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DEPTH OF POWERS OF EDGE IDEALS OF CYCLES AND STARLIKE TREES

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ABSTRACT. Let I be the edge ideal of a cycle of length $n \geq 5$ over a polynomial ring $S = \mathrm{k}[x_1,\dots,x_n]$. We prove that for $2 \leq t < \lceil (n+1)/2 \rceil$,

$$\operatorname{depth}(S/I^t) = \left\lceil \frac{n-t+1}{3} \right\rceil.$$

Also, we compute the depth of powers of the edge ideal of a starlike tree, i.e., the join of several path graphs at a common root.

1. Introduction

Let I be a homogeneous ideal in a standard graded polynomial ring $S = \mathrm{k}[x_1,\dots,x_n]$ over a field k. Brodmann [Br] proved that the depth function of powers $t \to \operatorname{depth}(S/I^t)$ is convergent. Ha, Nguyen, Trung, and Trung [HNTT] proved that the depth function of powers of monomial ideals could be any nonnegative integer-valued convergent function. On the other hand, when restricting to edge ideals of graphs, one expects that the depth function of their powers is nonincreasing. This phenomenon has been verified for several classes of graphs (see [HH, KTY, Mo]).

Let us now recall the notion of the edge ideals of graphs. Let G be a simple graph on the vertex set $V(G) = [n] = \{1,\dots,n\}$ and edge set $E(G) \subseteq V(G) \times V(G)$. The edge ideal of G, denoted by $I(G)$, is the squarefree monomial ideal of S generated by x_ix_j where $\{i,j\}$ is an edge of G. For a homogeneous ideal I, we denote by $\operatorname{dstab}(I)$ the *index of depth stability* of I, i.e., the smallest positive integer number k such that $\operatorname{depth}(S/I^\ell) = \operatorname{depth}(S/I^k)$ for all $\ell \geq k$. In [T], the second author found a combinatorial formula for $\operatorname{dstab}(I(G))$ for large classes of graphs, including unicyclic graphs. In particular, when G is a tree, $\operatorname{dstab}(I(G)) = n - \varepsilon_0(G)$ where $\varepsilon_0(G)$ is the number of leaves of G; when G is a cycle of length $n \geq 5$, $\operatorname{dstab}(I(G)) = \lceil \frac{n+1}{2} \rceil$. Although we know the limit depth and its index of depth stability, intermediate values for depth of powers of edge ideals were unknown even for cycles. The depth of powers of edge ideals of paths was only given recently by Bălănescu and Cimpoeaş [BC1]. For general graphs, Seyed Fakhari [SF] gave a sharp lower bound for the depth of the second power of their edge ideals. In this paper, we compute the depth of powers of edge ideals of cycles. For each $n \geq 3$, C_n denotes a cycle of length n, i.e., a graph on $V(G) = [n]$ and edge set $E(G) = \{\{1,2\},\{2,3\},\dots,\{n-1,n\},\{1,n\}\}$.

Dedicated to Professor Ngo Viet Trung on the occasion of his 70th birthday.

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Theorem 1.1. *Let $I(C_n)$ be the edge ideal of a cycle of length $n \ge 5$. Then*

$$\operatorname{depth}\left(S/I(C_n)^t\right) = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil, & \text{if } t = 1, \\ \left\lceil \frac{n-t+1}{3} \right\rceil, & \text{if } 2 \le t < \left\lceil \frac{n+1}{2} \right\rceil, \\ 1, & \text{if } n \text{ is even and } t \ge \frac{n}{2}+1, \\ 0, & \text{if } n \text{ is odd and } t \ge \frac{n+1}{2}. \end{cases}$$

In particular, the depth function of powers of edge ideals of cycles makes a big drop just before it stabilizes. The initial value and limiting values were well-known (see [Mo, C, T]), so our contribution is the computation of $\operatorname{depth}(S/I(C_n)^t)$ for $2 \le t < \lceil (n+1)/2 \rceil$. We now outline the ideas to carry out this computation. For simplicity of notation, we set $I = I(C_n)$ and $e_i = x_i x_{i+1}$ for $i = 1,\dots,n-1$, $e_n = x_1 x_n$. Denote

$$\varphi(n,t) = \left\lceil \frac{n-t+1}{3} \right\rceil.$$

(1) First, we show that $\operatorname{depth}(S/I^t) \le \operatorname{depth}(S/(I^t : (e_2 \cdots e_t))) \le \varphi(n,t)$.

(2) To establish the lower bound, by Lemma 2.3, we need to show that $\operatorname{depth}(S/(I^t : f)) \ge \varphi(n,t)$ and $\operatorname{depth}(S/(I^t, f)) \ge \varphi(n,t)$, where $f = x_1 \cdots x_{2t-2}$. For the first term, we use induction on t as $(I^t : f)$ is well-understood. For the second term, we note that $(I^t, f) = (I^t, x_1x_2) \cap (I^t, x_3 \cdots x_{2t-2})$. By repeated use of the Depth Lemma, we reduce to proving that $\operatorname{depth}(S/(I^t + I(H))) \ge \varphi(n,t)$ for all non-zero subgraphs H of C_n. We accomplish that by induction on t and downward induction on the number of edges of H.

In order to compute $\operatorname{depth}(S/I(G)^t)$ for an arbitrary graph G, according to [NV2, Theorem 1.1], we can reduce to the case when G is connected. We also note that the regularity of powers of edge ideals of graphs is known for many classes of graphs (see [MV] for a recent survey on the topic). This is partly due to the fact that the regularity of powers of these edge ideals behaves nicely. On the other hand, our next result shows that, in general, one cannot expect a simple formula for the depth of powers of the edge ideal of a tree in terms of its combinatorial invariants.

We now describe a formula for the depth of powers of edge ideals of starlike trees. We first introduce some notations. A path of length $n-1$, denoted by P_n, is a graph on $V(P_n) = [n]$ whose edge set is $\{\{1,2\},\{2,3\},\dots,\{n-1,n\}\}$. Assume that $k \ge 2$ is a natural number. Let $\mathbf{a} = (a_1,\dots,a_k) \in \mathbb{N}^k$ be a vector of positive integers such that $|\mathbf{a}| = a_1 + \cdots + a_k = n-1$. The starlike tree $T_{\mathbf{a}}$, which is the join of k paths of lengths $a_1,\dots,a_k$ at a common root 1, is the graph on $[n]$ with edge set

$$\begin{aligned} E(T_{\mathbf{a}}) = &\{\{1,2\},\dots,\{a_1,a_1+1\},\{1,a_1+2\},\dots,\{a_1+a_2,a_1+a_2+1\},\dots, \\ &\{1,a_1+\cdots+a_{k-1}+2\},\dots,\{a_1+\cdots+a_k,a_1+\cdots+a_k+1\}\}. \end{aligned}$$

Starlike trees are natural generalizations of paths, as the join of two paths of length a_1 and a_2 is a path of length $a_1 + a_2 + 1$.

For $i = 0,1,2$, let $\alpha_i(\mathbf{a})$ be the number of a_j such that $a_j \equiv i \pmod 3$. We define $g : \mathbb{N}^k \to \mathbb{N}$ by

$$g(\mathbf{a}) = \begin{cases} \sum_{i=1}^k \left\lceil \frac{a_i-1}{3} \right\rceil, \text{ if } \alpha_1(\mathbf{a}) = 0 \text{ and } \alpha_2(\mathbf{a}) \neq 0, \\ 1 + \sum_{i=1}^k \left\lceil \frac{a_i-1}{3} \right\rceil, \text{ otherwise.} \end{cases}$$

We may further assume that $a_j \equiv 2 \pmod 3$ for $j = 1, \ldots, \alpha_2(\mathbf{a})$, $a_j \equiv 0 \pmod 3$ for $j = \alpha_2(\mathbf{a}) + 1, \ldots, \alpha_2(\mathbf{a}) + \alpha_0(\mathbf{a})$ and $a_j \equiv 1 \pmod 3$ for $j = \alpha_0(\mathbf{a}) + \alpha_2(\mathbf{a}) + 1, \ldots, k$. Let

$$\begin{aligned} \beta_1 &= \min\{\alpha_2(\mathbf{a}), t-1\}, \\ \beta_2 &= \min\left\{\alpha_0(\mathbf{a}), \left\lfloor \frac{\max\{t-1-\alpha_2(\mathbf{a}), 0\}}{2} \right\rfloor \right\}, \\ \beta_3 &= \left\lfloor \frac{\max\{t-1-\beta_1-2\beta_2, 0\}}{3} \right\rfloor. \end{aligned}$$

We then define $\mathbf{b} \in \mathbb{N}^k$ as follows.

$$b_i = \begin{cases} a_i - 1, & \text{for } i = 1, \ldots, \beta_1, \\ a_i - 2, & \text{for } i = \alpha_2(\mathbf{a}) + 1, \ldots, \alpha_2(\mathbf{a}) + \beta_2 \\ a_i, & \text{otherwise.} \end{cases}$$

With the above notations, we have:

Theorem 1.2. *Assume that $k \geq 2$ is a natural number. Let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ be a vector of positive integers. Denote by $T_{\mathbf{a}}$ the starlike trees obtained by joining k paths of length $a_1, \ldots, a_k$ at the common root 1. Then for all t such that $1 \leq t \leq |\mathbf{a}| - k = s$, we have that*

$$\operatorname{depth}\left(S/I(T_{\mathbf{a}})^t\right) = g(\mathbf{b}) - \beta_3.$$

Example 1.3. Let $\mathbf{a} = (3,3,5)$. Then $\alpha_0 = 2$, $\alpha_1 = 0$ and $\alpha_2 = 1$. By Theorem 1.2, we see that the sequence $\{\operatorname{depth}(S/I(T_{\mathbf{a}})^t) \mid 1 \leq t \leq 9\}$ is $\{4,4,4,3,3,2,2,2,1\}$.

