ON TOPOLOGICAL GROUPS OF AUTOMORPHISMS ON UNIONS

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Abstract. We study groups of homeomorphic bijections on spaces that are finite unions of compact connected linearly ordered subsets. We prove that all such groups when endowed with the topology of point-wise convergence are topological groups.

1. Introduction

In this paper, a homeomorphic bijection of a topological space X with itself will be called an *automorphism* of X. The set of such automorphisms of X will be denoted by $Hom(X)$. This set is known to be an abstract group under function composition. A natural research direction involving this structure is finding topologies on $Hom(X)$ that make it a topological group. Recall that given a topology $\mathcal{T}$ on $Hom(X)$, the pair $G = \langle Hom(X), \mathcal{T}\rangle$ is a topological group if the operation of composition is continuous on $G \times G$ and the map $f \mapsto f^{-1}$ is continuous on G. Among notable facts in this direction is the theorem of Arens [1] that for any metric compactum X, the set $Hom(X)$ endowed with the compact open topology is a topological group. We refer the reader to [3] for a variety of natural topologies that can be introduced on $Hom(X)$. Among them is the topology of point-wise convergence. A natural basis for this topology consists of sets in the form $\{g \in Hom(X) : g(x_i) \in O_i\ i = 0, ..., n-1\}$, where x_i's are some fixed elements of X and O_i's are some fixed open non-empty subsets of X. The space of automorphisms of X endowed with the topology of point-wise convergence will be denoted by $Hom_p(X)$. It is left as an exercise in [3] that $Hom_p(\mathbb{R}^2)$ is not a topological group. An encouraging result was obtained by Sorin in [6], where he proved that $Hom_p(L)$ is a topological group for any connected LOTS L. A certain degree of connectedness is important. If one attempts to complete the mentioned exercise about $Hom_p(\mathbb{R}^2)$ one will find that the same argument shows that $Hom_p(Cantor\ Set)$ is not a topological group either. The author recently observed a more general statement that covers both these exercises, which will be published elsewhere.

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We will be concerned with the following general problem:

Problem. *Let $Hom_p(X)$ and $Hom_p(Y)$ be topological groups.*

1. *Is $Hom_p(X \times \{0,1\})$ a topological group?*
2. *Is $Hom_p(X \oplus Y)$ a topological group?*
3. *Is $Hom_p(X \cup Y)$ a topological group, where X and Y are closed subspaces of the union?*

In this work we give affirmative answers to all three questions of the problem if X and Y are connected and compact LOTS by proving the following more general statement.

Main Result (Theorem 3.4). *Let X be the union of a finite number of connected compact LOTS. Then, $Hom_p(X)$ is a topological group.*

In notation and terminology of the general topological nature, we will follow [4]. A space is always a topological space. Recall that an open subset U of a space X is canonical if U is the interior of $\overline{U}$. For general facts about topological groups, we refer the reader to [3]. A linearly ordered topological spaces (or simply ordered space), abbreviated as LOTS, is a linearly ordered set endowed with the topology generated by sets (a,b), $\{x \in L : x < a\}$, and $\{x \in L : x > a\}$. For general LOTS-related facts, we refer the reader to [5]. Finally, when dealing with several linearly ordered sets at the same time, we will distinguish their intervals via subscription as in $[a,b]_L$ (the same applies to other types of intervals).

2. Preliminary Observations

In this section, we will discuss the topological structure of spaces that can be written as the union of a finite number of compact connected linearly ordered topological subspaces. Given such a space X, we will identify a basis for the topology of X that will be used in the argument of our main result.

