ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

STABILITY AND SENSITIVITY ANALYSIS OF NON-STATIONARY α-FRACTAL FUNCTIONS

ANARUL ISLAM MONDAL AND SANGITA JHA*

ABSTRACT. This article presents a comprehensive investigation of fractal interpolation functions associated with a sequence of iterated function systems (IFSs). By selecting a suitable sequence of IFS parameters, the resulting non-stationary fractal function becomes a better approximant for the non-smooth function. To achieve this, we first construct the non-stationary interpolant within the Lipschitz space and examine key topological properties of the associated non-linear fractal operator. Furthermore, we explore the stability of the interpolant under small perturbations and analyze the sensitivity to perturbations in the IFS parameters. We provide an upper bound for the errors encountered during the approximation process. Finally, we study the continuous dependence of the proposed interpolant on various IFS parameters.

1. Introduction

The representation of arbitrary functions or data sets using simple classical functions like polynomials, trigonometric functions, or exponentials is a fundamental concept in numerical analysis and approximation theory. While traditional approaches have been successful in many cases, they often fall short in producing precise approximations for irregular forms encountered in real-world signals such as time series, financial data, climatic data, and bioelectric recordings. To address these irregularities, Barnsley [3] introduced the concept of fractal interpolation functions (FIFs), which provide a powerful framework for handling complex and irregular data. To further explore this field, interested readers are encouraged to look at the books [4, 14], and the additional sources cited therein.

FIFs offer several advantages compared to traditional interpolation functions. They are generally self-similar/affine, and their graph's Hausdorff-Besicovitch dimensions are non-integers. For an analysis of the dimensional properties of stationary fractal functions, readers can refer to [1, 2, 21, 25]. The primary advantage of fractal interpolants over classical interpolants lies in their ability to generate both smooth and non-smooth interpolations, depending on the chosen IFS parameters. It is worth noting that only a limited number of techniques, including subdivision schemes—a popular method can produce nonsmooth interpolants [9]. Recently, there have been attempts to establish a connection between subdivision methods and fractal interpolation [10, 13].

Motivated by the work of Barnsley, Navascués [17] explored a family of non-affine fractal functions, denoted as f^α or α-fractal functions. These functions correspond to a continuous function f defined on a closed and bounded interval I in $\mathbb{R}$. The function f^α approximates and interpolates f. The convergence analysis of the perturbation method was studied by Vijender, Chand, Navascués and Sebastian [27, 28]. This fractal perturbation

2000 *Mathematics Subject Classification.* Primary 28A80, 41A10; Secondary 43B15, 46A32.
Key words and phrases. Fractal functions (primary) and Non-stationary iterated function system, and Approximation and Stability.

approach gives rise to an operator that connects the theory of FIF with Operator Theory, Approximation Theory, Functional Analysis, and Harmonic Analysis.

Evidently, a lot of work has been done in the development of fractal approximations (stationary) [5, 8, 6, 7, 19, 20, 23], and it is still a subject of very active research, with an extensive list of connections and applications. But on the other hand, many questions remain to be settled in the non-stationary case, and more specifically in the case of a sequence of IFSs. The difficulty lies in the fact that the standard tools are not well developed in the setting of a sequence of IFSs. Consequently, advances in non-stationary IFS are required.

Using the fixed points of contractive operators for a specific type of IFS is a helpful way to build fractals [11, 22]. In the case of usual fractal functions, a common procedure of construction is to consider an operator on a complete space and define the fractal function as the unique fixed point. However, if the operator is substituted by a sequence of maps, then we face problems with similar construction. Levin, Dyn, Viswanathan [13] investigate the trajectory of contraction mappings which produces limit attractors at various scales with different features or shapes. It is known that various phenomena exhibit complex multiscale over a wide range of scales. Analysis of the geometry of complex phenomena has primarily been thought of constant fractal dimension. However, it has been observed by Takayasu [24] that more general descriptions are possible in terms of a scale-dependent extension of the fractal framework (fractal dimension may be a function of scale). As a consequence, the scale dependent behaviours can describe phenomena whose geometry may vary with scale, like "random walk trajectories, topographic surfaces, the galaxy distribution in the universe", etc. The use of mixed contraction maps in the construction of the non-stationary FIFs gives the flexibility to incorporate the scale and location-dependent features. In [15], Massopust advanced fractal function to a new and more adaptive environment by utilising the concept of forward and backward trajectories to offer new forms of fractal functions with the variable local and global behaviours. Recently, we have studied the non-stationary α-fractal functions in different function spaces [16].

In this article, we construct non-stationary α-fractal interpolation functions within the space of all Lipschitz functions on a closed and bounded interval I in $\mathbb{R}$. Our aim in presenting the non-stationary variant is not only to generalize the stationary situation but also to expand the applicability of non-stationary fractal functions. The inherent self-similarity of FIFs created by IFSs with constant parameters may lead to a loss of flexibility and noticeable inaccuracies when fitting and approximating complex curves and non-stationary data that exhibit less self-similarity. Non-stationary α-FIFs, on the other hand, do not depend on local or global data points, enabling greater flexibility and accuracy. We examine the stability and sensitivity of the analytical characteristics of FIFs generated by a class of IFSs with variable parameters. Furthermore, we calculate an upper bound for the errors that arise when comparing our proposed interpolant with the non-stationary interpolant derived from perturbed IFSs.

The rest of the article is organized as follows: Section 2 reviews trajectories and stationary/non-stationary iterated function systems (IFSs). Section 3 describes the construction of non-stationary α-fractal functions on the Lipschitz space $Lip_d(I)$, including the use of a sequence of IFSs for generating fractal functions. Section 4 explores a nonlinear fractal operator on $Lip_d(I)$ and discusses its topological properties. Section 5 investigates stability analysis by perturbing the ordinates and examines the sensitivity of the IFS parameters, providing an upper bound for approximation errors. Finally, Section 6 explores the continuous dependence of non-stationary fractal interpolation functions on the IFS parameters α and b.

2. Preparatory Results

In this section, we collect the necessary tools leading to the construction of non-stationary α-fractal functions in the Lipschitz space. For more details on this section, we invite the reader to study the paper of Massopust [15] and the work by Barnsley [3] and Navascués [17].

2.1. *Iterated Function System and Trajectories.* Let $(\mathbf{X}, d)$ be a complete metric space. Let $\mathscr{H}(\mathbf{X})$ denote the collection of all non-empty compact subsets of $\mathbf{X}$ and define the Hausdorff distance between sets A and B of $\mathscr{H}(\mathbf{X})$ as

$$d_H(A,B) = \inf\{\varepsilon \geq 0; A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\},$$

where $A_\varepsilon = \bigcup_{x \in A} \{z \in \mathbf{X};\ d(z,x) \leq \varepsilon\}$. The space $(\mathscr{H}(\mathbf{X}), d_H)$ is a complete metric space known as the space of fractals.

Definition 2.1. Let $(\mathbf{X}, d)$ be a complete metric space and $w_i : \mathbf{X} \to \mathbf{X}$ be N continuous maps. Then the system $\mathscr{I} = \{\mathbf{X}; w_i : i = 1, 2, \ldots, N\}$ is called an iterated function system (IFS). If each map w_i in $\mathscr{I}$ is a contraction, then the IFS $\mathscr{I}$ is hyperbolic.

The attractor of the IFS is the fixed point of the set-valued Hutchinson map $\mathscr{W} : \mathscr{H}(\mathbf{X}) \to \mathscr{H}(\mathbf{X})$,

$$\mathscr{W}(E) = \bigcup_{i=1}^{N} w_i(E),\ E \in \mathscr{H}(\mathbf{X}).$$

W is a contraction map on $\mathscr{H}(\mathbf{X})$ with the Lipschitz constant $Lip(\mathscr{W}) := \max\{Lip(w_i) : i = 1, 2, \ldots, N\}$. The uniqueness of the fixed point is guaranteed by the Banach fixed point theorem. The limit of the iterative process $A_k = \mathscr{W}(A_{k-1}); k \in \mathbb{N}$, where $A_0 \in \mathscr{H}(\mathbf{X})$ is any arbitrary set, also gives the attractor of the IFS.

Definition 2.2. Let $T : \mathbf{X} \to \mathbf{X}$ be a contraction map on a complete metric space $(\mathbf{X}, d)$. The forward iterates of T are transformations $T^{\circ n} : \mathbf{X} \to \mathbf{X}$ defined by $T^{\circ 0}(x) = x$ and $T^{\circ(n+1)}(x) = T(T^{\circ n}(x)) = \underbrace{T \circ T \circ \cdots \circ T}_{(n+1 \text{ times})}(x)$ for $n \in \mathbb{N} \cup \{0\}$.

Definition 2.3. (Forward and Backward Trajectories) Let $\mathbf{X}$ be a metric space and $\{T_r\}_{r \in \mathbb{N}}$ be a sequence of Lipschitz maps on $\mathbf{X}$. We define forward and backward trajectories respectively

$$\phi_r := T_r \circ T_{r-1} \circ \ldots \circ T_1 \text{ and } \psi_r := T_1 \circ T_2 \circ \ldots \circ T_r.$$

The subject now concerns the convergence of general trajectories, i.e., under what conditions the forward and backward trajectories will converge. In addition, we look for the trajectories producing new types of fractal sets. Recently, Levin et al. [13] observed that the backward trajectories converge under relatively mild conditions and may produce a new class of fractal sets. In [13], the authors use the assumption of compact invariant domain to guarantee the convergence of backward trajectories. In [18], Navascués and

Verma replace the compact invariant domain condition by a weaker condition. We now recall the result in the following.

Proposition 2.4. [18, Proposition 2.6] Let $\{T_r\}_{r\in\mathbb{N}}$ be a sequence of Lipschitz maps on a complete metric space $(\mathbf{X},d)$ with Lipschitz constants $c_r, r\in\mathbb{N}$ respectively. If $\exists\, x^*$ in the space such that the sequence $\{d(x^*,T_r(x^*))\}$ is bounded, and $\sum_{r=1}^{\infty}\prod_{i=1}^{r}c_i<\infty$, then the sequence $\{\psi_r(x)\}$ converges for all $x\in\mathbf{X}$ to a unique limit $\bar{x}$.