We structure the paper as follows. In Section 2, we set up the notation and provide some background. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Preliminaries

In this section, we recall some definitions and properties concerning the depth of monomial ideals and edge ideals of graphs. The interested readers are referred to [BH, D] for more details.

Throughout the paper, we let $S = \mathrm{k}[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal homogeneous ideal of S.

2.1. *Depth.* For a finitely generated graded S-module L, the depth of L is defined to be

$$\operatorname{depth}(L) = \min\{i \mid H^i_{\mathfrak{m}}(L) \neq 0\},$$

where $H^i_{\mathfrak{m}}(L)$ denotes the i-th local cohomology module of L with respect to $\mathfrak{m}$. We have the following well-known Depth Lemma (see [BH, Proposition 1.2.9]).

Lemma 2.1. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated graded S-modules. Then*

(1) $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(L), \operatorname{depth}(N)\}$,
(2) $\operatorname{depth}(L) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$,
(3) $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(L) - 1, \operatorname{depth}(M)\}$.

We make repeated use of the following results in the sequel. The first one is [R, Corollary 1.3]. The second one is a consequence of the Depth Lemma.

Lemma 2.2. *Let I be a monomial ideal and f a monomial such that $f \notin I$. Then*

$$\operatorname{depth}(S/I) \le \operatorname{depth}(S/(I:f)).$$

Lemma 2.3. *Let I be a homogeneous ideal and f be a homogeneous form of S. Then*

$$\operatorname{depth}(S/I) \ge \min\{\operatorname{depth}(S/(I:f)), \operatorname{depth}(S/(I,f))\}.$$

Proof. Applying the Depth Lemma to the short exact sequence

$$0 \to S/(I:f) \to S/I \to S/(I,f) \to 0,$$

we obtain the conclusion. □

Remark 2.4. When I is a monomial ideal and f is a monomial of S, Caviglia, Ha, Herzog, Kummini, Terai, and Trung [CHHKTT] proved a stronger result, namely

$$\operatorname{depth}(S/I) \in \{\operatorname{depth}(S/(I,f)), \operatorname{depth}(S/(I:f))\}.$$

But Lemma 2.3 is sufficient for us in this paper. We thank an anonymous referee for pointing this out.

As a consequence, we have

Corollary 2.5. *Let I be a monomial ideal and f be a monomial of S. Assume that* $\operatorname{depth}(S/(I,f)) \ge \operatorname{depth}(S/(I:f))$. *Then* $\operatorname{depth}(S/I) = \operatorname{depth}(S/(I:f))$.

Proof. By Lemma 2.2 and Lemma 2.3, we have that

$$\operatorname{depth}(S/(I:f)) \ge \operatorname{depth}(S/I) \ge \min\{\operatorname{depth}(S/(I:f)), \operatorname{depth}(S/(I,f))\} = \operatorname{depth}(S/(I:f)).$$

The conclusion follows. □

Finally, we also use the following simple result.

Lemma 2.6. *Let $S = \mathrm{k}[x_1,\dots,x_n]$, $R_1 = \mathrm{k}[x_1,\dots,x_a]$, and $R_2 = \mathrm{k}[x_{a+1},\dots,x_n]$ for some natural number a such that $1 \le a < n$. Let I and J be homogeneous ideals of R_1 and R_2, respectively. Then*

(1) $\operatorname{depth}(S/(I+J)) = \operatorname{depth}(R_1/I) + \operatorname{depth}(R_2/J)$.
(2) *Let $P = I + (x_{a+1},\dots,x_b)$. Then* $\operatorname{depth}(S/P) = \operatorname{depth}(R_1/I) + (n-b)$.

Proof. Part (1) is standard; for example, see [NV2, Lemma 2.3].

Part (2) follows from Part (1) and the fact that $\operatorname{depth}(R_2/(x_{a+1},\dots,x_b)) = (n-b)$. □

2.2. *Graphs and their edge ideals.* Let G be a finite simple graph over the vertex set $V(G) = [n] = \{1,2,\dots,n\}$ and the edge set $E(G)$. For a vertex $i \in V(G)$, let the neighbourhood of x be the subset $N_G(i) = \{j \in V(G) \mid \{i,j\} \in E(G)\}$, and set $N_G[i] = N_G(i) \cup \{i\}$. The degree of a vertex $i \in V(G)$ is the number of its neighbours. A leaf of G is a vertex of degree 1.

A simple graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an induced subgraph of G if it is a subgraph of G and $E(H) = E(G) \cap V(H) \times V(H)$.

The edge ideal of G is defined to be

$$I(G) = (x_i x_j \mid \{i,j\} \in E(G)) \subseteq S.$$

We now recall several classes of graphs that we study in this work. A path P_n of length $n-1$ is the graph on $[n]$ whose edges are $\{i,i+1\}$ for $i = 1,\ldots,n-1$.

A walk in G is a sequence of (possibly repeated) vertices $i_1,\ldots,i_\ell$ such that $\{i_j, i_{j+1}\} \in E(G)$ for all $j = 1,\ldots,\ell-1$. A walk is called an even (odd) walk if ℓ is even (odd). It is called closed if $i_1 = i_\ell$.

A cycle C_n of length $n \geq 3$ is the graph on $[n]$ whose edges are $\{i,i+1\}$ for $i = 1,\ldots,n-1$ and $\{1,n\}$.

A tree is a connected graph without any cycle. A starlike tree is a tree with at most one vertex of degree ≥ 3.

A subset $U \subset [n]$ is called an independent set of G if the induced subgraph of G on U has no edges.

A graph H is bipartite if there exists a bipartition of the vertex set of H, $V(H) = U \cup V$ such that $U \cap V = \emptyset$ and $E(H) \subseteq U \times V$. It is a complete bipartite graph if, furthermore, $E(H) = U \times V$.

2.3. *Colon ideals of monomial ideals.* We have the following simple result about colon ideals of monomial ideals.

Lemma 2.7. *Let I be an ideal of S generated by the monomials $f_1,\ldots,f_s$ and f be a monomial of S. Then $(I:f)$ is generated by $f_1/\gcd(f_1,f),\ldots,f_s/\gcd(f_s,f)$.*

Proof. Since $f_i \in I$, we have that $f_i/\gcd(f_i,f) \in (I:f)$. Now assume that g is any monomial in $(I:f)$. Then $fg \in I$. Since I is a monomial ideal, there exists $j \in \{1,\ldots,s\}$ such that $f_j \mid fg$. In particular, $(f_j/\gcd(f_j,f)) \mid (f/\gcd(f_j,f)) \cdot g$. Since $\gcd(f_j/\gcd(f_j,f), f/\gcd(f_j,g)) = 1$, we deduce that $f_j/\gcd(f_j,f)$ divides g. The conclusion follows. ☐

As a consequence, we have

Corollary 2.8. *Let I and J be monomial ideals and f be a monomial of S. We have that*

$$((I+J):f) = (I:f) + (J:f).$$

Proof. Assume that I and J are generated by monomials $f_1,\ldots,f_s$ and $g_1,\ldots,g_t$ respectively. Then $I+J$ is generated by $f_1,\ldots,f_s,g_1,\ldots,g_t$. The conclusion follows from Lemma 2.7. ☐

For each subset $U \subset [n]$, we set $x_U = \prod_{u\in U} x_u$, $N_G(U) = \cup_{u\in U} N_G(u)$, and $N_G[U] = \cup_{u \in U} N_G[u]$.

Lemma 2.9. *Let G be a simple graph. Assume that U is an independent set of G. We have that*

$$(I(G):x_U) = I(G) + (x_v \mid v \in N_G(U)) = I(G') + (x_v \mid v \in N_G(U)),$$

where G' is the induced subgraph of G on $V(G) \setminus N_G[U]$.

Proof. Let $\{i,j\}$ be an edge of G. Since U is an independent set, we deduce that $x_i x_j \nmid x_U$. If $\{i,j\} \cap U = \emptyset$, we have that $x_i x_j / \gcd(x_i x_j, x_U) = x_i x_j$. If $i \in U$, we have that $x_i x_j/\gcd(x_i x_j, x_U) = x_j$. Since $i \in U$, we have that $j \in N_G(U)$. By Lemma 2.7, we deduce that

$$(I(G):x_U) = I(G) + (x_v \mid v \in N_G(U)).$$

Now, for any edge $\{i,j\}$ of G such that $\{i,j\} \cap N_G[U] \neq \emptyset$, we must have $\{i,j\} \cap N_G(U) \neq \emptyset$. Thus, $x_i x_j \in (x_v \mid v \in N_G(U))$. The conclusion follows. ☐

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Finally, we have the following result [Mo, Lemma 2.10].