Definition 2.1. *A topological space X is locally ordered at $x \in X$ if there exists an open neighborhood of x which is a LOTS under some linear order. The set of all points at which X is locally ordered is denoted by $LO(X)$.*

Lemma 2.2. *Let X be the union of a finite number of closed linearly ordered spaces. Then $LO(X)$ is an open and dense subspace of X.*

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Proof. Let $X = \bigcup\{L_i : i = 1, ..., n\}$, where L_i is a closed subset of X and is a LOTS for each $i \in I$. Fix any non-empty set $U \subset X$. We need to find a non-empty open subset of U which is a LOTS. Let $k \leq n$ be the smallest such that U is a subset of $\bigcup\{L_i : i = 1, ..., k\}$. Therefore, $V = U \setminus \bigcup\{L_i : i = 1, ..., k-1\}$ is not empty and is open in X. Since $V \subset L_k$, V is an a non-empty open subset of U which is a LOTS. ☐

Definition 2.3. *Let X be the union of a finite number of compact connected LOTS. An open set $U \subset X$ is called orderly if the following properties hold:*

(1) *U is canonical.*
(2) *U is connected.*
(3) *$\overline{U} \setminus U$ is finite.*
(4) *$\overline{U} \setminus U \subset LO(X)$*

Lemma 2.4. *For any topological space X, let U and V be connected open subsets of X and $\overline{U} \setminus U = \overline{V} \setminus V$. Then either $U = V$ or $U \cap V = \emptyset$.*

Proof. Let $x \in V \cap U$. It suffices to show that $V \subset U$. Assume that $V \setminus U$ is not empty. Since V is connected, $V \setminus U$ is not open. Then there exists $y \in V \setminus U$ such that any neighborhood of y meets both U and V. Since $y \notin U$, we conclude that $y \in \overline{U} \setminus U$. Since $\overline{U} \setminus U = \overline{V} \setminus V$, we arrive at $y \notin V$, a contradiction. ☐

The following Lemma 2.5 will be referenced both for its argument and for the statement.

Lemma 2.5. *Let X be the union of a finite number of compact connected LOTS. Let U be an orderly neighborhood of $p \in X$, and let D be a dense subset of X. Then there exists an orderly neighborhood V of p such that $\overline{V} \subset U$ and $\overline{V} \setminus V \subset D$.*

Proof. If $p \in LO(X)$, then the conclusion is clear. We now assume that $p \notin LO(X)$. Put $B = \overline{U} \setminus U = \{b_i : i = 1, ..., n\}$. For each $k \in \{1, ..., n\}$, let $I_k \subset LO(X)$ be an open neighborhood of b_k such that $\overline{I}_k \cap \overline{I}_m = \emptyset$ for distinct k, m. Since U is canonical, we may assume that I_k is a connected LOTS without both maximum and minimum. We may also assume that $(b_k, \max I_k]_{I_k} \subset U$. For each $b_k \in B$, fix $c_k \in (b_k, \max I_k)_{I_k} \cap D$. Let $V_0 = U$ and $V_k = V_{k-1} \setminus (b_k, c_k]_{I_k}$ for $0 < k \leq n$. Put $V = V_n$. Let us show that V is as desired. Since $p \notin LO(X)$, we conclude that $p \in V$. To show that V is open it suffices to show that $(b_k, c_k]_{I_k}$ is closed in U for each $b_k \in B$. For this observe that $[b_k, c_k]_{I_k}$ is compact, and therefore, closed in X. Since $b_k \in \overline{U} \setminus U$, we conclude that $[b_k, c_k]_{I_k} \cap U = (b_k, c_k]_{I_k}$. Hence, $(b_k, c_k]_{I_k}$ is closed in U.

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Next let us show that V is orderly. The boundary of V_k is $\{b_i : i = 1, ..., k-1\} \cup \{c_i : i = k, ..., n\}$.This follows from the fact that c_k is an interior point of I_k for each k. Therefore, $\overline{V} \setminus V = \{c_k : k = 1, ..., n\}$ is a subset of $D \cap LO(X)$. It remains to show that V is connected and canonical.

To show that V is canonical, observe that any neighborhood of c_k has a smaller neighborhood that is a subset of I_k and is of the form $(x, y)_{I_k}$. Therefore, $(c_k, y)_{I_k}$ and $(x, c_k)_{I_k}$ are non-empty sets that meet V and the complement of V, respectively.