2.2. ***Stationary Fractal Interpolation Functions.*** For a fixed $k\in\mathbb{N}$, we denote by $\mathbb{N}_k$ the first k natural numbers and $\mathbb{N}_k^0=\mathbb{N}_k\cup\{0\}$. Let $I=[a,b]$ and define a partition Δ on I by

$$\Delta=\{(x_0,x_1,\dots,x_N):a=x_0<x_1<\dots<x_N=b\}.$$

For $i\in\mathbb{N}_N$, let $I_i=[x_{i-1},x_i]$. Suppose the affine maps $l_i:I\to I_i$ are such that

$$l_i(x_0)=x_{i-1}\ ,\ l_i(x_N)=x_i,\ \text{and}\ |l_i(z_1)-l_i(z_2)|\le l|z_1-z_2|\ \forall z_1,\ z_2\in I, \tag{2.1}$$

where $0\le l<1$. Set $\mathscr{M}=I\times\mathbb{R}$. Let the N continuous maps $F_i:\mathscr{M}\to\mathbb{R},\ i\in\mathbb{N}_N$ be such that

$$F_i(x_0,y_0)=y_{i-1},\qquad F_i(x_N,y_N)=y_i, \tag{2.2}$$

and F_i is a contraction in the second variable for $i\in\mathbb{N}_N$. The following are the most chosen maps for the formation of IFS

$$\begin{cases} l_i(x)=a_ix+e_i,\\ F_i(x,y)=\alpha_i(x)y+q_i(x),\quad i\in\mathbb{N}_N,\end{cases} \tag{2.3}$$

where a_i,e_i can be determined by conditions (2.1). $\alpha_i(x)$ are scaling functions satisfying $\|\alpha_i\|_\infty<1$ and $q_i(x)$ are suitable continuous functions such that condition (2.2) is satisfied. For $i\in\mathbb{N}_N$, we define $W_i:\mathscr{M}\to I_i\times\mathbb{R}$ by

$$W_i(x,y)=(l_i(x),F_i(x,y))\quad\forall(x,y)\in\mathscr{M}.$$

The complete metric space $(\mathscr{M},d_H)$ with the above N-maps $\{W_i:i\in\mathbb{N}_N\}$ forms an IFS. The uniqueness of the attractor of the IFS is given by Barnsley; mentioned below.

Theorem 2.5. [3] *The IFS $\mathscr{I}=\{\mathscr{M};W_i:i\in\mathbb{N}_N\}$ admits a unique attractor G, and G is the graph of a continuous function $f:I\to\mathbb{R}$ which interpolates the given data.*

2.3. ***Stationary α-fractal Functions.*** Let $\mathscr{C}(I)$ be the set of all continuous functions on a compact interval I. Let $f\in\mathscr{C}(I)$ be a prescribed function. Let us choose the function $q_i(x)$ in (2.3) as

$$q_i(x)=f(l_i(x))-\alpha_i(x)b(x),\ i\in\mathbb{N}_N,$$

where $\alpha_i:I\to\mathbb{R}$ are continuous scaling functions satisfying $\|\alpha_i\|_\infty<1$ and $b:I\to\mathbb{R}$ be continuous function such that $b\ne f$, $b(x_0)=f(x_0)$ and $b(x_N)=f(x_N)$. By Theorem 2.5, the corresponding IFS has a unique attractor G, which is the graph of a continuous map, namely $f^\alpha_{\Delta,b}:I\to\mathbb{R}$ that interpolates the given data set. The map is known as the α-fractal function.

Though the stationary fractal interpolant approximates irregular functions very well, it depends on local data points. To get a fractal interpolant that is independent of local data points and gets more flexibility, Massopust [15] introduced non-stationary fractal interpolation by taking a sequence of IFSs. In [18], the authors studied the parameterized non-stationary FIFs in $\mathscr{C}(I)$.

2.4. *Non-stationary α-fractal Functions.* Let $f \in \mathscr{C}(I)$ and $r \in \mathbb{N}$. We use the following notation:

$$\alpha_r := (\alpha_{1,r}, \alpha_{2,r}, \ldots, \alpha_{N,r}), \ \ \alpha := \{\alpha_r\}_{r\in\mathbb{N}} \ \text{ and } \ b := \{b_r\}_{r\in\mathbb{N}}.$$

Let $\alpha_{i,r} : I \to \mathbb{R}$ be continuous functions such that

$$\|\alpha\|_\infty = \sup\{\|\alpha_r\|_\infty : r \in \mathbb{N}\} < 1, \ \text{ where } \ \|\alpha_r\|_\infty = \sup\{\|\alpha_{i,r}\|_\infty : i \in \mathbb{N}_N\},$$

and $b_r \in \mathscr{C}(I)$ such that

$$b_r \neq f, \ b_r(x_0) = f(x_0) \text{ and } b_r(x_N) = f(x_N). \tag{2.4}$$

To define a sequence of IFSs, we use the following sequence of continuous maps

$$\begin{cases} l_i(x) = a_i x + e_i = \frac{x_i - x_{i-1}}{x_N - x_0} x + \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}, \\ F_{i,r}(x,y) = \alpha_{i,r}(x) y + f(l_i(x)) - \alpha_{i,r}(x) b_r(x), \quad i \in \mathbb{N}_N. \end{cases} \tag{2.5}$$

For each $i \in \mathbb{N}_N$, we define

$$W_{i,r} : \mathscr{M} \to I_i \times \mathbb{R} \text{ by } W_{i,r}(x,y) = (l_i(x), F_{i,r}(x,y)).$$

Now we have a sequence of IFSs $\mathscr{I}_r = \{\mathscr{M}; W_{i,r} : i \in \mathbb{N}_N\}$. Let

$$\mathscr{C}_f(I) = \{g \in \mathscr{C}(I) : g(x_i) = f(x_i), \ i = 0, N\}.$$

Then $\mathscr{C}_f(I)$ is a complete metric space. For $r \in \mathbb{N}$, we define a sequence of Read-Bajraktarević (RB) operators $T^{\alpha_r} : \mathscr{C}_f(I) \to \mathscr{C}_f(I)$ by

$$\begin{aligned} (T^{\alpha_r} g)(x) &= F_{i,r}(l_i^{-1}(x), \ g(l_i^{-1}(x))) \\ &= f(x) + \alpha_{i,r}(Q_i(x)) \cdot g(Q_i(x)) - \alpha_{i,r}(Q_i(x)) \cdot b_r(Q_i(x)), \end{aligned} \tag{2.6}$$

for $x \in I_i, \ i \in \mathbb{N}_N$, where $Q_i(x) = l_i^{-1}(x)$.
The above operator is well defined and for any function $h \in \mathscr{C}_f(I)$, the sequence of backward trajectories $\{T^{\alpha_1} \, o \, T^{\alpha_2} \, o \ldots o \, T^{\alpha_r} h\}$ converges to a map f^α of $C_f(I)$ [18]. The map f^α is the unique map that satisfies the following equation

$$f^\alpha(x) = f(x) + \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \ldots \alpha_{i,j}(Q_i^j(x))(f - b_j)(Q_i^j(x)), \tag{2.7}$$

where Q_i^j is a suitable finite composition of maps Q_i. The map f^α is called the non-stationary α-fractal interpolation function.

3. Non-stationary α-fractal function on Lipschitz Space

Let $g : I \to \mathbb{R}$ be a function. For $0 < d \leq 1$, define

$$Lip_d(g) \ = \ \sup\left\{ \frac{|g(x) - g(y)|}{|x - y|^d} : x, y \in I \text{ and } x \neq y \right\}.$$

The Lipschitz space is defined as $Lip_d(I) = \{g : I \to \mathbb{R} : Lip_d(g) < \infty\}$. Define $\|g\|_d = \max\{\|g\|_\infty, Lip_d(g)\}$. It is routine to show that $(Lip_d(I), \|.\|_d)$ is a Banach space. For more details of the Lipschitz functions in an arbitrary Banach space, please refer to [12]. Let

$$Lip_{d,f}(I) = \{g \in Lip_d(I) : g(x_0) = f(x_0), g(x_N) = f(x_N)\}.$$

Then $Lip_{d,f}(I)$ is a Banach space.

Theorem 3.1. *Let $f \in Lip_d(I)$. Let $r \in \mathbb{N}, b_r \in Lip_{d,f}(I)$ be such that $\|b\|_d := \sup_{r\in\mathbb{N}} \|b_r\|_d < \infty$ and the scaling functions $\alpha_{i,r} \in Lip_d(I)$ are chosen such that $\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right) < \frac{1}{2}$. We define a sequence of RB operators $T^{\alpha_r} : Lip_{d,f}(I) \to Lip_{d,f}(I)$ by*

$$(T^{\alpha_r} g)(x) = f(x) + \alpha_{i,r}(Q_i(x)) \cdot g(Q_i(x)) - \alpha_{i,r}(Q_i(x)) \cdot b_r(Q_i(x)), \tag{3.1}$$

for $x \in I_i$, $i \in \mathbb{N}_N$. Then the following hold.