Lemma 2.10. *Suppose that G is a graph, i is a leaf of G and j is the unique neighbour of i. Then for any $t \geq 2$, we have that*

$$(I(G)^t : (x_i x_j)) = I(G)^{t-1}.$$

2.4. *Even-connection and a colon of powers of edge ideals.* Let $I = I(G)$ be the edge ideal of a simple graph G. In this subsection, we first recall the notion of even-connection via a collection of edges of G introduced by Banerjee [B]. We then describe the colon ideals of powers of I by a monomial corresponding to a collection of edges of G. We use the following notation. For an edge $e = \{i, j\}$ of G, x^e denotes the monomial $x_i x_j$. Assume that t is a positive integer. For a collection $\mathbf{e} = (e_1, \ldots, e_t)$ of t edges of G, $x^{\mathbf{e}}$ denotes the monomial $x^{e_1} \cdots x^{e_t}$.

Definition 2.11. Let $\mathbf{e} = (e_1, \ldots, e_t)$ be a collection of t (possibly repeated) edges of G. We say that two vertices u and v of G are **e**-even connected if there exist (possibly repeated) vertices $i_1, \ldots, i_{2k}$ of G such that

(1) $\{u, i_1\}, \{i_1, i_2\}, \ldots, \{i_{2k-1}, i_{2k}\}, \{i_{2k}, v\} \in E(G)$,
(2) $\{i_{2j+1}, i_{2j+2}\} = e_\ell$ for some $\ell \in \{1, \ldots, t\}$ and all $j = 0, \ldots, k-1$,
(3) for all $j = 1, \ldots, t$, $\left|\{p \mid \{i_{2p+1}, i_{2p+2}\} = e_j\}\right| \leq \left|\{q \mid e_q = e_j\}\right|$.

Note that if $\{u, v\} \in E(G)$ then u and v are **e**-even connected for arbitrary collections **e**. We call the walk $u, i_1, i_2, \ldots, i_{2k}, v$ in the Definition 2.11 an **e**-even walk connecting u and v. The following is [B, Theorem 6.7].

Theorem 2.12. *Let $I = I(G)$ be the edge ideal of a simple graph G and $\mathbf{e} = (e_1, \ldots, e_t)$ be a collection of t (possibly repeated) edges of G. Then $(I^{t+1} : x^{\mathbf{e}})$ is generated by quadratic monomials $x_u x_v$ such that u and v are* **e***-even connected.*

Example 2.13. Let G be the graph on $V(G) = \{1, \ldots, 6\}$ and edge set

$$E(G) = \{\{1,3\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{4,6\}\}.$$

Let $\mathbf{e} = (\{3,4\}, \{3,4\}, \{5,6\})$. Then 1 and 2 are **e**-even connected via the sequence of vertices $3, 4, 5, 6, 4, 3$. Indeed, we have

$$\left(I(G)^4 : (x_3^2 x_4^2 x_5 x_6)\right) = I(G) + (x_1^2, x_1x_2, x_1x_5, x_1x_6, x_2^2, x_2x_5, x_2x_6).$$

We now prove

Lemma 2.14. *Let $\mathbf{e} = (e_1, \ldots, e_t)$ be a collection of t distinct edges of G. Assume that $e_j = \{i_{2j-1}, i_{2j}\}$ for $j = 1, \ldots, t$. Note that the vertices $i_1, \ldots, i_{2t}$ are not necessarily distinct. Furthermore, assume that the induced subgraph of G on $N_G[\mathbf{e}] = \cup_{j=1}^{2t} N_G[i_j]$ does not contain an odd cycle. For each $s = 1, \ldots, t+1$, we define the graph G_s on the vertex set $[n]$ recursively as follows*

$$G_1 = G, E(G_{s+1}) = E(G_s) \cup \{\{u, v\} \mid u \in N_{G_s}(i_{2s-1}), v \in N_{G_s}(i_{2s})\}.$$

Then, for all $s = 1, \ldots, t$, we have that

$$\left(I(G)^{s+1} : (x^{e_1} \cdots x^{e_s})\right) = I(G_{s+1}) = \left(I(G_s) : x_{i_{2s-1}}\right) \cap \left(I(G_s) : x_{i_{2s}}\right).$$

Proof. For each $s = 1, \ldots, t$, we denote by $\mathbf{e}_s$ the collection of edges $\mathbf{e}_s = (e_1, \ldots, e_s)$. Let $J_1 = I(G)$ and $J_{s+1} = \left(I(G)^{s+1} : (x^{e_1} \cdots x^{e_s})\right)$. We prove by induction on s that

$$J_{s+1} = I(G_{s+1}) = \left(I(G_s) : x_{i_{2s-1}}\right) \cap \left(I(G_s) : x_{i_{2s}}\right). \tag{2.1}$$

The base case $s = 1$ follows immediately from Theorem 2.12.

Now, assume that the statement holds for $s - 1 \geq 1$. First, we prove that $J_{s+1} \subseteq I(G_{s+1})$. By Theorem 2.12, a minimal generator of J_{s+1} is of the form $x_u x_v$ such that u and v are $\mathbf{e}_s$-even connected. By definition, there exist vertices $j_1, \ldots, j_{2k}$ such that

(1) $\{u, j_1\}, \{j_1, j_2\}, \ldots, \{j_{2k-1}, j_{2k}\}, \{j_{2k}, v\}$ are edges of G,
(2) $\{j_1, j_2\}, \ldots, \{j_{2k-1}, j_{2k}\}$ are among $e_1, \ldots, e_s$,
(3) $\{j_1, j_2\}, \ldots, \{j_{2k-1}, j_{2k}\}$ are distinct edges of G.

If e_s does not appear among the edges $\{j_1, j_2\}, \ldots, \{j_{2k-1}, j_{2k}\}$ then u and v are $\mathbf{e}_{s-1}$-even connected. Hence, $x_u x_v \in I(G_s) \subseteq I(G_{s+1})$. Thus, we may assume that $j_{2\ell-1} = i_{2s-1}$ and $j_{2\ell} = i_{2s}$ for some $\ell \in \{1, \ldots, s\}$. In particular, u and i_{2s-1} and i_{2s} and v are $\mathbf{e}_{s-1}$-even connected. Hence, by induction, $u \in N_{G_s}(i_{2s-1})$ and $v \in N_{G_s}(i_{2s})$. Thus, $\{u, v\} \in E(G_{s+1})$.

Now, we prove that

$$N_{G_s}(i_{2s-1}) \cap N_{G_s}(i_{2s}) = \emptyset. \tag{2.2}$$

Indeed, assume that $u \in N_{G_s}(i_{2s-1}) \cap N_{G_s}(i_{2s})$. By Theorem 2.12 and induction hypothesis, u is even-connected to i_{2s-1} and i_{2s}. Hence, the concatenation of even-walks from u to i_{2s-1} and u to i_{2s} forms a closed odd walk in $N_G[\mathbf{e}_s]$, a contradiction to the assumption.

Now, we prove that if $u \in N_{G_s}(i_{2s-1})$ and $v \in N_{G_s}(i_{2s})$ then $x_u x_v \in J_{s+1}$. By induction, there exist $\mathbf{e}_{s-1}$-even walks $u, j_1, \ldots, j_{2k}, i_{2s-1}$ and $i_{2s}, p_1, \ldots, p_{2l}, v$. If $\{j_1, \ldots, j_{2k}\} \cap \{p_1, \ldots, p_{2l}\} = \emptyset$ then the concatenation of the two even-walks form an $\mathbf{e}_s$-even walk connecting u and v. The conclusion follows from Theorem 2.12. If $j_{2\ell} = p_{2m+1}$ for some ℓ and m then the walk $j_{2\ell}, \ldots, j_{2k}, i_{2s-1}, i_{2s}, p_1, \ldots, p_{2m+1}$ is a closed odd walk in $N_G[\mathbf{e}_s]$, a contradiction. Finally, assume that $j_{2\ell} = p_{2m}$ for some ℓ and m. Now along the even walk $u, j_1, \ldots, j_{2k}, i_{2s-1}, i_{2s}, p_1, \ldots, p_{2l}, v$ we can omit the middle part from $j_{2\ell}$ to p_{2m} and obtain a shorter even walk. We can repeat this until there is no further repetition of the vertices on the walk to obtain an $\mathbf{e}_s$-even walk connecting u and v.

Finally, the equality

$$I(G_{s+1}) = \left(I(G_s) : x_{i_{2s-1}}\right) \cap \left(I(G_s) : x_{i_{2s}}\right)$$

follows from Lemma 2.9 and Eq. (2.2). □

2.5. ***Strongly disjoint families of complete bipartite subgraphs.*** In this subsection, we recall the result of Kimura [K] bounding the projective dimension of edge ideals of graphs in terms of the notion of strongly disjoint families of complete bipartite subgraphs. A strongly disjoint family of complete bipartite subgraphs of a graph G is a family of (non-induced) subgraphs $B_1, \ldots, B_g$ of G such that

(1) each B_i is a complete bipartite graph for $1 \leq i \leq g$,
(2) the graphs $B_1, \ldots, B_g$ have pairwise disjoint vertex sets,
(3) there exists an induced matching $e_1, \ldots, e_g$ of G for each $e_i \in E(B_i)$ for $1 \leq i \leq g$.

We have the following result [K, Theorem 1.1].