To show that V is connected, let us show that V_k is connected for each $k = 0, ..., n$. Since U is orderly, V_0 is connected. Assume that V_k is connected for $0 \leq k < n$. To show that V_{k+1} is connected let us argue by contradiction. Then, there exist disjoint open sets O and O' such that $V_{k+1} = O \cup O'$. Then $V_k \setminus \{c_{k+1}\}$ is the union of disjoint open sets $(b_{k+1}, c_{k+1})_{I_{k+1}}$, O, and O'. Since V_{k+1} is connected c_{k+1} is a boundary point of each of the three sets contradicting the fact that c_{k+1} has an open neighborhood which is a connected LOTS.

Our proof is complete and let us finish with a remark for future use.

Remark. Using the constructions of V we can create an orderly set W by taking c_k anywhere in I_k not necessarily to the right of b_k and by letting $W = U \cup \bigcup\{I_k \setminus (minI_k, c_k]_{I_k} : k = 1, ..., n\}$. This set is also orderly by a similar argument but need not be a subset of U if at least one c_k is taken outside of U. $\square$

Remark 2.6. *The sets described in the final remark of Lemma 2.5 will be useful in our future arguments. Let us refer to the described W as being determined by $U, \{I_k\}_k, \{c_k\}$.*

If in the statement of In Lemma 2.5, we put additional restrictions on D and the boundary of U, the same argument leads to the following statement.

Lemma 2.7. *Let X be the union of a finite number of compact connected LOTS. Let U be an orderly neighborhood of $p \in X$, and let D be an open dense subset of X. Suppose that $\overline{U} \setminus U \subset D$. Then there exists an orderly neighborhood V of p such that $\overline{V} \subset U$ and $U \setminus V \subset D$.*

Lemma 2.8. *Let X be the union of a finite number of compact connected LOTS. Then X has a basis consisting of orderly open sets.*

Proof. Let $X = \bigcup_{i=1}^{n} L_i$, where L_i is a connected compact LOTS. We will induct on n. If $n = 1$, then X is a LOTS and the conclusion follows. Assume now that the conclusion of our lemma holds for $n = k$, where $k \geq 1$. We now assume that $n = k+1$. Fix $p \in X$ and an open neighborhood O of p. Our goal is to find an orderly neighborhood of p which is a subset of O. Let $X' = \bigcup_{i=1}^{n-1} L_i$ and $O' = O \cap X'$. We may assume that $p \in X'$. By our assumption there exists an orderly open neighborhood U of p in X' which is a subset of O'.

Claim 1. $LO(X) \cap O'$ is open and dense in O'.

To prove the claim, it suffices to show that the interior $Int_X(X')$ of X' in X is dense in X'. Pick $x \in X' \setminus Int_X(X')$. Since L_n is closed in X, we have $x \in L_n$. Since $x \notin Int_X(X')$, we have $x \notin LO(X)$. Therefore, any neighborhood of x contais points that are not in L_n. Since L_n is closed , all such points are in the interior of X' in X.

By Claim 1 and Lemmas 2.2 and 2.5, we may assume that $\overline{U} \setminus U \subset LO(X)$. By Lemma 2.7, we can find an orderly neighborhood V of p in X' such that $\overline{V} \subset U$ and $U \setminus V \subset LO(X)$.

Now let us color the points of $G = \overline{V}$ in green color and the points of $R = X \setminus U$ in red color.

Claim 2. There exists a finite collection $\mathcal{J}$ of convex open sets in L_n such that the following hold:

1. *$\bigcup \mathcal{J}$ is a subset of O.*
2. *$\bigcup \mathcal{J}$ contains all green points of L_n, and misses all red points of L_n.*
3. *Each $I \in \mathcal{J}$ has green points.*
4. *The closures of distinct members of $\mathcal{J}$ are disjoint.*

The conclusion of the claim follows from the fact that the sets of green and red points in L_n are disjoint compacta and all green points are in O.

Remark: Due to density and openness of $LO(X)$ in X, we may assume that for each $J \in \mathcal{J}$, the set $\overline{J} \setminus J \subset LO(X)$.