1. *The RB operator T^{α_r} defined in equation (3.1) is well defined on $Lip_{d,f}(I)$.*
2. *In fact, $T^{\alpha_r} : Lip_{d,f}(I) \to Lip_{d,f}(I) \subset Lip_d(I)$ is a contraction map.*
3. *There exists a unique function $f^{\alpha}_{b,Lip_d} \in Lip_{d,f}(I)$ such that the sequence $\{T^{\alpha_1} \, o \, T^{\alpha_2} \, o \ldots o \, T^{\alpha_r} g\}$ converges to the map f^{α}_{b,Lip_d} for every $g \in Lip_{d,f}(I)$.*

Proof. (1) The norm defined on $Lip_{d,f}(I)$ is $\|f\|_d = \max\{\|f\|_\infty, Lip_d(f)\}$ so that, $\|T^{\alpha_r} f\|_d = \max\{\|T^{\alpha_r} f\|_\infty, Lip_d(T^{\alpha_r} f)\}$. From the definition of RB operators, we have

$$(T^{\alpha_r} g)(x) = f(x) + \alpha_{i,r}(Q_i(x)) \cdot (g - b_r)(Q_i(x)).$$

Now,

$$\begin{aligned}
Lip_d(T^{\alpha_r} g) &= \max_{i\in\mathbb{N}_N} \sup_{\substack{x,y\in I_i \\ x\neq y}} \frac{|T^{\alpha_r} g(x) - T^{\alpha_r} g(y)|}{|x-y|^d} \\
&= \max_{i\in\mathbb{N}_N} \sup_{\substack{x,y\in I_i \\ x\neq y}} \frac{|f(x) - f(y) + \alpha_{i,r}(Q_i(x)).(g-b_r)(Q_i(x)) - \alpha_{i,r}(Q_i(y)).(g-b_r)(Q_i(y))|}{|x-y|^d} \\
&\leq \max_{i\in\mathbb{N}_N} \sup_{\substack{x,y\in I_i \\ x\neq y}} \left[\frac{|f(x)-f(y)|}{|x-y|^d} + \frac{|\alpha_{i,r}(Q_i(x)).[(g-b_r)(Q_i(x)) - (g-b_r)(Q_i(y))]|}{|x-y|^d}\right. \\
&\qquad \left. + \frac{|(g-b_r)(Q_i(y))[\alpha_{i,r}(Q_i(x)) - \alpha_{i,r}(Q_i(y))]|}{|x-y|^d}\right] \\
&\leq Lip_d(f) + \max_{i\in\mathbb{N}_N} \|\alpha_{i,r}\|_\infty \sup_{\substack{x,y\in I_i \\ x\neq y}} \frac{|(g(Q_i(x)) - g(Q_i(y)))| + |(b_r(Q_i(x)) - b_r(Q_i(y)))|}{|x-y|^d} \\
&\qquad + \|g-b_r\|_\infty \max_{i\in\mathbb{N}_N} \sup_{\substack{x,y\in I_i \\ x\neq y}} \frac{|(\alpha_{i,r}(Q_i(x)) - \alpha_{i,r}(Q_i(y)))|}{|x-y|^d} \\
&= Lip_d(f) + \max_{i\in\mathbb{N}_N} \|\alpha_{i,r}\|_\infty \sup_{\substack{x,y\in I_i \\ x\neq y}} \frac{|(g(Q_i(x)) - g(Q_i(y)))| + |(b_r(Q_i(x)) - b_r(Q_i(y)))|}{a_i^d |Q_i(x) - Q_i(y)|^d} \\
&\qquad + \|g-b_r\|_\infty \max_{i\in\mathbb{N}_N} \sup_{\substack{x,y\in I_i \\ x\neq y}} \frac{|(\alpha_{i,r}(Q_i(x)) - \alpha_{i,r}(Q_i(y)))|}{a_i^d |Q_i(x) - Q_i(y)|^d} \\
&= Lip_d(f) + \max_{i\in\mathbb{N}_N} \left(\frac{\|\alpha_{i,r}\|_\infty}{a_i^d}\right) \sup_{\substack{\tilde{x},\tilde{y}\in I \\ \tilde{x}\neq\tilde{y}}} \frac{|g(\tilde{x}) - g(\tilde{y})| + |b_r(\tilde{x}) - b_r(\tilde{y}))|}{|\tilde{x}-\tilde{y}|^d} \\
&\qquad + \|g-b_r\|_\infty \max_{i\in\mathbb{N}_N} \sup_{\substack{\tilde{x},\tilde{y}\in I \\ \tilde{x}\neq\tilde{y}}} \frac{|(\alpha_{i,r}(\tilde{x}) - \alpha_{i,r}(\tilde{y}))|}{a_i^d |\tilde{x}-\tilde{y}|^d}
\end{aligned}$$

$$\leq Lip_d(f)+\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_\infty}{a_i^d}\right)(Lip_d(g)+Lip_d(b_r))+\max_{i\in\mathbb{N}_N}\left(\frac{Lip_d(\alpha_{i,r})}{a_i^d}\right)(\|g\|_\infty+\|b_r\|_\infty)$$
$$\leq Lip_d(f)+\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right)(Lip_d(g)+Lip_d(b_r)+\|g\|_\infty+\|b_r\|_\infty).$$

As $f,g,b_r\in Lip_{d,f}(I)$, the above estimation ensures that $Lip_d(T^{\alpha_r}g)<\infty$ and so that $T^{\alpha_r}g\in Lip_d(I)$. Also $T^{\alpha_r}g(x_0)=f(x_0)$ and $T^{\alpha_r}g(x_N)=f(x_N)$. Hence $T^{\alpha_r}g\in Lip_{d,f}(I)$ and the RB operator T^{α_r} defined in equation (2.6) is well defined on $Lip_{d,f}(I)$.

(2) For $x\in I_i$,

$$|(T^{\alpha_r}g_1-T^{\alpha_r}g_2)(x)|=|\alpha_{i,r}(Q_i(x))||(g_1-g_2)(Q_i(x))|$$
$$\leq\max_{i\in\mathbb{N}_N}(\|\alpha_{i,r}\|_\infty)\|g_1-g_2\|_\infty,$$

and hence

$$\|(T^{\alpha_r}g_1-T^{\alpha_r}g_2)\|_\infty\leq\max_{i\in\mathbb{N}_N}(\|\alpha_{i,r}\|_\infty)\|g_1-g_2\|_\infty. \tag{3.2}$$

Using similar steps in the estimation of $Lip_d(T^{\alpha_r}g)$, we obtain

$$Lip_d(T^{\alpha_r}g_1-T^{\alpha_r}g_2)\leq\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right)(Lip_d(g_1-g_2)+\|g_1-g_2\|_\infty). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\|(T^{\alpha_r}g_1-T^{\alpha_r}g_2)\|_d$$
$$=\max\{\|(T^{\alpha_r}g_1-T^{\alpha_r}g_2)\|_\infty,Lip_d(T^{\alpha_r}g_1-T^{\alpha_r}g_2)\}$$
$$\leq\max\left\{\max_{i\in\mathbb{N}_N}(\|\alpha_{i,r}\|_\infty)\|g_1-g_2\|_\infty,\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right)(Lip_d(g_1-g_2)+\|g_1-g_2\|_\infty)\right\}$$
$$\leq\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right)\max\{\|g_1-g_2\|_d,2\|g_1-g_2\|_d\}$$
$$=2\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right)\|g_1-g_2\|_d.$$

By assumptions on the sequence of scaling functions, we can ensure that T^{α_r} is a contraction.

(3) Let $g\in Lip_d(I)$ be an arbitrary function. We have to check if the sequence $\{\|T^{\alpha_r}g-g\|_d\}$ is bounded. Now, by a similar calculation as in item (1), we get

$$Lip_d(T^{\alpha_r}g-g)\leq Lip_d(T^{\alpha_r}g)+Lip_d(g)$$
$$\leq Lip_d(f)+\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_{i,r}\|_d}{a_i^d}\right)(Lip_d(g)+Lip_d(b_r)+\|g\|_\infty+\|b_r\|_\infty)+Lip_d(g)$$
$$\leq Lip_d(f)+\left(1+\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_i\|_d}{a_i^d}\right)\right)Lip_d(g)+2\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_i\|_d}{a_i^d}\right)\|b_r\|_d$$
$$\leq Lip_d(f)+\left(1+\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_i\|_d}{a_i^d}\right)\right)Lip_d(g)+2\max_{i\in\mathbb{N}_N}\left(\frac{\|\alpha_i\|_d}{a_i^d}\right)\|b\|_d, \tag{i}$$

where $\|\alpha_i\|_d:=\sup_{r\in\mathbb{N}}\|\alpha_{i,r}\|_d$. Also,

$$|(T^{\alpha_r}g-g)(x)|=|(f-g)(x)|+|\alpha_{i,r}(Q_i(x))|\cdot|(g-b_r)(Q_i(x))|$$
$$\leq\|f-g\|_\infty+\|\alpha\|_\infty\|g-b_r\|_\infty$$

$$\leq \|f-g\|_\infty + \|\alpha\|_\infty(\|g\|_\infty + \|b_r\|_\infty)$$
$$\leq \|f-g\|_d + \|\alpha\|_\infty(\|g\|_d + \|b_r\|_d).$$

Hence,

$$\|T^{\alpha_r}g - g\|_\infty \leq \|f-g\|_d + \|\alpha\|_\infty(\|g\|_d + \|b\|_d). \tag{ii}$$

Combining (i) and (ii), we get that the bound of $\|T^{\alpha_r}g - g\|_d$ is independent of r. Using Proposition 2.4, $\exists$ a unique $f^\alpha_{b,Lip_d} \in Lip_{d,f}(I)$ such that $f^\alpha_{b,Lip_d} = \lim_{r\to\infty} T^{\alpha_1}\, o\, T^{\alpha_2}\, o \ldots o\, T^{\alpha_r} g$ for any $g \in Lip_{d,f}(I)$. This completes the proof of the theorem. □

Definition 3.2. The function f^α_{b,Lip_d} is called a Lipschitz non-stationary α-fractal function with respect to f, α, b and the partition Δ.

Remark 3.3. As each T^{α_r} is a contraction, there exists a unique stationary α-fractal function f^α_r such that $T^{\alpha_r}(f^\alpha_r) = f^\alpha_r$ and it satisfies the functional equation:

$$f^\alpha_r(x) = F_{i,r}\left(\, Q_i(x),\ f^\alpha_r(Q_i(x))\right)\ \ \forall\ \ x \in I_i,$$

where $Q_i(x) := l_i^{-1}(x)$. That is,

$$f^\alpha_r(x) = f(x) + \alpha_{i,r}(Q_i(x)).f^\alpha_r(Q_i(x)) - \alpha_{i,r}(Q_i(x))b_r(Q_i(x)).$$

4. A Nonlinear Fractal Operator on $Lip_d(I)$

Suppose $L_r : Lip_d(I) \to Lip_d(I)$ is a sequence of operators such that $\|L\|_\infty := \sup_{r\in\mathbb{N}} \|L_r\|_\infty < \infty$ and satisfy $(L_r(f))(x_0) = f(x_0)$ and $(L_r(f))(x_N) = f(x_N)$. We set $b_r = L_r f$. The corresponding non-stationary α-fractal function will be denoted by f^α_b.