Theorem 2.15. *Let $\mathscr{B} = \{B_1, \dots, B_g\}$ be a strongly disjoint family of complete bipartite subgraphs of a graph G. Then*

$$\operatorname{pd}(S/I(G)) \geq \left(\sum_{i=1}^{g} |V(B_i)| \right) - g.$$

We now deduce the following bound.

Lemma 2.16. *Let G be a simple graph on $V(G) = [n]$. Let B_1 be a complete bipartite subgraph of G with $e_1 \in B_1$. Let H be a subgraph of G on $V(H) = [n] \setminus V(B_i)$. Assume that H is a forest and $N_G[e_1] \cap V(H) = \emptyset$. Then*

$$\operatorname{depth}(S/I(G)) \leq 1 + \operatorname{depth}(R/I(H)),$$

where R is the polynomial ring on $V(H)$.

Proof. Since H is a forest, by [NV1, Theorem 7.7], there exists a strongly disjoint family of complete bipartite graphs $B_2, \dots, B_g$ of H such that

$$\operatorname{pd}(R/I(H)) = \left(\sum_{i=2}^{g} |V(B_i)| \right) - (g-1).$$

By the assumption of the lemma, we see that $B_1, \dots, B_g$ form a strongly disjoint family of complete bipartite graphs of G. By Theorem 2.15, we have that

$$\operatorname{pd}(S/I(G)) \geq \left(\sum_{i=1}^{g} |V(B_i)| \right) - g.$$

The conclusion follows from the Auslander-Buchsbaum formula. □

Remark 2.17. Lemma 2.16 is a special case of [HHV, Lemma 1.2]. We keep this simple version to avoid introducing too many terminologies.

3. Depth of powers of edge ideals of cycles

In this section, we compute the depth of powers of edge ideals of paths and cycles. We fix the following notation throughout the rest of the paper. For each n, P_n and C_n denote the path of length $n-1$ and the cycle of length n, respectively, on the vertex set $[n]$. $S = \mathrm{k}[x_1, \dots, x_n]$ is a standard graded polynomial ring over a field k. For a real number a, denote by $\lceil a \rceil$ the least integer at least a, $\lfloor a \rfloor$ the largest integer at most a. First, we have two simple lemmas.

Lemma 3.1. *Let a, b be integers. Then*

$$\left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil \geq \left\lceil \frac{a+b}{3} \right\rceil.$$

Proof. There exist unique integers k, a_1, l, b_1 such that $a = 3k + a_1$ and $b = 3l + b_1$ with $1 \leq a_1, b_1 \leq 3$. By definition, $\lceil \frac{a}{3} \rceil = k+1$ and $\lceil \frac{b}{3} \rceil = l+1$. Since $a_1, b_1 \leq 3$, $a_1 + b_1 \leq 6$, hence $\left\lceil \frac{a_1+b_1}{3} \right\rceil \leq 2$. The conclusion follows. □

Lemma 3.2. *Let I,J,K be homogeneous ideals of S such that $I \subseteq K$. Then, for any positive integer t, we have*

$$(I+J)^t + K = J^t + K.$$

Proof. We have $(I+J)^t = J^t + I(I+J)^{t-1}$. Since $I(I+J)^{t-1} \subseteq I \subseteq K$, the conclusion follows. □

Furthermore, the depth of $I(P_n)$ and $I(C_n)$ is well known [Mo, C]. By convention, P_1 is the graph on $[1]$ with no edge and $I(P_1)$ is the zero ideal in $S = k[x_1]$.

Lemma 3.3. *We have*

(1) $\operatorname{depth}(S/I(P_n)) = \left\lceil \frac{n}{3} \right\rceil$ *for* $n \geq 1$,
(2) $\operatorname{depth}(S/I(C_n)) = \left\lceil \frac{n-1}{3} \right\rceil$ *for* $n \geq 3$.

We now come to a crucial step in computing the depth of powers of edge ideals of paths and cycles. For the rest of this section, we denote $e_i = x_i x_{i+1}$ for $i = 1,\dots,n-1$ and $e_n = x_1 x_n$. To avoid complicated notation, we assume that e_i also denotes the corresponding edge $\{i, i+1\}$ of P_n for $i = 1,\dots,n-1$, and e_n also denotes the edge $\{1,n\}$ of C_n. It is clear from the context when e_i is a monomial in S or when e_i is an edge of a graph. We define

$$\varphi(n,t) = \left\lceil \frac{n-t+1}{3} \right\rceil.$$

Lemma 3.4. *Let H be any subgraph of P_n. Then, for any positive integer t with $t < n$, we have that*

$$\operatorname{depth}\left(S/(I(P_n)^t + I(H))\right) \geq \varphi(n,t).$$

Proof. We use downward induction on the number of edges of H, denoted by $|E(H)|$, induction on n, and induction on t. If $|E(H)| = n-1$ or $t = 1$, then $I(P_n)^t + I(H) = I(P_n)$. By Lemma 3.3, we have that

$$\operatorname{depth}(S/I(P_n)) = \left\lceil \frac{n}{3} \right\rceil \geq \varphi(n,t).$$

Assume that $|E(H)| < n-1$ and $t \geq 2$. Since, $\sqrt{I(P_n)^t + I(H)} = I(P_n)$, $\mathfrak{m}$ is not an associated prime of $I(P_n)^t + I(H)$. Hence, if $n \leq 4$ and $t \geq 2$ then

$$\operatorname{depth}\left(S/(I(P_n)^t + I(H))\right) \geq 1 \geq \varphi(n,t).$$

Thus, we may assume that $n \geq 5$ and $t \geq 2$.

Let i be the smallest index such that $e_i \notin H$, i.e., $e_1,\dots,e_{i-1} \in H$. Let $J = I(P_n)^t + I(H)$. We have that $(J, e_i) = I(P_n)^t + I(H')$ with H' is a subgraph of P_n with $E(H') = E(H) \cup \{e_i\}$. Since $|E(H')| > |E(H)|$, by induction on $|E(H)|$, $\operatorname{depth}(S/(J,e_i)) \geq \varphi(n,t)$.

By Lemma 2.3, it suffices to prove that

$$\operatorname{depth}(S/(J : e_i)) \geq \varphi(n,t). \tag{3.1}$$

There are four cases to consider.

Case 1. $i = 1$ and $e_2 \in H$. We claim that

$$(J : e_1) = L^{t-1} + (x_3) + I(H'), \tag{3.2}$$

where $L = (x_1x_2, x_4x_5, \dots, x_{n-1}x_n)$ and H' is the induced subgraph of H on $V(H) \setminus \{1,2,3\}$.

Proof of Eq. (3.2). By assumption, $e_1 \notin H$ and $e_2 \in H$. Let $Q = L + (x_3x_4)$. Then $I(P_n) = Q + (e_2) = Q + I(H)$. By Lemma 3.2, $J = I(P_n)^t + I(H) = Q^t + I(H)$. By Corollary 2.8 and Lemma 2.10, we have that

$$(J : e_1) = Q^{t-1} + (I(H) : e_1) = Q^{t-1} + (x_3) + I(H').$$

Since $Q = L + (x_3x_4)$ and $(x_3x_4) \subset (x_3)$, Eq. (3.2) follows from Lemma 3.2. □

For each $\ell \geq 1$, we set $K_\ell = L^\ell + (x_3) + I(H')$. In particular, $(J : e_1) = K_{t-1}$. We prove by induction on ℓ that $\operatorname{depth}(S/K_\ell) \geq \varphi(n,t)$ for all $1 \leq \ell \leq t-1$. For $\ell = 1$, note that $K_1 = L + (x_3) + I(H') = L + (x_3)$. Let G be the induced subgraph of P_n on $\{4, \ldots, n\}$. By Lemma 2.6, we have that

$$\operatorname{depth}(S/K_1) = \operatorname{depth}(R/I(G)) + 1 = 1 + \varphi(n-3,1) \geq \varphi(n,t), \tag{3.3}$$

where $R = \mathrm{k}[x_4, \ldots, x_n]$.

Now assume that $\operatorname{depth}(S/K_\ell) \geq \varphi(n,t)$. First, we prove that $\operatorname{depth}(S/(K_{\ell+1} + (e_1))) \geq \varphi(n,t)$. Since $L = I(G) + (e_1)$, by Lemma 3.2, we have that

$$K_{\ell+1} + (e_1) = I(G)^{\ell+1} + I(H') + (e_1) + (x_3). \tag{3.4}$$

By Lemma 2.6, we have that

$$\operatorname{depth}\left(S/(I(G)^{\ell+1} + I(H') + (e_1) + (x_3))\right) = \operatorname{depth}\left(R/(I(G)^{\ell+1} + I(H'))\right) + 1. \tag{3.5}$$

Since $G \cong P_{n-3}$ and H' is a subgraph of G, by induction on n, we deduce that

$$\operatorname{depth}\left(R/(I(G)^{\ell+1} + I(H'))\right) \geq \varphi(n-3, \ell+1). \tag{3.6}$$

From Eq. (3.4), Eq. (3.5), Eq. (3.6), we deduce that

$$\operatorname{depth}(S/(K_{\ell+1} + (e_1))) \geq \varphi(n-3, \ell+1) + 1 = \varphi(n, \ell+1) \geq \varphi(n,t),$$

for $\ell \leq t-1$. By Lemma 3.2 and Lemma 2.10, we have that

$$(K_{\ell+1} : e_1) = K_\ell. \tag{3.7}$$

By the Depth Lemma and induction on ℓ, we deduce that $\operatorname{depth}(S/K_{\ell+1}) \geq \varphi(n,t)$. That concludes the proof of inequality (3.1) for Case 1.