Put $W = U \cup \bigcup \mathcal{J}$. Let us show that W is an orderly neighborhood of p and is a subset of O. Since U contains p so does W. Since both U and $\bigcup \mathcal{J}$ are subsets of O so is W. To prove that W is open, fix an arbitrary $x \in W$. We have two cases:

$x \in L_n$: Let U_x be an open neighborhood of x in X that misses R (all read points of X) as well as $L_n \setminus \bigcup \mathcal{J}$. Then $U_x \cap L_n \subset \bigcup \mathcal{J}$ and $U_x \cap X' \subset U$. Hence, $U_x \subset W$.

$x \notin L_n$: Then there exists a neighborhood U_x of x in X that misses L_n and $X' \setminus U$. Hence $U_x \subset W$.

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The set W is connected since U and J's are and each J contains some green points of U. The boundary of W is finite since every point on the boundary of W is either on the boundary of U or on the boundary of some $J \in \mathcal{J}$, which are finite. Finally, if W is not canonical we can simply replace it with the interior of $\overline{W}$. The proof is complete. □

3. Study

We are now ready to prove our main result that if X is the union of a finite number of compact connected LOTS, then $Hom_p(X)$ is a topological group. For this we need to verify that the operation of function composition is a continuous map from $Hom_p(X) \times Hom_p(X)$ to $Hom_p(X)$ and that the correspondence $f \mapsto f^{-1}$ is a continuous map from $Hom_p(X)$ to $Hom_p(X)$.

Lemma 3.1. *Let X be the union of a finite number of connected compact LOTS. Then $f \mapsto f^{-1}$ is a continuous map from $Hom_p(X)$ to $Hom_p(X)$.*

Proof. Fix $f \in Hom_p(X)$. Let $V_{f^{-1}}$ be any open neighborhood of f^{-1}. We need to find an open U_f containing f such that $g^{-1} \in V_{f^{-1}}$ whenever $g \in U_f$. We may assume that there exist $x \in X$ and an orderly neighborhood O_x of x such that $V_{f^{-1}} = \{h \in Hom_p(X) : h(y) \in O_x\}$, where $y = f(x)$. Let O'_x be an orderly neighborhood of x such that $\overline{O'_x} \subset O_x$. For each $b \in \overline{O'_x} \setminus O'_x$, fix an open neighborhood I_b of b in $LO(X) \cap O_x$ such that $x \notin \overline{I}_b$ and $\{\overline{I}_b : b \in \overline{O'_x} \setminus O'_x\}$ is disjoint. Since O_x is canonical, we may assume that I_b is a connected LOTS without both maximum and minimum. Let O''_x be an open neighborhood of x that misses each I_b. Let

$$U_f = \{h \in Hom_p(X) : h(b) \in f(I_b) \ for\ every\ b \in \overline{O'_x} \setminus O'_x \ and\ h(x) \in f(O''_x)\}.$$

Since the border of O'_x is finite and the images of open sets under f are open, the set is an open neighborhood of f. To show that U_f is as desired, fix an arbitrary $h \in U_f$.

Claim. $h(O'_x)$ contains y.

To prove the claim first note that $h(O'_x)$ is an orderly set containing $h(x)$. According to Remark 2.6, the orderly set B determined by $f(O'_x), \{f(I_b) : b \in \overline{O'_x} \setminus O'_x\}$, and $\{h(b) : b \in \overline{O'_x} \setminus O'_x\}$ also contains $h(x)$. By Lemma 2.4, the sets coincides. By Remark 2.6, B contains y since $f(I_b)$ misses $f(x) = y$ for each b in the border of O'_x.