Definition 4.1. Let $f \in Lip_d(I)$ and Δ be fixed. We define the α-fractal operator $\mathfrak{F}^\alpha_b \equiv \mathfrak{F}^\alpha_{\Delta,b}$ as

$$\mathfrak{F}^\alpha_b : Lip_d(I) \subset \mathscr{C}(I) \to \mathscr{C}(I),\ \ \mathfrak{F}^\alpha_b(f) = f^\alpha_b.$$

Remark 4.2. In the case of a stationary fractal function, a similar construction is well studied in literature [17]. If we take $\alpha_{i,r} = \alpha_i \forall\ r \in \mathbb{N}, i \in \mathbb{N}_N$ and $b_r = Lf\ \forall\ r \in \mathbb{N}$, where $L : Lip_d(I) \to Lip_d(I)$ is an operator such that $(L(f))(x_0) = f(x_0)$ and $(L(f))(x_N) = f(x_N)$. Then the non-stationary α-fractal function will coincide with the stationary one.

Our next concern is to study the error approximation in the non-stationary perturbation process. The error bound in the different fractal approximations is well-studied in the stationary case [17].

Proposition 4.3. Let f^α_b be the non-stationary FIF corresponding to the seed function $f \in Lip_d(I)$. Then we have the following error bound

$$\|f^\alpha_b - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \sup_{r\in\mathbb{N}}\{\|f - L_r(f)\|_\infty\}. \tag{4.1}$$

Proof. The proof is similar to that given in Theorem 4.1. of [18]. □

Corollary 4.4. Let $f \in Lip_d(I)$ be the germ function and f^α_b be the corresponding non-stationary FIF. Then for any $j \in \mathbb{N}$, we have the following inequality

$$\|f^\alpha_b - L_j(f)\|_\infty \leq \frac{1}{1-\|\alpha\|_\infty} \sup_{r\in\mathbb{N}}\{\|f - L_r(f)\|_\infty\}.$$

Proof. Let $j \in \mathbb{N}$. Using inequality (4.1), we get

$$\begin{aligned}\|f_b^\alpha - L_j(f)\|_\infty &= \|f_b^\alpha - f + f - L_j(f)\|_\infty \\ &\le \|f_b^\alpha - f\|_\infty + \|f - L_j(f)\|_\infty \\ &\le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \sup_{r\in\mathbb{N}}\{\|f - L_r(f)\|_\infty\} + \|f - L_j(f)\|_\infty \\ &\le \frac{1}{1-\|\alpha\|_\infty} \sup_{r\in\mathbb{N}}\{\|f - L_r(f)\|_\infty\}.\end{aligned}$$

□

Based on the same arguments used in [18], we know if L_r is linear, then $\mathfrak{F}_b^\alpha$ is a linear operator. In order to keep track of this, let us write it down in the next proposition:

Proposition 4.5. The fractal operator $\mathfrak{F}_b^\alpha$ is a linear operator, provided that the sequence of operators $L_r : Lip_d(I) \to Lip_d(I)$ are linear for each $r \in \mathbb{N}$.

Unless otherwise specified, note that we do not assume that L_r is linear. As a result, the fractal operator is typically nonlinear (not necessarily linear). With regard to the conventional setting of fractal operators spread throughout the literature, the present findings abandon the general assumption of linearity and boundedness of the map L_r. Consequently, the research presented here may uncover possible applications of the fractal operator within the theory of unbounded and nonlinear operators.

Let us now collect some standard definitions of operators of interest in nonlinear functional analysis and perturbation theory. Let $(A, \|.\|_A)$ and $(B, \|.\|_B)$ be two normed linear spaces.

Definition 4.6. If an operator $\mathscr{T} : A \to B$ maps bounded sets to bounded sets, then it is said to be topologically bounded.

Definition 4.7. Let $\mathscr{T}_1 : D(\mathscr{T}_1) \subset A \to B$ and $\mathscr{T}_2 : D(\mathscr{T}_2) \subset A \to B$ be two operators such that $D(\mathscr{T}_2) \subset D(\mathscr{T}_1)$. If $\mathscr{T}_1, \mathscr{T}_2$ satisfy the following inequality

$$\|\mathscr{T}_1(u)\|_B \le t_1\|u\|_A + t_2\|\mathscr{T}_2(u)\|_B \;\; \forall\, u \in D(\mathscr{T}_2),$$

where t_1 and t_2 are some non-negative constants, then $\mathscr{T}_1$ is said to be relatively (norm) bounded with respect to $\mathscr{T}_2$ or simply $\mathscr{T}_2$-bounded. The $\mathscr{T}_2$-bound of $\mathscr{T}_1$ is defined as the infimum of all possible values of t_2 satisfying the aforementioned inequality.

Definition 4.8. An operator $\mathscr{T} : A \to B$ is said to be Lipschitz if there exists a constant $q > 0$ such that

$$\|\mathscr{T}(u) - \mathscr{T}(v)\|_B \le q\|u - v\|_A \;\forall\, u, v \in A.$$

For a Lipschitz operator $\mathscr{T} : A \to B$, the Lipschitz constant of $\mathscr{T}$ is denoted by $|\mathscr{T}|$.

Definition 4.9. Let $\mathscr{T}_1 : D(\mathscr{T}_1) \subset A \to B$ and $\mathscr{T}_2 : D(\mathscr{T}_2) \subset A \to B$ be two operators such that $D(\mathscr{T}_2) \subset D(\mathscr{T}_1)$. If $\mathscr{T}_1, \mathscr{T}_2$ satisfies the following inequality

$$\|\mathscr{T}_1(u) - \mathscr{T}_1(v)\|_B \le M_1\|u - v\|_A + M_2\|\mathscr{T}_2(u) - \mathscr{T}_2(v)\|_B \;\forall u, v \in D(\mathscr{T}_1),$$

where M_1 and M_2 are non-negative constants, then we say that $\mathscr{T}_1$ is relatively Lipschitz with respect to $\mathscr{T}_2$ or simply $\mathscr{T}_2$-Lipschitz. The infimum of all such values of M_2 is called the $\mathscr{T}_2$-Lipschitz constant of $\mathscr{T}_1$.

Proposition 4.10. The non-stationary fractal operator $\mathfrak{F}_b^\alpha : Lip_d(I) \to \mathscr{C}(I)$ is continuous whenever $L_r : Lip_d(I) \to Lip_d(I)$ is continuous for each $r \in \mathbb{N}$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a convergent sequence in $Lip_d(I)$, converges to $f \in Lip_d(I)$. We have,

$$f_b^\alpha(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(f - L_j f)(Q_i^j(x)).$$

Now,

$$\begin{aligned}
&|(f_n)_b^\alpha(x) - f_b^\alpha(x)| \\
&\leq |f_n(x) - f(x)| + |\lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(f_n - f - L_j f_n + L_j f)(Q_i^j(x))| \\
&\leq \|f_n - f\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j(\|f_n - f\|_\infty + \|L_j f_n - L_j f\|_\infty) \\
&\leq \|f_n - f\|_d + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j(\|f_n - f\|_d + \|L_j f_n - L_j f\|_d).
\end{aligned}$$

Since the inequality holds for all $x \in I$, we have

$$\|(f_n)_b^\alpha - f_b^\alpha\|_\infty \leq \|f_n - f\|_d + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j(\|f_n - f\|_d + \|L_j f_n - L_j f\|_d).$$

As the sequence (f_n) converges to f, we get our desired result using continuity of $L_j, j \in \mathbb{N}$. □

Proposition 4.11. If for each $r \in \mathbb{N}$, the operator $L_r : Lip_d(I) \to Lip_d(I)$ is a Lipschitz operator with Lipschitz constant $|L_r|$, then the non-stationary fractal operator $\mathfrak{F}_b^\alpha : Lip_d(I) \to \mathscr{C}(I)$ is also a Lipschitz operator, and $|\mathfrak{F}_b^\alpha| \leq \dfrac{1 + |L|\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}$, where $|L| := \sup_{r\in\mathbb{N}} |L_r| < \infty$.

Proof. Let $f, g \in Lip_d(I)$. Then

$$f_b^\alpha(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(f - L_j f)(Q_i^j(x)),$$

$$g_b^\alpha(x) = g(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(g - L_j g)(Q_i^j(x)).$$

Therefore,

$$\begin{aligned}
|f_b^\alpha(x) - g_b^\alpha(x)| &= |f(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(f - L_j f)(Q_i^j(x)) \\
&\qquad - g(x) - \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(g - L_j g)(Q_i^j(x))| \\
&\leq |f(x) - g(x)| + |\lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\ldots\alpha_{i,j}(Q_i^j(x))(f - g - L_j f + L_j g)(Q_i^j(x))| \\
&\leq \|f - g\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j(\|f - g\|_\infty + \|L_j f - L_j g\|_\infty)
\end{aligned}$$

$$\begin{aligned}
&\le \|f-g\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r}\|\alpha\|_\infty^j(\|f-g\|_\infty + |L_j|\cdot\|f-g\|_\infty)\\
&\le \left(1+\sum_{j=1}^{\infty}\|\alpha\|_\infty^j(1+|L|)\right)\cdot\|f-g\|_\infty\\
&= \left(1+\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}(1+|L|)\right)\cdot\|f-g\|_\infty\\
&\le \frac{1+|L|\cdot\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\cdot\|f-g\|_d.
\end{aligned}$$

This holds for every $x\in I$, hence

$$\|\mathfrak{F}_b^\alpha(f)-\mathfrak{F}_b^\alpha(g)\|_\infty = \|f_b^\alpha - g_b^\alpha\|_\infty \le \frac{1+|L|\cdot\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f-g\|_d.$$

This concludes the proof. □

Proposition 4.12. The non-stationary fractal operator $\mathfrak{F}_b^\alpha : Lip_d(I)\to\mathscr{C}(I)$ is topologically bounded provided that $L_r : Lip_d(I)\to Lip_d(I)$ is uniformly bounded.