Case 2. $i = 1$ and $e_2 \notin H$. Since $e_1, e_2 \notin H$, we have that $I(H) : e_1 = I(H)$. By Lemma 3.2 and Lemma 2.10, we have that

$$(J : e_1) = I(P_n)^{t-1} + I(H).$$

The inequality (3.1) follows from induction on t.

Case 3. $i > 1$ and $e_{i+1} \in H$. We claim that

$$(J : e_i) = L^{t-1} + (x_{i-1}, x_{i+2}) + I(H') + I(P_{i-2}), \tag{3.8}$$

where $L = (x_ix_{i+1}, x_{i+3}x_{i+4}, \ldots, x_{n-1}x_n)$, P_{i-2} is the path on $1, \ldots, i-2$ and H' is the induced subgraph of H on $\operatorname{supp} H \setminus \{1, \ldots, i+2\}$.

Proof of Eq. (3.8). By assumption, $I(P_n) = L + I(H)$. By Lemma 3.2, $J = L^t + I(H)$. By Lemma 2.10, Corollary 2.8, and Lemma 2.9, the conclusion follows. □

For each $\ell = 1,\dots,t-1$, we set $K_\ell = L^\ell + (x_i, x_{i+2}) + I(H') + I(P_{i-2})$. With an argument similar to Case 1, we reduce to prove that

$$\operatorname{depth}(S/(K_\ell + (e_i))) \geq \varphi(n,t),$$

for all $\ell = 1,\dots,t-1$. Let G' be the induced subgraph of P_n on $\{i+3,\dots,n\}$. Then $G' \cong P_{n-i-2}$ and H' is a subgraph of G'. Note that G', e_i, and P_{i-2} have support on different sets of variables. By Lemma 2.6 and Lemma 3.3, we have that

$$\operatorname{depth}(S/(K_\ell + (e_i))) = \operatorname{depth}\left(R/(I(G')^\ell + I(H'))\right) + 1 + \left\lceil \frac{i-2}{3} \right\rceil, \tag{3.9}$$

where $R = \mathrm{k}[x_{i+3},\dots,x_n]$. By induction on n, we have that $\operatorname{depth}\left(R/(I(G')^\ell + I(H'))\right) \geq \varphi(n-i-2,\ell)$. Hence,

$$\operatorname{depth}(S/(K_\ell + (e_i))) \geq \varphi(n-i-2,\ell) + 1 + \left\lceil \frac{i-2}{3} \right\rceil \geq \varphi(n,t).$$

Case 4. $i > 1$ and $e_{i+1} \notin H$. By Lemma 3.2, Lemma 2.10, Corollary 2.8, and Lemma 2.9, we have that

$$(J : e_i) = I(G')^{t-1} + (x_{i-1}) + I(P_{i-2}) + I(H'), \tag{3.10}$$

where G' is the induced subgraph of P_n on $\{i,\dots,n\}$, P_{i-2} is the path $1,\dots,i-2$ and H' is the induced subgraph of H on $\operatorname{supp} H \setminus \{1,\dots,i-1\}$. By Lemma 2.6, we have that

$$\operatorname{depth}(S/(J : e_i)) = \operatorname{depth}\left(R/(I(G')^{t-1} + I(H'))\right) + \left\lceil \frac{i-2}{3} \right\rceil.$$

Since $G' \cong P_{n-i+1}$ and H' is a subgraph of G', by induction on n, we have that $\operatorname{depth}(R/(I(G')^{t-1} + I(H'))) \geq \varphi(n-i+1,t-1)$. Hence,

$$\operatorname{depth}(S/(J : e_i)) \geq \varphi(n-i+1,t-1) + \left\lceil \frac{i-2}{3} \right\rceil \geq \varphi(n,t).$$

The conclusion follows. $\square$

To obtain an upper bound for $\operatorname{depth}(S/I(P_n)^t)$, we prove

Lemma 3.5. *Let $e_i = x_i x_{i+1}$ for all $i = 1,\dots,n-1$ and $I = I(P_n) = (x_1x_2,\dots,x_{n-1}x_n)$. Then, for any $t \in \{1,\dots,n-2\}$, we have that*

$$\operatorname{depth}(S/(I^t : (e_2 \dots e_t))) = \varphi(n,t).$$

Proof. By Lemma 2.14, we have that $(I^t : (e_2 \dots e_t)) = I(G_{n,t})$, where $G_{n,t}$ is the graph on $V(G_{n,t}) = [n]$ and edge set

$$E(G_{n,t}) = E(P_n) \cup \{\{i,j\} \mid i < j \leq t+2 \text{ is of different parity}\}.$$

We prove by induction on n and downward induction on $t \leq n-2$ that

$$\operatorname{depth}(S/I(G_{n,t})) = \varphi(n,t) = \left\lceil \frac{n-t+1}{3} \right\rceil.$$

If $t = n-2$, then $G_{n,t}$ is a complete bipartite graph, hence $\operatorname{depth}(S/I(G_{n,t})) = 1$. Thus, we may assume that $t \leq n-3$. Hence, $I(G_{n,t}) = I(G_{n-1,t}) + (e_{n-1})$. Furthermore, this decomposition is a Betti splitting by [NV1, Corollary 4.12]. Since $I(G_{n-1,t}) \cap (e_{n-1}) = e_{n-1}((x_{n-2}) + I(G_{n-3,t}))$, by [NV1, Corollary 4.8] and induction on n, we have that

$$\begin{aligned}\operatorname{pd}(S/I(G_{n,t})) &= \max\{\operatorname{pd}(S/I(G_{n-1,t})), 1, \operatorname{pd}(S/I(G_{n-3,t})) + 1\} \\ &= \max\{n-1-\varphi(n-1,t), 1, n-\varphi(n-3,t)-1\} = n-\varphi(n,t).\end{aligned}$$

The conclusion follows from the Auslander-Buchsbaum formula. □

Theorem 3.6. *Let $I(P_n)$ be the edge ideal of a path of length $n-1$. Then*

$$\operatorname{depth}(S/I^t) = \max\left\{\left\lceil\frac{n-t+1}{3}\right\rceil, 1\right\},$$

for all $t \geq 1$.

Proof. By Lemma 3.3 and [T], we may assume that $2 \leq t \leq n-3$. By Lemma 3.4, take H be the empty graph, we deduce that $\operatorname{depth}(S/I^t) \geq \varphi(n,t)$. The conclusion then follows from Lemma 3.5 and Lemma 2.2. □

Remark 3.7. Note that for any integer n, $\lceil \frac{n}{3} \rceil = n+1-\lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n+1}{3} \rceil$. In particular, Theorem 3.6 is a special case of [BC1, Theorem 1]. We include a simple argument here because Lemma 3.4 will be critical to deduce the formula for depth of powers of edge ideals of cycles. Also, Ştefan [St] proved a similar formula for Stanley depth of $I(P_n)^t$.

We now turn to the edge ideals of cycles C_n. The depth of powers of $I(C_n)$ in the case $n \leq 4$ is clear. Thus, we may assume that $n \geq 5$. By [T], we know that $\operatorname{dstab}(I(C_n)) = \lceil \frac{n+1}{2} \rceil$. Thus, we may assume that $2 \leq t < \lceil \frac{n+1}{2} \rceil$. First, we note that $f = x_1 \cdots x_{2t-2}$ is a product of distinct variables. By the Depth Lemma, to establish the lower bound for $\operatorname{depth}(S/I(C_n)^t)$, it suffices to prove that $\operatorname{depth}(S/(I^t : f)) \geq \varphi(n,t)$ and $\operatorname{depth}(S/(I^t, f)) \geq \varphi(n,t)$. We establish the first inequality in the following lemma.