By Claim, $h^{-1}(y) \in O'_x \subset O_x$. Hence, $h^{-1} \in V_{f^{-1}}$. □

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Lemma 3.2. *Let X be the union of a finite number of compact connected LOTS. Let $f \in Hom_p(X)$ and U_y an open neighborhood of $y = f(x)$. Then there exist open neighborhoods U_x and U_f of x and f, respectively such that $h(U_x) \subset U_y$ whenever $h \in U_f$*

Proof. Let O_x be an orderly neighborhood of x such that $f(O_x) \subset U_y$. Let O'_x be an orderly neighborhood of x such that $\overline{O'_x} \subset O_x$. For each $b \in \overline{O'_x} \setminus O'_x$, fix an open neighborhood I_b of b in $LO(X) \cap O_x$ such that $x \notin \overline{I}_b$ and $\{\overline{I}_b : b \in \overline{O'_x} \setminus O'_x\}$ is disjoint. We can assume that each I_b is a connected LOTS without both maximum and minimum. Let

$$U_f = \{h \in Hom_p(X) : h(b) \in f(I_b) \ for \ every \ b \in \overline{O'_x} \setminus O'_x \ and \ h(x) \in f(O'_x)\}.$$

Since our construction is very similar to that in Lemma 3.1, by the argument identical to that of Claim of Lemma 3.1, $h(O'_x)$ contains $f(a)$ for every $a \in O'_x$ which is not in $\cup\{\overline{I}_b : b \in \overline{O'_x} \setminus O'_x\}$ whenever $h \in U_f$. Therefore, $U_x = O'_x \setminus \cup\{\overline{I}_b : b \in \overline{O'_x} \setminus O'_x\}$ is as desired. □

Lemma 3.3. *Let X be the union of a finite number of compact connected LOTS. Then $\langle f, g\rangle \mapsto f \circ g$ is a continuous map from $Hom_p(X) \times Hom_p(X)$ to $Hom_p(X)$.*

Proof. Pick $f, g \in Hom_p(X)$ and an open neighborhood $W_{f\circ g}$ of $f \circ g$. We need to find open neighborhoods U_f and V_g of f and g, respectively, such that $f' \circ g' \in W_{f\circ g}$ whenever $f' \in U_f$ and $g' \in V_g$.We may assume that there exists an open set $O_z \subset X$ of $z = f((g(x))$ such that $W_{f\circ g} = \{h \in Hom_p(X) : h(x) \in O_z\}$.

By Lemma 3.2, there exist a neighborhood U_f of f and a neighborhood O_y of y such that $h(O_y) \subset O_z$ whenever $h \in U_f$. Next, By Lemma 3.2, there exist a neighborhood V_g of g and a neighborhood O_x of x such that $h(O_x) \subset O_y$ whenever $h \in V_f$. Clearly, U_f and V_g are as desired. □

Lemmas 3.1 and 3.3 imply our main result.

Theorem 3.4. *Let X be the union of a finite number of connected compact LOTS. Then, $Hom_p(X)$ is a topological group.*

Remark 3.5. *Our argument only needs from X that each point of X has a basis consisting of neighborhoods with compact closure and properties 1-4 in the definition of "orderly" (Definition 2.3). This implies, in particular, that if X is a locally finite collection of subspaces that are compact connected LOTS, then $Hom_p(X)$ is a topological group. For example, any locally connected GO-space is such.*

In connection with the remark, the following question may present an interest.

Question 3.6. *Let $\mathcal{C}$ be a class of spaces such that $Hom_p(X)$ is a topological group for any X that is the union of a finite number of closed subspaces that are members of $\mathcal{C}$. Is it true that $Hom_p(X)$ is a topological group for any X that is a locally finite collection of closed subspaces that are members of $\mathcal{C}$?*

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References

[1] R. Arens, *Topologies for Homeomorphism Groups*, American Journal of Mathematics, vol. 68, no. 4, 1946, pp. 593–610

[2] A. Arhangelskii, *Topological Function Spaces*, Math. Appl., vol. 78, Kluwer Academic Publishers, Dordrecht, 1992

[3] A. Arhangelskii and M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Press/World Scientific, 2008

[4] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[5] D. J. Lutzer, *On Generalized Ordered Spaces*, Dissertationes Mathematicae, Warszawa, 1970, pp. 33.

[6] B. V. Sorin, *Compactifications of Homeomorphism Groups of Linearly Ordered Compacta*, Mathematical Notes volume 112, pages 126–141 (2022), https://doi.org/10.1134/S0001434622070148

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