Proof. Let f be a function in $Lip_d(I)$. We have,

$$\begin{aligned}
|f_b^\alpha(x)| &\le |f(x)| + |\lim_{r\to\infty}\sum_{j=1}^{r}\alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f-L_jf)(Q_i^j(x))|\\
&\le \|f\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r}\|\alpha\|_\infty^j\|f-L_jf\|_\infty\\
&\le \|f\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r}\|\alpha\|_\infty^j(\|f\|_\infty + \|L_jf\|_\infty)\\
&\le (1+\sum_{j=1}^{\infty}\|\alpha\|_\infty^j)\|f\|_\infty + \sum_{j=1}^{\infty}\|\alpha\|_\infty^j\|L_jf\|_\infty\\
&= \frac{1}{1-\|\alpha\|_\infty}\|f\|_\infty + \sum_{j=1}^{\infty}\|\alpha\|_\infty^j\|L_jf\|_\infty\\
&\le \frac{1}{1-\|\alpha\|_\infty}\|f\|_d + \sum_{j=1}^{\infty}\|\alpha\|_\infty^j\|L_jf\|_\infty.
\end{aligned}$$

Hence

$$\|\mathfrak{F}_b^\alpha(f)\|_\infty = \|f_b^\alpha\|_\infty \le \frac{1}{1-\|\alpha\|_\infty}\|f\|_d + \sum_{j=1}^{\infty}\|\alpha\|_\infty^j\|L_jf\|_\infty.$$

Since $L_j (j\in\mathbb{N})$ is uniformly bounded, it follows from the above inequality that the operator $\mathfrak{F}_b^\alpha$ is topologically bounded. □

In the following propositions of this section, we assume that L_r be a sequence of linear operators such that there exists a linear operator L satisfying $\|Lf\|_\infty = \sup_{r\in\mathbb{N}}\|L_rf\|_\infty$. Let us move on to the following proposition using this presumption.

Proposition 4.13. The non-stationary fractal operator $\mathfrak{F}_b^\alpha : Lip_d(I)\to\mathscr{C}(I)$ is relatively Lipschitz with respect to L with L-Lispchitz constant of $\mathfrak{F}_b^\alpha$ not exceeding $\dfrac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$.

Proof. Let $f, g \in Lip_d(I)$. Then the functions satisfy the following equations:

$$f_b^\alpha(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f - L_jf)(Q_i^j(x)),$$

and

$$g_b^\alpha(x) = g(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(g - L_jg)(Q_i^j(x)).$$

Now,

$$\begin{aligned}
&|f_b^\alpha(x) - g_b^\alpha(x)| \\
&= |f(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f - L_jf)(Q_i^j(x)) \\
&\qquad - g(x) - \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(g - L_jg)(Q_i^j(x))| \\
&\le |f(x) - g(x)| + |\lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f - g - L_jf + L_jg)(Q_i^j(x))| \\
&\le \|f-g\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j(\|f-g\|_\infty + \|L_jf - L_jg\|_\infty) \\
&\le \|f-g\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j(\|f-g\|_\infty + \|L_j(f-g)\|_\infty) \\
&\le \left(1 + \sum_{j=1}^{\infty}\|\alpha\|_\infty^j\right)\|f-g\|_\infty + \left(\sum_{j=1}^{\infty}\|\alpha\|_\infty^j\right)\|L(f-g)\|_\infty \\
&\le \left(\frac{1}{1-\|\alpha\|_\infty}\right)\|f-g\|_d + \left(\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\right)\|Lf - Lg\|_\infty.
\end{aligned}$$

For all x, the above mentioned inequality is true, hence

$$\|\mathfrak{F}_b^\alpha(f) - \mathfrak{F}_b^\alpha(g)\|_\infty = \|f_b^\alpha - g_b^\alpha\|_\infty \le \left(\frac{1}{1-\|\alpha\|_\infty}\right)\|f-g\|_d + \left(\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\right)\|Lf - Lg\|_\infty.$$

This completes the proof. □

Proposition 4.14. The non-stationary fractal operator $\mathfrak{F}_b^\alpha : Lip_d(I) \to \mathscr{C}(I)$ is relatively bounded with respect to L with L-bound is less than or equal to $\dfrac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$.

Proof. Let f be an arbitrary function in $Lip_d(I)$. From (2.6), we have

$$f_b^\alpha(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f - L_jf)(Q_i^j(x)).$$

Therefore,

$$\begin{aligned}
|f_b^\alpha(x)| &= |f(x)| + |\lim_{r\to\infty}\sum_{j=1}^{r} \alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f - L_jf)(Q_i^j(x))| \\
&\le \|f\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r} \|\alpha\|_\infty^j\|f - L_jf\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\leq \|f\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r}\|\alpha\|_\infty^j(\|f\|_\infty + \|L_j f\|_\infty)\\
&\leq \|f\|_\infty + \lim_{r\to\infty}\sum_{j=1}^{r}\|\alpha\|_\infty^j(\|f\|_\infty + \|L f\|_\infty)\\
&= \|f\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}(\|f\|_\infty + \|Lf\|_\infty)\\
&= \frac{1}{1-\|\alpha\|_\infty}\|f\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|Lf\|_\infty\\
&\leq \frac{1}{1-\|\alpha\|_\infty}\|f\|_d + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|Lf\|_\infty.
\end{aligned}$$

The aforementioned inequality holds for all x, hence

$$\|\mathfrak{F}_b^\alpha(f)\|_\infty = \|f_b^\alpha\|_\infty \leq \frac{1}{1-\|\alpha\|_\infty}\|f\|_d + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|Lf\|_\infty. \tag{4.2}$$

This proves our claim. □

5. Stability and sensitivity analysis

Let us now investigate the stability of the FIF with changeable parameters produced by IFS $\mathscr{I}_r = \{\mathscr{M}; W_{i,r}(x,y) = (l_i(x), F_{i,r}(x,y)), i \in \mathbb{N}_N\}$, where the maps are defined in (2.5) and $\mathscr{M} = I \times [k_1,k_2] \subset \mathbb{R}^2$. The similar results for the stationary case can be observed in [29]. Let $\bar{\mathbf{D}} := \{(x_i,\bar{y}_i) : i \in \mathbb{N}_N^0\}$ be another set of interpolation points in $\mathscr{M}$ which can be considered as the perturbations of ordinates of the points in $\mathbf{D} := \{(x_i,y_i) \in I \times [k_1,k_2] : i \in \mathbb{N}_N^0\}$. For the data set $\bar{\mathbf{D}}$, an IFS can be defined by $\bar{\mathscr{I}}_r = \{\mathscr{M}; \bar{W}_{i,r}(x,y) = (l_i(x), \bar{F}_{i,r}(x,y)), i \in \mathbb{N}_N\}$, where $l_i(x),\ i \in \mathbb{N}_N$, are the maps defined in (2.5), and $\bar{F}_{i,r}$ are defined as

$$\bar{F}_{i,r} = \alpha_{i,r}(x)y + \hat{f}(l_i(x)) - \alpha_{i,r}(x)\hat{b}_r(x), \quad i \in \mathbb{N}_N, \quad r \in \mathbb{N}. \tag{5.1}$$

Here we consider the base functions b_r and perturbed base functions $\bar{b}_r$ in $\mathscr{C}_f(I)$ such that $\sup_{r\in\mathbb{N}}\|b_r\|_\infty < \infty$ and $\sup_{r\in\mathbb{N}}\|\bar{b}_r\|_\infty < \infty$.

Theorem 5.1. *Let $\mathbf{D} := \{(x_i,y_i) : i \in \mathbb{N}_N^0\}$ and $\bar{\mathbf{D}} := \{(x_i,\bar{y}_i) : i \in \mathbb{N}_N^0\}$ be two data sets in $\mathscr{M}$. Let f_b^α be the non-stationary FIF for $\mathbf{D}$ generated by the sequence of IFSs $\mathscr{I}_r = \{\mathscr{M}; W_{i,r}(x,y) = (l_i(x), F_{i,r}(x,y)), i \in \mathbb{N}_N\}$ defined in (2.5) and $\bar{f}_b^\alpha$ be the non-stationary FIF for $\bar{\mathbf{D}}$ generated by the sequence of IFSs $\bar{\mathscr{I}}_r = \{\mathscr{M}; \bar{W}_{i,r}(x,y) = (l_i(x), \bar{F}_{i,r}(x,y)), i \in \mathbb{N}_N\}$ defined through (5.1). Then we have,*

$$\|f_b^\alpha - \bar{f}_b^\alpha\|_\infty \leq \frac{\|f-\hat{f}\|_\infty + \|\alpha\|_\infty \cdot \sup_{r\in\mathbb{N}}\{\|b_r - \hat{b}_r\|_\infty\}}{1-\|\alpha\|_\infty}. \tag{5.2}$$

Proof. From (2.7), we have

$$f_b^\alpha(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r}\alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f-b_j)(Q_i^j(x)).$$

Therefore,

$$\begin{aligned}
&|f_b^\alpha(x) - \hat{f}_b^\alpha(x)| \\
&= |f(x) + \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(f - b_j)(Q_i^j(x)) - \hat{f}(x) \\
&\qquad - \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(\hat{f} - \hat{b}_j)(Q_i^j(x))| \\
&\le |f(x) - \hat{f}(x)| + |\lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(f - \hat{f} - b_j + \hat{b}_j)(Q_i^j(x))| \\
&\le \|f - \hat{f}\|_\infty + \lim_{r\to\infty} \sum_{j=1}^{r} \|\alpha\|_\infty^j (\|f - \hat{f}\|_\infty + \|b_j - \hat{b}_j\|_\infty) \\
&\le \left(1 + \sum_{j=1}^{\infty} \|\alpha\|_\infty^j\right) \|f - \hat{f}\|_\infty + \left(\lim_{r\to\infty} \sum_{j=1}^{r} \|\alpha\|_\infty^j\right) \sup_{r\in\mathbb{N}} \{\|b_r - \hat{b}_r\|_\infty\} \\
&= \left(1 + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\right) \|f - \hat{f}\|_\infty + \left(\frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\right) \sup_{r\in\mathbb{N}} \{\|b_r - \hat{b}_r\|_\infty\} \\
&= \frac{\|f - \hat{f}\|_\infty + \|\alpha\|_\infty \cdot \sup_{r\in\mathbb{N}} \{\|b_r - \hat{b}_r\|_\infty\}}{1 - \|\alpha\|_\infty}.
\end{aligned}$$

The above inequality holds for all $x \in I$; hence inequality (5.2) follows. □

Remark 5.2. Let $f, \hat{f}$ be two piecewise linear functions through the interpolation data sets $\mathbf{D}$ and $\bar{\mathbf{D}}$ respectively. Also assume that $b_r = \hat{b}_r = b$ is a linear function passing through the points (x_0, y_0) and (x_N, y_N). Then we have

$$\|f_b^\alpha - \bar{f}_b^\alpha\|_\infty \le \frac{1 + \|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \max_{i\in\mathbb{N}_N^0} \{|y_i - \bar{y}_i|\},$$

which is the same result for stationary FIF given in [29]. So, our result can be treated as a generalisation of the existing result.