Lemma 3.8. *Assume that $n \geq 5$ and $2 \leq t < \lceil \frac{n+1}{2} \rceil$. Then*

$$\operatorname{depth}(S/(I(C_n)^t : (x_1 \cdots x_{2t-2}))) \geq \varphi(n,t).$$

Proof. For each $t = 1, \ldots, \lceil \frac{n+1}{2} \rceil - 1$, let $J_t = (I^t : (x_1 \cdots x_{2t-2}))$. By Lemma 2.14,

$$J_{t+1} = (J_t : x_{2t-1}) \cap (J_t : x_{2t}). \tag{3.11}$$

Note that $\operatorname{depth}(S/J_1) = \operatorname{depth}(S/I(C_n)) = \lceil \frac{n-1}{3} \rceil = \varphi(n,2)$. First, consider the base case $t = 2$. By Lemma 2.9, $(I : x_1) + (I : x_2) = (x_1, x_2, x_3, x_n) + I(G)$, where G is the induced subgraph of C_n on $\{4, \ldots, n-1\}$. In particular, $G \cong P_{n-4}$. By Lemma 2.6 and Lemma 3.3, we have that

$$\operatorname{depth}(S/((I : x_1) + (I : x_2))) = \left\lceil \frac{n-4}{3} \right\rceil = \varphi(n,2) - 1. \tag{3.12}$$

By Lemma 2.3,

$$\operatorname{depth}(S/J_2) \geq \min\{\operatorname{depth}(S/J_1), \operatorname{depth}(S/((I : x_1) + (I : x_2)) + 1)\} = \varphi(n,2). \tag{3.13}$$

Now, consider the induction step. By Lemma 2.14 and Lemma 2.9, we have that

$$(J_t : x_{2t-1}) + (J_t : x_{2t}) = (x_n, x_2, x_4, \ldots, x_{2t-4}, x_{2t-2}, x_{2t-1}, x_{2t}, x_{2t+1}) + I(H) \tag{3.14}$$

where H is the path from $2t+2$ to $n-1$. Note that $x_1, x_3, \ldots, x_{2t-3}$ are variables that do not appear in $(J_t : x_{2t-1}) + (J_t : x_{2t})$. By Lemma 2.6 and Lemma 3.3, we have that

$$\text{depth}(S/((J_t : x_{2t-1}) + (J_t : x_{2t}))) = t - 1 + \left\lceil \frac{n-2t-2}{3} \right\rceil \geq \varphi(n,t+1) - 1. \tag{3.15}$$

By Lemma 2.2 and induction, we have that

$$\min\{\text{depth}(S/(J_t : x_{2t-1})), \text{depth}(S/(J_t : x_{2t}))\} \geq \text{depth}(S/J_t) \geq \varphi(n,t).$$

Together with equation (3.11) and Lemma 2.3, we have that

$$\begin{aligned}\text{depth}(S/J_{t+1}) &\geq \min\{\varphi(n,t), \text{depth}(S/((J_t : x_{2t-1}) + (J_t : x_{2t}))) + 1\} \\ &\geq \varphi(n,t+1).\end{aligned}$$

The conclusion follows. □

The second inequality is established in the following lemma.

Lemma 3.9. *Assume that $t \geq 2$ and $f = x_1 \cdots x_{2t-2}$. Then* $\text{depth}(S/(I^t, f)) \geq \varphi(n,t)$.

Proof. For each $j = 1, \ldots, t-2$, let $f_j = x_{2j-1} \cdots x_{2t-2}$. Then $f = f_1$ and $f_j = (x_{2j-1}x_{2j}) \cdot f_{j+1}$. In other words, for any subgraph H of G consisting of edges which are subsets of $\{e_1, e_3, \ldots, e_{2j-3}\}$, we have

$$I^t + I(H) + (f_j) = (I^t + I(H) + (x_{2j-1}x_{2j})) \cap (I^t + I(H) + (f_{j+1})). \tag{3.16}$$

The conclusion follows from Lemma 2.1 and the following lemma. □

Lemma 3.10. *Let H be a non-empty subgraph of C_n. Then for $t \geq 2$, we have that*

$$\text{depth}(S/(I(C_n)^t + I(H))) \geq \varphi(n,t).$$

Proof. Since H is non-empty, we may assume that $e_n = x_1x_n \in H$. We prove by downward induction on the number of edges of H. If $|E(H)| = n$, then $I(C_n)^t + I(H) = I(C_n)$. By Lemma 3.3, we have that

$$\text{depth}\left(S/(I(C_n)^t + I(H))\right) = \varphi(n,2) \geq \varphi(n,t).$$

Let i be the smallest index such that $e_i \notin H$, i.e., $e_0 = e_n, e_1, \ldots, e_{i-1} \in H$. Let $J = I(C_n)^t + I(H)$. Since $J + (e_i) = I(C_n)^t + I(H')$ with $|E(H')| > |E(H)|$, thus, by induction on $|E(H)|$, we deduce that $\text{depth}(S/(J + (e_i))) \geq \varphi(n,t)$. By Lemma 2.3, it suffices to prove that

$$\text{depth}(S/(J : e_i)) \geq \varphi(n,t). \tag{3.17}$$

Note that $e_{i-1} \in H$. Hence $x_{i-1} \in I(H) : e_i$. There are two cases to consider.

Case 1. $e_{i+1} \notin H$. By Corollary 2.8 and Lemma 2.10, we have that

$$(J : e_i) = (x_{i-1}) + I(G)^{t-1} + I(H'), \tag{3.18}$$

where G is the induced subgraph of C_n on $[n] \setminus \{i-1\}$ and H' is the induced subgraph of H on $V(H) \setminus \{i-1\}$. In particular, $G \cong P_{n-1}$ and H' is a subgraph of G. By Lemma 2.6 and Lemma 3.4, we deduce that

$$\text{depth}(S/(J : e_i)) = \text{depth}\left(R/(I(G)^{t-1} + I(H'))\right) \geq \varphi(n-1,t-1) = \varphi(n,t),$$

where $R = \mathrm{k}[x_1,\dots,x_{i-2},x_i,\dots,x_n]$.

Case 2. $e_{i+1} \in H$. By Corollary 2.8, Lemma 2.10 and Lemma 2.9, we have that

$$(J : e_i) = (x_{i-1},x_{i+2}) + ((e_i) + I(G))^{t-1} + I(H'), \tag{3.19}$$

where G is the induced subgraph of C_n on $[n] \setminus \{i-1,i,i+1,i+2\}$ and H' is the induced subgraph of H on $V(H) \setminus \{i-1,i,i+1,i+2\}$. For each $\ell = 1,\dots,t-1$, let

$$K_\ell = (x_{i-1},x_{i+2}) + ((e_i) + I(G))^{\ell} + I(H'). \tag{3.20}$$

We prove by induction on ℓ that $\mathrm{depth}(S/K_\ell) \geq \varphi(n,t)$ for all $1 \leq \ell \leq t-1$. When $\ell = 1$, we have that $K_\ell = (x_{i-1},x_{i+2}) + (e_i) + I(G)$. Let $R = \mathrm{k}[x_1,\dots,x_{i-2},x_{i+3},\dots,x_n]$. Since $G \cong P_{n-4}$, by Lemma 2.6 and Lemma 3.3, we have that

$$\mathrm{depth}(S/K_\ell) = \mathrm{depth}(R/I(G)) + 1 = \left\lceil \frac{n-4}{3} \right\rceil + 1 \geq \varphi(n,t).$$

Now, assume that $\mathrm{depth}(S/K_\ell) \geq \varphi(n,t)$. By Corollary 2.8 and Lemma 2.10, we have that $(K_{\ell+1} : e_i) = K_\ell$. By Lemma 2.3 and induction, it suffices to prove that

$$\mathrm{depth}(S/(K_{\ell+1} + (e_i))) \geq \varphi(n,t), \tag{3.21}$$

for $\ell \leq t-2$. By Lemma 3.2, we have that $K_{\ell+1} + (e_i) = (x_{i-1},x_{i+2}) + I(G)^{\ell+1} + I(H') + (e_i)$. Note that H' is a subgraph of G. By Lemma 2.6 and Lemma 3.4, we have that

$$\mathrm{depth}(S/(K_{\ell+1} + (e_i))) = 1 + \mathrm{depth}\left(R/(I(G)^{\ell+1} + I(H'))\right) \geq 1 + \varphi(n-4,\ell+1) \geq \varphi(n,t), \tag{3.22}$$

for all $\ell \leq t-2$.

The conclusion follows. ∎

We now give an upper bound for the depth of powers of edge ideals of cycles.

Lemma 3.11. *Assume that $I = I(C_n)$ and $t \leq n-2$. Then*

$$\mathrm{depth}(S/(I^t : (e_2 \cdots e_t))) \leq \varphi(n,t).$$

Proof. Let $J = (I^t : (e_2 \cdots e_t))$. By Lemma 2.14, we have that $J = I(G_{n,t})$, where $G_{n,t}$ is the graph on $V(G_{n,t}) = [n]$ and edge set

$$E(G_{n,t}) = E(C_n) \cup \{\{i,j\} \mid 1 \leq i < j \leq t+2 \text{ is of different parity}\}.$$

First, assume that $t = n-2$. If n is even, then $G_{n,t}$ is a complete bipartite graph, hence $\mathrm{depth}(S/J) = 1$. If n is odd, let H be the restriction of $G_{n,t}$ to $[n] \setminus \{1\}$. Then, H is a complete bipartite graph. Furthermore, we have $J = x_1(x_2,x_4,\dots,x_{n-1},x_n) + I(H)$. By [NV1, Corollary 4.12], this is a Betti splitting. Furthermore, $x_1(x_2,x_4\dots,x_{n-1},x_n) \cap I(H) = x_1 I(H)$. Hence, by [NV1, Corollary 4.8], we have that

$$\mathrm{pd}(S/J) = \mathrm{pd}(S/(x_1 I(H))) + 1 = n-1.$$

Now, assume that $t = n-3$. By Lemma 2.9, we have that

$$(J : x_n) = (x_1,x_{n-1}) + I(K_{U,V}),$$

where $K_{U,V}$ is the complete bipartite graph on U and V are the partition of $\{2,\dots,n-2\}$ into odd and even numbers. By Lemma 2.2, we deduce that

$$\operatorname{depth}(S/J) \le \operatorname{depth}(S/(J : x_n)) \le 2.$$

Finally, assume that $t \le n-4$. By Lemma 2.9, we have that

$$(J : x_{n-1}) = (x_n, x_{n-2}) + (I(P_{n-3})^t : (e_2 \cdots e_t)). \tag{3.23}$$

By Lemma 2.2 and Lemma 3.5, we deduce that

$$\operatorname{depth}(S/J) \le \operatorname{depth}(S/(J : x_{n-1})) = 1 + \varphi(n-3,t) = \varphi(n,t).$$

The conclusion follows. □

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3, Lemma 3.8 and Lemma 3.9, we get that $\operatorname{depth}(S/I^t) \ge \varphi(n,t)$. By Lemma 2.2 and Lemma 3.11, we get that $\operatorname{depth}(S/I^t) \le \varphi(n,t)$. The conclusion follows. □

Remark 3.12. In [BC2, BC3], Bălănescu and Cimpoeaş considered the path ideals of cycles; they obtained a sharp upper bound for the depth of powers of these path ideals and exact values for some special classes of these path ideals. The overlap of their results with our results presented in the current paper is minimal.

