega^n\backslash x_{n-1})^t,x_{n-1})$ and $(I^t,x_n)=(I(P_\omega^n\backslash x_n)^t,x_n)$. Applying Lemma 2.5, Theorem 4.9, Theorem 4.17, and the inductive hypothesis, it follows that

$$\begin{aligned}\operatorname{depth}(S/((I^t:x_n):x_{n-1}))\geq 2,\quad \operatorname{depth}(S/I)=2,\\ \operatorname{depth}(S/((I^t:x_n),x_{n-1}))\geq 2,\quad \operatorname{depth}(S/(I^t,x_n))\geq 2.\end{aligned}$$

By Lemma 2.6 and the exact sequences (20) mentioned above, we obtain $\operatorname{depth}(S/I^t)\geq 2$, which means that $\operatorname{depth}\big(\frac{S}{I(P_\omega^n)^t}\big)\geq\max\Big\{\lceil\frac{n-t}{3}\rceil,2\Big\}$.

(ii) If $n\geq 8$, then $\operatorname{depth}(S/I)=\min\{\lceil\frac{i}{3}\rceil+\lceil\frac{n-i-1}{3}\rceil,\lceil\frac{i-2}{3}\rceil+\lceil\frac{n-i-2}{3}\rceil+1\}\geq\lceil\frac{n-1}{3}\rceil$ by Theorem 4.9. Using Lemma 2.5, Theorem 4.17, and the inductive hypothesis, we obtain

$$\begin{aligned}\operatorname{depth}(S_1/(J^t:x_{n-1}x_{n-2}))&\geq\max\big\{\lceil\frac{n-t}{3}\rceil,2\big\},\\ \operatorname{depth}(S_1/((J^t:x_{n-1}),x_{n-2}))&\geq\max\big\{\lceil\frac{n-t}{3}\rceil,2\big\},\\ \operatorname{depth}(S/(I^t:x_{n-1}x_n))&\geq\max\big\{\lceil\frac{n-t+1}{3}\rceil,2\big\},\\ \operatorname{depth}(S/(I^t,x_{n-1}))&\geq\max\big\{\lceil\frac{n-t+1}{3}\rceil,2\big\}.\end{aligned}$$

Applying Lemma 2.6 to the exact sequences (19) mentioned above, we obtain $\operatorname{depth}(S/I^t)\geq\max\big\{\lceil\frac{n-t}{3}\rceil,2\big\}$. □

The following four examples provide instances in which the lower bounds of powers of the edge ideal of a non-trivial edge-weighted integrally closed path in Theorems 4.17 and 4.18 are attained.

Example 4.19. *$I_1 = (x_1^2x_2^2, x_2x_3, x_3x_4, x_4x_5)$, $I_2 = (x_1^2x_2^2, x_2x_3, x_3x_4, x_4x_5, x_5x_6)$ and $I_3 = (x_1^2x_2^2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7)$ are edge ideals of integrally closed paths P_ω^5, P_ω^6 and P_ω^7, respectively. Using CoCoA, we obtain that $\mathrm{depth}(R/I_1^2) = \mathrm{depth}(R/I_2^2) = \mathrm{depth}(R/I_3^2) = 2$, which is the lower bound given by Theorem 4.17.*

Example 4.20. *$I_1 = (x_1^2x_2^2, x_2x_3, x_3^3x_4^3, x_4x_5)$, $I_2 = (x_1^4x_2^4, x_2x_3, x_3^2x_4^2, x_4x_5, x_5x_6)$ and $I_3 = (x_1^2x_2^2, x_2x_3, x_3^3x_4^3, x_4x_5, x_5x_6, x_6x_7)$ are edge ideals of integrally closed paths P_ω^5, P_ω^6 and P_ω^7, respectively. Using CoCoA, we obtain that $\mathrm{depth}(R/I_1^2) = \mathrm{depth}(R/I_2^2) = \mathrm{depth}(R/I_3^2) = 2$, which is the lower bound given by Theorem 4.17.*

Example 4.21. *$I_1 = (x_1x_2, x_2^2x_3^2, x_3x_4, x_4x_5)$, $I_2 = (x_1x_2, x_2^2x_3^2, x_3x_4, x_4x_5, x_5x_6)$, $I_3 = (x_1x_2, x_2^2x_3^2, x_3x_4, x_4x_5, x_5x_6, x_6x_7)$ and $I_4 = (x_1x_2, x_2x_3, x_3x_4, x_4x_5^2, x_5x_6, x_6x_7, x_7x_8)$ are edge ideals of integrally closed paths P_ω^5, P_ω^6, P_ω^7 and P_ω^8, respectively. Using CoCoA, we obtain that $\mathrm{depth}(R/I_1^2) = 1$, $\mathrm{depth}(R/I_2^2) = \mathrm{depth}(R/I_3^2) = 2$ and $\mathrm{depth}(R/I_4^2) = 3$, which are the lower bounds given by Theorem 4.18.*

Example 4.22. *$I_1 = (x_1x_2, x_2^2x_3^2, x_3x_4, x_4^3x_5^3, x_5x_6)$, $I_2 = (x_1x_2, x_2^2x_3^2, x_3x_4, x_4^3x_5^3, x_5x_6, x_6x_7)$ and $I_3 = (x_1x_2, x_2^2x_3^2, x_3x_4, x_4^3x_5^3, x_5x_6, x_6x_7, x_7x_8)$ are edge ideals of integrally closed paths P_ω^6, P_ω^7 and P_ω^8, respectively. By using CoCoA, we obtain that $\mathrm{depth}(R/I_1^2) = \mathrm{depth}(R/I_2^2) = \mathrm{depth}(R/I_3^2) = 2$, which is the lower bound given by Theorem 4.18.*

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School of Mathematical Sciences, Soochow University, Suzhou, Jiangsu, 215006, P. R. China
E-mail address: zhuguangjun@suda.edu.cn(Corresponding author:Guangjun Zhu), 3136566920@qq.com(Shiya Duan), 237546805@qq.com(Yijun Cui), lijiaxinworking@163.com(Jiaxin Li).