Remark 5.3. Perturbations of abscissas of interpolation points can be taken to affect the values of the non-stationary FIFs associated with the interpolation points. Also, perturbations of both abscissas and ordinates may be considered to examine the stability of the non-stationary FIF. For more details, the reader is invited to read the paper of Wang and Yu [29].

Next, we discuss the sensitivity of the non-stationary α-FIF defined by the IFS $\mathscr{I}_r$. Let $f, b_r, \alpha_{i,r}$ be as defined before and $T_{i,r} : \mathscr{M} \to \mathbb{R}, i \in \mathbb{N}_N, r \in \mathbb{N}$, be a sequence of continuous functions on $\mathscr{M}$ such that for all $(x,y) \in \mathscr{M}$,

$$T_{i,r}(x,y) = f(x) + [\alpha_{i,r}(Q_i(x)) + t_{i,r}\theta_{i,r}(Q_i(x))](g - b_r)(Q_i(x)) + s_{i,r}\phi_{i,r}(Q_i(x)),$$

where $t_{i,r}, s_{i,r}$ are parameters of perturbation satisfying $0 < t_{i,r} < 1$ and $0 < s_{i,r} < 1$, $\phi_{i,r}, \theta_{i,r}(> 0) \in Lip_d(I)$ satisfying $\sup_r \max_i \|\alpha_{i,r}\|_\infty + \sup_r \max_i \|t_{i,r}\theta_{i,r}\|_\infty < 1$ and $\phi_{i,r}(x_0) = \phi_{i,r}(x_N) = 0$. The function $T_{i,r}$ is a perturbation of the function $F_{i,r}$ for each $i \in \mathbb{N}_N, r \in \mathbb{N}$. Thus the IFS $\mathscr{I}'_r = \{\mathscr{M}; (l_i(x), T_{i,r}(x,y)), i \in \mathbb{N}_N\}$ may be treated as the perturbation IFS of the IFS $\mathscr{I}_r =$

$\{\mathscr{M};(l_i(x),F_{i,r}(x,y)),i\in\mathbb{N}_N\}$. For each $r\in\mathbb{N},i\in\mathbb{N}_N,T_{i,r}$ is also contractive in the second variable and it satisfy

$$T_{i,r}(x_0,y_0)=y_{i-1},\ \ T_{i,r}(x_N,y_N)=y_i.$$

Therefore the IFS $\mathscr{I'}_r=\{\mathscr{M};(l_i(x),T_{i,r}(x,y)),i\in\mathbb{N}_N\}$ determines a unique non-stationary FIF, denoted by $f^{\alpha,t}_{b,s}$.

Theorem 5.4. *Let* $\mathbf{D}:=\{(x_i,y_i):i\in\mathbb{N}^0_N\}$ *be a data set in* $\mathscr{M}$. *Let* f^α_b *be the non-stationary FIFs corresponding to the sequence of IFSs* $\mathscr{I}_r=\{\mathscr{M};(l_i(x),F_{i,r}(x,y)),i\in\mathbb{N}_N\}$ *defined in (2.5) and* $f^{\alpha,t}_{b,s}$ *be the non-stationary FIF generated by the sequence of IFSs* $\mathscr{I'}_r=\{\mathscr{M};(l_i(x),T_{i,r}(x,y)),i\in\mathbb{N}_N\}$. *Then*

$$\|f^{\alpha,t}_{b,s}-f^\alpha_b\|_\infty\le\frac{\|\phi\|_\infty}{1-\|\alpha\|_\infty-|t|_\infty\|\theta\|_\infty}|s|_\infty+\frac{\|\theta\|_\infty\sup\{\|f-b_r\|_\infty\}}{(1-\|\alpha\|_\infty)(1-\|\alpha\|_\infty-|t|_\infty\|\theta\|_\infty)}|t|_\infty, \tag{5.3}$$

where

$$\begin{aligned}\|\phi\|_\infty&=\sup_{r\in\mathbb{N}}\{\max_{i\in\mathbb{N}_N}\|\phi_{i,r}\|_\infty\},\ \ \|\theta\|_\infty=\sup_{r\in\mathbb{N}}\{\max_{i\in\mathbb{N}_N}\|\theta_{i,r}\|_\infty\},\\ |s|_\infty&=\sup_{r\in\mathbb{N}}\{\max_{i\in\mathbb{N}_N}s_{i,r}\},\ \ |t|_\infty=\sup_{r\in\mathbb{N}}\{\max_{i\in\mathbb{N}_N}t_{i,r}\}.\end{aligned}$$

Proof. From (2.7), we have

$$f^\alpha_b(x)-f(x)=\lim_{r\to\infty}\sum_{j=1}^r\alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q^j_i(x))(f-b_j)(Q^j_i(x)). \tag{5.4}$$

For $r\in\mathbb{N}$, let us define an RB operator V^{α_r} on $\mathscr{M}$ by

$$\begin{aligned}(V^{\alpha_r}g)(x)&=T_{i,r}(l_i^{-1}(x),\ g(l_i^{-1}(x)))\\ &=f(x)+[\alpha_{i,r}(Q_i(x))+t_{i,r}\theta_{i,r}(Q_i(x))](g-b_r)(Q_i(x))+s_{i,r}\phi_{i,r}(Q_i(x))\\ &=f(x)+[\mathbf{a}_r(x)+\mathbf{c}_r(x)](g-b_r)(Q_i(x))+s_{i,r}\phi_{i,r}(Q_i(x)),\end{aligned}$$

where $\mathbf{a}_r(x)=\alpha_{i,r}(Q_i(x))$ and $\mathbf{c}_r(x)=t_{i,r}\theta_{i,r}(Q_i(x))$.

$$\begin{aligned}&V^{\alpha_1}\circ V^{\alpha_2}\circ\dots\circ V^{\alpha_r}f(x)-f(x)\\ &=\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big(V^{\alpha_2}\circ V^{\alpha_3}\circ\dots\circ V^{\alpha_r}f-b_1\Big)(Q_i(x))+s_{i,1}\phi_{i,1}(Q_i(x))\end{aligned}$$

Using induction, we obtain

$$\begin{aligned}&V^{\alpha_1}\circ V^{\alpha_2}\circ\dots\circ V^{\alpha_r}f(x)-f(x)\\ &=\sum_{j=1}^r\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_j(x)+\mathbf{c}_j(x)\Big](f-b_j)(Q^j_i(x))\\ &\quad+\sum_{j=1}^r s_{i,j}\phi_{i,j}(Q^j_i(x))\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_{j-1}(x)+\mathbf{c}_{j-1}(x)\Big],\end{aligned}$$

where Q_i^j is a suitable finite composition of mappings Q_i. Now, taking the limit as $r \to \infty$, we get

$$\begin{aligned} & f_{b,s}^{\alpha,t}(x) - f(x) \\ &= \lim_{r\to\infty} \sum_{j=1}^{r} \Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_j(x)+\mathbf{c}_j(x)\Big](f-b_j)(Q_i^j(x)) \\ &\quad + \lim_{r\to\infty}\sum_{j=1}^{r} s_{i,j}\phi_{i,j}(Q_i^j(x))\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_{j-1}(x)+\mathbf{c}_{j-1}(x)\Big]. \end{aligned} \tag{5.5}$$

Subtracting (5.4) from (5.5), we get

$$\begin{aligned} & f_{b,s}^{\alpha,t}(x) - f_b^{\alpha}(x) \\ &= \lim_{r\to\infty}\sum_{j=1}^{r} s_{i,j}\phi_{i,j}(Q_i^j(x))\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_{j-1}(x)+\mathbf{c}_{j-1}(x)\Big] \\ &\quad + \lim_{r\to\infty}\sum_{j=1}^{r}\Bigg[\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_j(x)+\mathbf{c}_j(x)\Big] \\ &\qquad\qquad - \mathbf{a}_1(x)\mathbf{a}_2(x)\dots\mathbf{a}_j(x)\Bigg](f-b_j)(Q_i^j(x)) \\ &= \lim_{r\to\infty}\sum_{j=1}^{r} s_{i,j}\phi_{i,j}(Q_i^j(x))\Big[\mathbf{a}_1(x)+\mathbf{c}_1(x)\Big]\Big[\mathbf{a}_2(x)+\mathbf{c}_2(x)\Big]\dots\Big[\mathbf{a}_{j-1}(x)+\mathbf{c}_{j-1}(x)\Big] \\ &\quad + \lim_{r\to\infty}\sum_{j=1}^{r}\Bigg[\mathbf{a}_1(x)\cdot\mathbf{a}_2(x)\dots\mathbf{a}_{j-1}(x)\cdot\mathbf{c}_j(x)+\mathbf{a}_1(x)\cdot\mathbf{a}_2(x)\dots\mathbf{a}_{j-2}(x)\cdot\mathbf{c}_{j-1}(x) \\ &\qquad\qquad \times[\mathbf{a}_j(x)+\mathbf{c}_j(x)] \\ &\quad +\mathbf{a}_1(x)\cdot\mathbf{a}_2(x)\dots\mathbf{a}_{j-3}(x)\cdot\mathbf{c}_{j-2}(x)[\mathbf{a}_{j-1}(x)+\mathbf{c}_{j-1}(x)][\mathbf{a}_j(x)+\mathbf{c}_j(x)] \\ &\quad +\dots+\mathbf{c}_1(x)[\mathbf{a}_2(x)+\mathbf{c}_2(x)][\mathbf{a}_3(x)+\mathbf{c}_3(x)]\dots[\mathbf{a}_j(x)+\mathbf{c}_j(x)](f-b_j)(Q_i^j(x))\Bigg]. \end{aligned}$$