Remark 3.13. Our arguments extend to compute the depth of symbolic powers of edge ideals of cycles. We cover that in subsequent work [MTV].

4. Depth of powers of edge ideals of starlike trees

In this section, we compute the depth of powers of edge ideals of starlike trees. We first introduce some notations. Assume that $k \ge 2$ is a natural number. We use bold letters for vectors in $\mathbb{R}^k$. The vectors $\mathbf{e}_1,\dots,\mathbf{e}_k$ are the canonical unit vectors of $\mathbb{R}^k$; $\mathbf{1}$ denotes the vector whose all components are 1. Let $\mathbf{a} = (a_1,\dots,a_k) \in \mathbb{N}^k$ be a vector of positive integers such that $|\mathbf{a}| = a_1 + \cdots + a_k = n-1$. The starlike tree $T_{\mathbf{a}}$, which is the join of k paths of lengths $a_1,\dots,a_k$ at a common root 1, is the graph on $[n]$ with edge set

$$\begin{aligned} E(T_{\mathbf{a}}) = &\{\{1,2\},\dots,\{a_1,a_1+1\},\{1,a_1+2\},\dots,\{a_1+a_2,a_1+a_2+1\},\dots,\\ &\{1,a_1+\cdots+a_{k-1}+2\},\dots,\{a_1+\cdots+a_k,a_1+\cdots+a_k+1\}\}. \end{aligned}$$

For $i = 0,1,2$, let $\alpha_i(\mathbf{a})$ be the number of a_j such that $a_j \equiv i \pmod 3$. Let $g : \mathbb{N}^k \to \mathbb{N}$ be defined by

$$g(\mathbf{a}) = \begin{cases} \sum_{i=1}^k \left\lceil \frac{a_i - 1}{3} \right\rceil, & \text{if } \alpha_1(\mathbf{a}) = 0 \text{ and } \alpha_2(\mathbf{a}) \ne 0,\\ 1 + \sum_{i=1}^k \left\lceil \frac{a_i - 1}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

The following properties of g follow immediately from the definition.

Lemma 4.1. *Let $\mathbf{a} \in \mathbb{N}^k$ be a vector of positive integers.*

(1) *Assume that* $\mathbf{a}' \in \mathbb{N}^k$ *is another vector obtained by permuting the coordinates of* $\mathbf{a}$ *then* $g(\mathbf{a}) = g(\mathbf{a}')$.
(2) *Assume that* $a_i \geq 3$ *then* $g(\mathbf{a}) = g(\mathbf{a} - 3\mathbf{e}_i) + 1$.
(3) *Assume that* $a_i > b > 0$ *then* $g(\mathbf{a}) \leq \lceil \frac{b}{3} \rceil + g(\mathbf{a} - b\mathbf{e}_i)$.

First, we compute the depth of the edge ideal of a starlike tree.

Lemma 4.2. *Let* $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ *be a vector of positive integers. Let* $T_{\mathbf{a}}$ *be the starlike trees obtained by joining paths of length* $a_1, \ldots, a_k$ *at the root* 1*. Then*

$$\operatorname{depth}(S/I(T_{\mathbf{a}})) = g(\mathbf{a}).$$

Proof. We prove by induction on $s = |\mathbf{a} - \mathbf{1}|$. For simplicity of notation, we denote $I = I(T_{\mathbf{a}})$. When $s = 0$, $T_{\mathbf{a}}$ is a star graph, which is a complete bipartite graph. Thus, $\operatorname{depth}(S/I) = 1$. Assume that $s > 0$ and a_1 is largest, so $a_1 > 1$. First, consider the case $a_1 \geq 3$. By induction, Lemma 2.6, Lemma 2.9, and Lemma 4.1, we have that

$$\operatorname{depth}(S/(I, x_{a_1})) = 1 + g(\mathbf{a} - 2\mathbf{e}_1) \geq g(\mathbf{a})$$
$$\operatorname{depth}(S/(I : x_{a_1})) = 1 + g(\mathbf{a} - 3\mathbf{e}_1) = g(\mathbf{a}).$$

By Corollary 2.5, we have that $\operatorname{depth}(S/I) = \operatorname{depth}(S/(I : x_{a_1})) = g(\mathbf{a})$.

Now, assume that $a_1 = 2$. Since a_1 is largest, $1 \leq a_j \leq 2$ for all $j = 1, \ldots, k$. We may assume that $a_1 \geq a_2 \geq \cdots \geq a_k$. There are two cases to consider.

Case 1. $a_k = 2$. By Lemma 2.9, we have that

$$(I : x_2) = (x_1, x_3) + (x_4x_5, x_6x_7, \ldots, x_{2k}x_{2k+1}).$$

By Lemma 2.6, $\operatorname{depth}(S/(I : x_2)) = k$. Furthermore, (I, x_2) is isomorphic to the starlike tree $T_{\mathbf{a}'}$ with $\mathbf{a}' = (2, \ldots, 2) \in \mathbb{N}^{k-1}$ and x_3 is a free variable of (I, x_2). By Lemma 2.6 and induction, we have that $\operatorname{depth}(S/(I, x_2)) = k$. By Corollary 2.5, we deduce that $\operatorname{depth}(S/I) = k = g(\mathbf{a})$.

Case 2. $a_k = 1$. Let ℓ be the largest index such that $a_\ell = 2$. Then $\ell < k$. By Lemma 2.6 and Lemma 2.9, we have that

$$\operatorname{depth}(S/(I, x_1)) = k$$
$$\operatorname{depth}(S/(I : x_1)) = 1 + \ell.$$

By Corollary 2.5, we deduce that $\operatorname{depth}(S/I) = 1 + \ell = g(\mathbf{a})$. ∎

Before studying the depth of powers of starlike trees, we introduce some more notation. Without loss of generality, we assume for now that $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Let $p_0 = 0$ and $p_i = a_1 + \cdots + a_i$ for $i = 1, \ldots, k$. We order the edges of $T_{\mathbf{a}}$ by going from the leaf of the first branch to the root, then from the leaf of the second branch to the root, and so on. In the formula, the order is

$$\begin{aligned} &\{a_1, a_1+1\} > \{a_1 - 1, a_1\} > \cdots > \{1, 2\} > \\ &\{p_2, p_2+1\} > \{p_2 - 1, p_2\} > \cdots > \{1, a_1+2\} > \cdots > \\ &\{p_k, p_k+1\} > \cdots > \{1, p_{k-1}+2\}. \end{aligned} \tag{4.1}$$

We label the edges in this order by $e_1,\dots,e_{n-1}$. For each $i=1,\dots,n-1$, let H_i and T_i be the graphs whose edge sets are $\{e_1,\dots,e_i\}$ and $\{e_i,\dots,e_{n-1}\}$, respectively. We also have that

$$
\begin{aligned}
N_{T_{\mathbf{a}}}(1) &= \{p_0+2, p_1+2,\dots,p_{k-1}+2\},\\
N_{T_{\mathbf{a}}}(p_i+1) &= \{p_i\}, \text{ for } i=1,\dots,k,\\
N_{T_{\mathbf{a}}}(p_i+2) &= \{1,p_i+3\}, \text{ for i } =0,\dots,k-1,\\
N_{T_{\mathbf{a}}}(u) &= \{u-1,u+1\}, \text{ if } p_i+2<u\le p_{i+1} \text{ for some } i\in\{0,\dots,k-1\}.
\end{aligned} \tag{4.2}
$$

First, we prove

Lemma 4.3. *With the above notations, for all $t\ge 2$, we have that*

$$\operatorname{depth}(S/I(T_{\mathbf{a}})^t) \ge \min_{i=0,\dots,n-2}\{\operatorname{depth}(S/(I(T_{i+1})^{t-1}+(I(H_i):x^{e_{i+1}})))\}.$$

Proof. For each $i=0,\dots,n-1$, let $L_i = I(T_{\mathbf{a}})^t + I(H_i)$, where H_0 is the empty graph. By Lemma 2.3, we have that

$$\operatorname{depth}(S/L_i) \ge \min\{\operatorname{depth}(S/L_{i+1}), \operatorname{depth}(S/(L_i : x^{e_{i+1}}))\}. \tag{4.3}$$

Since $L_0 = I(T_{\mathbf{a}})^t$, $L_{n-1} = I(T_{\mathbf{a}})$ and $(L_0 : x^{e_1}) = I(T_{\mathbf{a}})^{t-1}$, we have that

$$\operatorname{depth}(S/I(T_{\mathbf{a}})^t) \ge \min_{i=0,\dots,n-2}\{\operatorname{depth}(S/(L_i : x^{e_{i+1}}))\}.$$

Since $I(T_{\mathbf{a}}) = I(T_{i+1}) + I(H_i)$, by Lemma 3.2, we have that $L_i = I(T_{i+1})^t + I(H_i)$. By Lemma 2.10, Corollary 2.8, and the fact that e_{i+1} is a leaf of T_{i+1}, we have that

$$(L_i : x^{e_{i+1}}) = I(T_{i+1})^{t-1} + (I(H_i) : x^{e_{i+1}}). \tag{4.4}$$

The conclusion follows. ☐

Let $\mathbf{t},\mathbf{b}\in\mathbb{N}^k$ be vectors of natural numbers. We write $\mathbf{t}\ll\mathbf{b}$ if $t_i\le b_i$ for all i. We define

$$\Gamma(\mathbf{a},t) = \{\mathbf{t}\in\mathbb{N}^k \mid \mathbf{t}\ll\mathbf{a}-\mathbf{1} \text{ and } |\mathbf{t}| = t-1\}.$$

We now prove a lower bound for the depth of powers of the edge ideals of starlike trees.