Let $\mathbf{a} = \sup_{r\in\mathbb{N}}\{\|\mathbf{a}_r\|_\infty\} = \|\alpha\|_\infty$ and $\mathbf{c} = \sup_{r\in\mathbb{N}}\{\|\mathbf{c}_r\|_\infty\} = |t|_\infty\|\theta\|_\infty$. Therefore,

$$\begin{aligned} & |f_{b,s}^{\alpha,t}(x) - f_b^{\alpha}(x)| \\ &\le \lim_{r\to\infty}\sum_{j=1}^{r}|s|_\infty\|\phi\|_\infty(\mathbf{a}+\mathbf{c})^{j-1} + \sup\{\|f-b_r\|_\infty\} \\ &\quad\times\lim_{r\to\infty}\sum_{j=1}^{r}\Big[\mathbf{a}^{j-1}\cdot\mathbf{c}+\mathbf{a}^{j-2}\cdot\mathbf{c}(\mathbf{a}+\mathbf{c})+\mathbf{a}^{j-3}\cdot\mathbf{c}(\mathbf{a}+\mathbf{c})^2+\dots+\mathbf{c}(\mathbf{a}+\mathbf{c})^{j-1}\Big], \\ &= |s|_\infty\|\phi\|_\infty\lim_{r\to\infty}\sum_{j=1}^{r}(\mathbf{a}+\mathbf{c})^{j-1} + \sup\{\|f-b_r\|_\infty\} \\ &\quad\times\mathbf{c}\lim_{r\to\infty}\sum_{j=1}^{r}\Big[\mathbf{a}^{j-1}+\mathbf{a}^{j-2}\cdot(\mathbf{a}+\mathbf{c})+\mathbf{a}^{j-3}\cdot(\mathbf{a}+\mathbf{c})^2+\dots+(\mathbf{a}+\mathbf{c})^{j-1}\Big] \\ &= |s|_\infty\|\phi\|_\infty\frac{1}{1-\mathbf{a}-\mathbf{c}} + \sup\{\|f-b_r\|_\infty\} \\ &\quad\times\mathbf{c}\sum_{j=1}^{\infty}(\mathbf{a}+\mathbf{c})^{j-1}\left[1+\left(\frac{\mathbf{a}}{\mathbf{a}+\mathbf{c}}\right)+\left(\frac{\mathbf{a}}{\mathbf{a}+\mathbf{c}}\right)^2+\dots+\left(\frac{\mathbf{a}}{\mathbf{a}+\mathbf{c}}\right)^{j-1}\right] \end{aligned}$$

$$= \frac{\|\phi\|_\infty |s|_\infty}{1-\mathbf{a}-\mathbf{c}} + \sup\{\|f-b_r\|_\infty\}\mathbf{c}\sum_{j=1}^{\infty}(\mathbf{a}+\mathbf{c})^{j-1} \times \left(\frac{1-\left(\frac{\mathbf{a}}{\mathbf{a}+\mathbf{c}}\right)^j}{1-\left(\frac{\mathbf{a}}{\mathbf{a}+\mathbf{c}}\right)} \right)$$

$$= \frac{\|\phi\|_\infty |s|_\infty}{1-\mathbf{a}-\mathbf{c}} + \sup\{\|f-b_r\|_\infty\}\sum_{j=1}^{\infty}\left((\mathbf{a}+\mathbf{c})^j-(\mathbf{a})^j\right)$$

$$= \frac{\|\phi\|_\infty |s|_\infty}{1-\mathbf{a}-\mathbf{c}} + \sup\{\|f-b_r\|_\infty\}\left(\frac{\mathbf{a}+\mathbf{c}}{1-\mathbf{a}-\mathbf{c}}-\frac{\mathbf{a}}{1-\mathbf{a}}\right)$$

$$= \frac{\|\phi\|_\infty}{1-\mathbf{a}-\mathbf{c}}|s|_\infty + \frac{\|\theta\|_\infty \sup\{\|f-b_r\|_\infty\}}{(1-\mathbf{a})(1-\mathbf{a}-\mathbf{c})}|t|_\infty.$$

The above inequality holds for each $x \in \mathscr{M}$, hence

$$\|f_{b,s}^{\alpha,t} - f_b^\alpha\|_\infty \le \frac{\|\phi\|_\infty}{1-\|\alpha\|_\infty - |t|_\infty\|\theta\|_\infty}|s|_\infty + \frac{\|\theta\|_\infty \sup\{\|f-b_r\|_\infty\}}{(1-\|\alpha\|_\infty)(1-\|\alpha\|_\infty-|t|_\infty\|\theta\|_\infty)}|t|_\infty.$$

□

6. Continuous dependence on parameters b, α.

In this section, we will investigate the continuous dependence of the non-stationary α-fractal function on different IFS parameters. The reader can refer to [26] for the same study in the stationary case. We will start with the continuous dependence of $f^\alpha_{\Delta,b}$ on the sequence of base functions $b := \{b_r\}$.

Theorem 6.1. *Let $f \in \mathscr{C}(I)$, and the partition $\triangle$, sequence of scale functions $\alpha_r \in \mathscr{C}(I)$, $r \in \mathbb{N}$ with $\|\alpha\|_\infty < 1$ be fixed. Let $A = \{b_r \in \mathscr{C}(I) : b_r(x) = f(x)\ \forall\, x = x_0, x_N\}$. Then, the map $\mathscr{A} : A \to \mathscr{C}(I)$ defined by*

$$\mathscr{A}(b) = f^\alpha_{\triangle,b}$$

is Lipschitz continuous.

Proof. From Section 2.4, we obtain that $f^\alpha_{\triangle,b}$ is unique for a fixed sequence of scale function α_r, a partition $\triangle$, and a suitable sequence of base function $b_r \in \mathscr{C}(I)$. Further, $f^\alpha_{\triangle,b}$ satisfies the functional equation: for all $x \in I_i,\ i \in \mathbb{N}_N$, we have

$$f^\alpha_{\Delta,b}(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r}\alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f-b_j)(Q_i^j(x)).$$

Let $b_r, c_r \in A,$ for $r \in \mathbb{N}$. Then

$$\mathscr{A}(b)(x) = f^\alpha_{\Delta,b}(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r}\alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f-b_j)(Q_i^j(x)),$$

and

$$\mathscr{A}(c)(x) = f^\alpha_{\Delta,c}(x) = f(x) + \lim_{r\to\infty}\sum_{j=1}^{r}\alpha_{i,1}(Q_i(x))\dots\alpha_{i,j}(Q_i^j(x))(f-c_j)(Q_i^j(x)).$$

On subtraction, we get for $x \in I_i$,

$$\mathscr{A}(b)(x) - \mathscr{A}(c)(x) = \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(c_j - b_j)(Q_i^j(x)).$$

Therefore,

$$\begin{aligned}\left|\mathscr{A}(b)(x) - \mathscr{A}(c)(x)\right| &\leq \lim_{r\to\infty} \sum_{j=1}^{r} \left|\alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(c_j - b_j)(Q_i^j(x))\right| \\ &\leq \sum_{j=1}^{\infty} \|\alpha\|_\infty^j \|c_j - b_j\|_\infty \\ &= \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \|b-c\|_\infty.\end{aligned}$$

For all $x \in I$, the aforementioned inequality holds. Hence,

$$\|\mathscr{A}(b) - \mathscr{A}(c)\|_\infty \leq \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \|b-c\|_\infty.$$

This shows that $\mathscr{A}$ is a Lipschitz continuous map with Lipschitz constant $\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$. □

Let $f = (f_1, f_2, \dots, f_k) \in (\mathscr{C}(I))^k$($k$-fold cartesian product) and we consider the product norm

$$\|f\|_\infty = \max\{\|f_1\|_\infty, \|f_2\|_\infty, \dots, \|f_k\|_\infty\}.$$

Theorem 6.2. *Let $f \in \mathscr{C}(I)$, sequence of base functions b_r and the partition $\triangle$, be fixed. Let $B = \{\alpha = \{\alpha_r\} : \alpha_{i,r} \in \mathscr{C}(I)$ and $\|\alpha\|_\infty \leq s < 1$, where s is a fixed number$\}$. Then the map $\mathscr{B} : B \to \mathscr{C}(I)$, defined by*

$$\mathscr{B}(\alpha) = f^\alpha_{\triangle,b}$$

is continuous.

Proof. For a fixed partition $\triangle$, a scale function α, and a suitable sequence of base functions b_r, the map $f^\alpha_{\triangle,b}$ is unique. Further, $f^\alpha_{\triangle,b}$ satisfies the functional equation: for all $x \in I_i$, $i \in \mathbb{N}_N$ we have

$$f^\alpha_{\Delta,b}(x) = f(x) + \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(f - b_j)(Q_i^j(x)). \tag{6.1}$$

Let $\alpha, \beta \in B$, then from the above functional equation, we have

$$\mathscr{B}(\alpha)(x) = f(x) + \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(f - b_j)(Q_i^j(x)),$$

and

$$\mathscr{B}(\beta)(x) = f(x) + \lim_{r\to\infty} \sum_{j=1}^{r} \beta_{i,1}(Q_i(x)) \dots \beta_{i,j}(Q_i^j(x))(f - b_j)(Q_i^j(x)).$$

To show that $\mathscr{B}$ is continuous at α, we subtract one from the other of the above two equations, for $x \in I_i,\ i \in \mathbb{N}_N$, we have

$$\begin{aligned}
&\mathscr{B}(\alpha)(x) - \mathscr{B}(\beta)(x)\\
&= \lim_{r\to\infty} \sum_{j=1}^{r} \alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x))(f-b_j)(Q_i^j(x))\\
&\quad - \lim_{r\to\infty} \sum_{j=1}^{r} \beta_{i,1}(Q_i(x)) \dots \beta_{i,j}(Q_i^j(x))(f-b_j)(Q_i^j(x))\\
&= \lim_{r\to\infty} \sum_{j=1}^{r} \Big(\alpha_{i,1}(Q_i(x)) \dots \alpha_{i,j}(Q_i^j(x)) - \beta_{i,1}(Q_i(x)) \dots \beta_{i,j}(Q_i^j(x))\Big)(f-b_j)(Q_i^j(x))\\
&= \lim_{r\to\infty} \sum_{j=1}^{r} \Big((\alpha_{i,1}-\beta_{i,1})(Q_i(x))\beta_{i,2}(Q_i^2(x)) \dots \beta_{i,j}(Q_i^j(x))\\
&\quad + \alpha_{i,1}(Q_i(x))(\alpha_{i,2}-\beta_{i,2})(Q_i^2(x))\beta_{i,3}(Q_i^3(x)) \dots \beta_{i,j}(Q_i^j(x))\\
&\quad + \dots + \alpha_{i,1}(Q_i(x))\alpha_{i,2}(Q_i^2(x)) \dots \alpha_{i,j-1}(Q_i^{j-1}(x))(\alpha_{i,j}-\beta_{i,j})(Q_i^j(x))\Big)(f-b_j)(Q_i^j(x)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big|\mathscr{B}(\alpha)(x) - \mathscr{B}(\beta)(x)\Big|\\
&\le \lim_{r\to\infty} \sum_{j=1}^{r} \Big|(\alpha_{i,1}-\beta_{i,1})(Q_i(x))\beta_{i,2}(Q_i^2(x)) \dots \beta_{i,j}(Q_i^j(x))\Big|\\
&\quad + \Big|\alpha_{i,1}(Q_i(x))(\alpha_{i,2}-\beta_{i,2})(Q_i^2(x))\beta_{i,3}(Q_i^3(x)) \dots \beta_{i,j}(Q_i^j(x))\Big| + \dots\\
&\quad + \Big|\alpha_{i,1}(Q_i(x))\alpha_{i,2}(Q_i^2(x)) \dots \alpha_{i,j-1}(Q_i^{j-1}(x))(\alpha_{i,j}-\beta_{i,j})(Q_i^j(x))\Big|\Big|(f-b_j)(Q_i^j(x))\Big|\\
&\le \lim_{r\to\infty} \sum_{j=1}^{r} \Big(\|\alpha_1-\beta_1\|_\infty \|\beta_{i,2}\|_\infty \dots \|\beta_{i,j}\|_\infty + \|\alpha_{i,1}\|_\infty \|\alpha_2-\beta_2\|_\infty \|\beta_{i,3}\|_\infty \dots \|\beta_{i,j}\|_\infty + \dots\\
&\quad + \|\alpha_{i,1}\|_\infty \|\alpha_{i,2}\|_\infty \dots \|\alpha_{i,j-1}\|_\infty \|\alpha_j - \beta_j\|_\infty\Big)\|f-b_j\|_\infty\\
&\le \lim_{r\to\infty} \sum_{j=1}^{r} \Big(\|\alpha-\beta\|_\infty \|\beta\|_\infty^{j-1} + \|\alpha\|_\infty \|\alpha-\beta\|_\infty \|\beta\|_\infty^{j-2} + \dots + \|\alpha\|_\infty^{j-1}\|\alpha-\beta\|_\infty\Big)\|f-b\|_\infty.
\end{aligned}$$

Without loss of generality, let $0 < \|\beta\|_\infty < \|\alpha\|_\infty < 1$. Then

$$\begin{aligned}
&\Big|\mathscr{B}(\alpha)(x) - \mathscr{B}(\beta)(x)\Big|\\
&\le \|f-b\|_\infty \|\alpha-\beta\|_\infty \lim_{r\to\infty} \sum_{j=1}^{r} \|\alpha\|_\infty^{j-1}\left\{1 + \left(\frac{\|\beta\|_\infty}{\|\alpha\|_\infty}\right) + \left(\frac{\|\beta\|_\infty}{\|\alpha\|_\infty}\right)^2 + \dots + \left(\frac{\|\beta\|_\infty}{\|\alpha\|_\infty}\right)^{j-1}\right\}\\
&\le \|f-b\|_\infty \|\alpha-\beta\|_\infty \sum_{j=1}^{\infty} \|\alpha\|_\infty^{j-1}\left(\frac{1-\left(\frac{\|\beta\|_\infty}{\|\alpha\|_\infty}\right)^j}{1-\left(\frac{\|\beta\|_\infty}{\|\alpha\|_\infty}\right)}\right)
\end{aligned}$$

$$= \|f-b\|_\infty \frac{\|\alpha-\beta\|_\infty}{\|\alpha\|_\infty - \|\beta\|_\infty} \sum_{j=1}^{\infty} \left(\|\alpha\|_\infty^j - \|\beta\|_\infty^j\right)$$

$$= \|f-b\|_\infty \frac{\|\alpha-\beta\|_\infty}{(1-\|\alpha\|_\infty)\cdot(1-\|\beta\|_\infty)}$$

$$\leq \|\alpha-\beta\|_\infty \frac{\|f-b\|_\infty}{(1-s)^2}.$$

The aforementioned inequality holds for all $x \in I$, therefore

$$\left\|\mathscr{B}(\alpha)-\mathscr{B}(\beta)\right\|_\infty \leq \|\alpha-\beta\|_\infty \frac{\|f-b\|_\infty}{(1-s)^2}.$$

Since $f, b \in \mathscr{C}(I)$ and α is fixed, we have $\mathscr{B}$ is continuous at α. As α was taken arbitrarily, $\mathscr{B}$ is continuous on B. □

Acknowledgments. We would like to thank the anonymous reviewers for the valuable and constructive suggestions that helped to improve the manuscript.

Declaration

Funding. Not applicable

Conflicts of interest. We do not have any conflict of interest.

Availability of data and material. Not applicable

Code availability. Not applicable

Authors' contributions. Both authors contributed equally in this manuscript.

References

1. Md. N. Akhtar, M. G. P. Prasad, and M. A. Navascués, *Box dimensions of α-fractal functions*, Fractals **24** (2016), no. 3, 1650037, 13.
2. Md. N. Akhtar, M. G. P Prasad, and M. A. Navascués, *Box dimension of α-fractal function with variable scaling factors in subintervals*, Chaos Solitons Fractals **103** (2017), 440–449.
3. M. F. Barnsley, *Fractal functions and interpolation*, Constr. Approx. **2** (1986), no. 4, 303–329.
4. ______, *Fractals everywhere*, Academic Press, Inc., Boston, MA, 1988.
5. P. Bouboulis and L. Dalla, *Fractal interpolation surfaces derived from fractal interpolation functions*, J. Math. Anal. Appl. **336** (2007), no. 2, 919–936.
6. A. K. B. Chand and G. P. Kapoor, *Generalized cubic spline fractal interpolation functions*, SIAM J. Numer. Anal. **44** (2006), no. 2, 655–676.
7. A. K. B. Chand and K. R. Tyada, *Constrained shape preserving rational cubic fractal interpolation functions*, Rocky Mountain J. Math. **48** (2018), no. 1, 75–105.
8. A. K. B. Chand, N. Vijender, P. Viswanathan, and A. V. Tetenov, *Affine zipper fractal interpolation functions*, BIT **60** (2020), no. 2, 319–344.
9. N. Dyn and D. Levin, *Subdivision schemes in geometric modelling*, Acta Numer. **11** (2002), 73–144.
10. N. Dyn, D. Levin, and P. Massopust, *Attractors of trees of maps and of sequences of maps between spaces with applications to subdivision*, J. Fixed Point Theory Appl. **22** (2020), no. 1, Paper No. 14, 24.
11. J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747.
12. J.A. Johnson, *Banach spaces of lipschitz functions and vector-valued lipschitz functions*, Trans. Amer. Math. Soc **148** (1970), 147–169.
13. D. Levin, N. Dyn, and P. Viswanathan, *Non-stationary versions of fixed-point theory, with applications to fractals and subdivision*, J. Fixed Point Theory Appl. **21** (2019), no. 1, Paper No. 26, 25.

14. P. R. Massopust, *Fractal functions, fractal surfaces, and wavelets*, second ed., Elsevier/Academic Press, London, 2016.
15. ———, *Non-stationary fractal interpolation*, Mathematics **7** (2019), no. 8, 666.
16. A. I. Mondal and S. Jha, *Non-stationary α-fractal functions and their dimensions in various function spaces*, Indag. Math. (N.S.) **35** (2024), no. 1, 159–180.
17. M. A. Navascués, *Fractal approximation*, Complex Analysis and Operator Theory **4** (2010), no. 4, 953–974.
18. M. A. Navascués and S. Verma, *Non-Stationary α-Fractal Surfaces*, Mediterr. J. Math. **20** (2023), no. 1, Paper No. 48.
19. B. V. Prithvi and S. K. Katiyar, *Interpolative operators: fractal to multivalued fractal*, Chaos Solitons Fractals **164** (2022), Paper No. 112449, 12.
20. S. Ri, *A new idea to construct the fractal interpolation function*, Indag. Math. (N.S.) **29** (2018), no. 3, 962–971.
21. H. Ruan, W. Su, and K. Yao, *Box dimension and fractional integral of linear fractal interpolation functions*, J. Approx. Theory **161** (2009), no. 1, 187–197.
22. M. Samuel and A. V. Tetenov, *On attractors of iterated function systems in uniform spaces*, Sib. Èlektron. Mat. Izv. **14** (2017), 151–155.
23. N. Secelean, *The fractal interpolation for countable systems of data*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **14** (2003), 11–19.
24. H. Takayasu, *Fractals in the physical sciences*, Manchester University Press, 1990.
25. S. Verma and P. R. Massopust, *Dimension preserving approximation*, Aequationes Math. **96** (2022), no. 6, 1233–1247.
26. S. Verma and P. Viswanathan, *A revisit to α-fractal function and box dimension of its graph*, Fractals **27** (2019), no. 6, 1950090, 15.
27. N. Vijender, *Fractal perturbation of shaped functions: convergence independent of scaling*, Mediterr. J. Math. **15** (2018), no. 6, Paper No. 211, 16.
28. N. Vijender, A. K. B. Chand, M. A. Navascués, and M. V. Sebastián, *Quantum α-fractal approximation*, Int. J. Comput. Math. **98** (2021), no. 12, 2355–2368.
29. H. Wang and J. Yu, *Fractal interpolation functions with variable parameters and their analytical properties*, J. Approx. Theory **175** (2013), 1–18.

DEPARTMENT OF MATHEMATICS, NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA, ROURKELA 769008
E-mail address: anarulmath96@gmail.com

DEPARTMENT OF MATHEMATICS, NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA, ROURKELA 769008
E-mail address: jhasa@nitrkl.ac.in, *Corresponding Author