Lemma 4.4. *Assume that $k\ge 2$. Let $\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{N}^k$ be a vector of positive integers. Assume that $2\le t<|\mathbf{a}-\mathbf{1}|$, then*

$$\operatorname{depth}(S/I(T_{\mathbf{a}})^t) \ge \min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t}\in\Gamma(\mathbf{a},t)\}.$$

Proof. We keep the notation as in the proof of Lemma 4.3, i.e., $L_i = I(T_{\mathbf{a}})^t + I(H_i)$. Furthermore, for $i=0,\dots,n-2$, we set $J_i = (L_i : x^{e_{i+1}}) = I(T_{i+1})^t + (I(H_i) : x^{e_{i+1}})$. We prove by induction on t and $|\mathbf{a}|$ that for each $i=0,\dots,n-2$, we have that

$$\operatorname{depth}(S/J_i) \ge \min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t}\in\Gamma(\mathbf{a},t)\}. \tag{4.5}$$

By Eq. (4.3), once we have Eq. (4.5), we also have

$$\operatorname{depth}(S/L_i) \ge \min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t}\in\Gamma(\mathbf{a},t)\}. \tag{4.6}$$

The base case $t=1$ is clear, as then $L_i = I(T_{\mathbf{a}})$ and $J_i = (1)$. By Eq. (4.1), there are four cases to consider.

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(1) e_{i+1} is a leaf of the ℓth branch for some $\ell \le k$ and this leaf also contains the root, i.e., $a_\ell = 1$.
(2) e_{i+1} is a leaf of the ℓth branch for some $\ell \le k$ and this leaf does not contain the root, i.e., $a_\ell > 1$.
(3) e_{i+1} is on the ℓth branch for some $\ell \le k$ and e_{i+1} is neither a leaf nor contains the root, i.e., $a_\ell > 2$.
(4) e_{i+1} is on the ℓth branch for some $\ell \le k$ and e_{i+1} is not a leaf but e_{i+1} contains the root, i.e., $a_\ell \ge 2$.

Case 1. $e_{i+1} = \{1, p_\ell + 2\}$ for some $\ell \le k$ and $a_\ell = 1$. By Lemma 2.9, we have that

$$J_i = I(T_{\ell+1})^{t-1} + (x_1, x_2, x_{a_1+2}, \ldots, x_{a_1+\cdots+a_\ell+2}) + K,$$

where K is the edge ideal of the induced graph of $T_{\mathbf{a}}$ on $\{1, \ldots, p_\ell + 2\} \setminus \{1, p_0 + 2, p_1 + 2, \ldots, p_\ell + 2\}$. In particular, K is the edge ideal of the disjoint union of paths of lengths $a_j - 2$ for $j = 1, \ldots, \ell - 1$. With our assumption, $a_j = 1$ for all $j \ge \ell$. Hence, by Lemma 2.6 and Lemma 3.3, we have that

$$\operatorname{depth}(S/J_i) = 1 + \sum_{j=1}^{\ell-1} \left\lceil \frac{a_j - 1}{3} \right\rceil = g(\mathbf{a}) \ge \min\{g(\mathbf{a} - \mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a}, t)\}. \tag{4.7}$$

Case 2. $e_{i+1} = \{p_\ell, p_\ell + 1\}$ for some $\ell \le k$ and $a_\ell > 1$. By Lemma 2.9, we have that

$$J_i = I(T_{\mathbf{a}})^{t-1} + I(H_\ell).$$

By induction on t, Eq. (4.6), and the fact that $g(\mathbf{a}) \le g(\mathbf{a}')$ if $\mathbf{a} \ll \mathbf{a}'$, the conclusion follows.

Case 3. $e_{i+1} = \{u, u+1\}$ for some u such that $p_{\ell-1} + 2 \le u < p_\ell$. In particular, $a_\ell > 2$. By Eq. (4.4) and Lemma 2.9, we have that

$$J_i = I(T_{\ell+1})^{t-1} + (x_{u+2}) + I(T_{\mathbf{c}}) + I(K_2)$$

where $\mathbf{c} = (a_1, \ldots, a_{\ell-1})$ and K_2 is the path from $u+3$ to $p_\ell + 1$. By Lemma 2.6 and Lemma 3.3, we have that

$$\operatorname{depth}(S/J_i) = \left\lceil \frac{p_\ell - u - 1}{3} \right\rceil + \operatorname{depth}\left(S'/(I(T_{\ell+1})^{t-1} + I(T_{\mathbf{c}}))\right), \tag{4.8}$$

where S' is the polynomial ring on $V(T_{\ell+1}) \cup V(T_{\mathbf{c}})$. Note that $I(T_{\ell+1})^{t-1} + I(T_{\mathbf{c}}) = I(G)^{t-1} + I(T_{\mathbf{c}})$ where G is the induced subgraph of $T_{\mathbf{a}}$ on $V(T_{\mathbf{a}}) \setminus \{u-1, u, \ldots, p_\ell + 1\}$ which is isomorphic to the starlike tree $T_{\mathbf{b}}$ with $\mathbf{b} = (a_1, \ldots, a_{\ell-1}, a_\ell - (p_\ell - u), a_{\ell+1}, \ldots, a_k)$. Since $I(G)^{t-1} + I(T_{\mathbf{c}})$ has the form $L_i = I(T_{\mathbf{a}})^{t-1} + I(H_i)$ with smaller exponent and smaller starlike tree, by induction on t there exists $\mathbf{t}_0 \in \mathbb{N}^k$ such that $\mathbf{t}_0 \ll \mathbf{b} - \mathbf{1}$ and $|\mathbf{t}_0| = t - 2$ and $\operatorname{depth}(S'/(I(T_{\ell+1})^{t-1}) + I(T_{\mathbf{c}})) \ge g(\mathbf{b} - \mathbf{t}_0)$. Together with Lemma 4.1, we deduce that

$$\operatorname{depth}(S/J_i) \ge \left\lceil \frac{p_\ell - u - 1}{3} \right\rceil + g(\mathbf{b} - \mathbf{t}_0) \ge g(\mathbf{a} - \mathbf{t}_1) \ge \min\{g(\mathbf{a} - \mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a}, t)\},$$

where $\mathbf{t}_1 = \mathbf{t}_0 + e_\ell$.

Case 4. $e_{i+1} = \{1, p_\ell + 2\}$ for some $\ell \le k$ and $a_\ell \ge 2$. By Lemma 2.9, we have that

$$J_i = T_{\ell+1}^{t-1} + (x_{p_0+2}, x_{p_1+2}, \ldots, x_{p_{\ell-1}+2}, x_{p_\ell+3}) + I(K),$$

where K is the union of paths of length a_j-2 for $j=1,\dots,\ell-1$ and a path of length $\max(a_\ell-3,0)$. By Lemma 2.6 and Lemma 3.3, we have that

$$\operatorname{depth}(S/J_i)=\sum_{j=1}^{\ell-1}\left\lceil\frac{a_j-1}{3}\right\rceil+\left\lceil\frac{a_\ell-2}{3}\right\rceil+\operatorname{depth}\left(S'/I(T_{\ell+1})^{t-1}\right). \tag{4.9}$$

where S' is the polynomial ring on the variables corresponding to $V(T_{\ell+1})$. Note that $T_{\ell+1}$ is isomorphic to the starlike tree $T_{\mathbf{a}'}$ with $\mathbf{a}'=(a_{\ell+1},\dots,a_k)\in N^{k-\ell}$. By induction on t applied to $T_{\mathbf{a}'}$, there exists $\mathbf{t}'\ll\mathbf{a}'-\mathbf{1}$ with $|\mathbf{t}'|=t-2$ such that $\operatorname{depth}(S'/T_{\ell+1}^{t-1})\ge g(\mathbf{a}'-\mathbf{t}')$. Let $\mathbf{t}_0=\mathbf{t}'+\mathbf{e}_\ell$. By Eq. (4.9) and Lemma 3.1, we deduce that

$$\operatorname{depth}(S/J_i)\ge g(\mathbf{a}-\mathbf{t}_0)\ge\min\{g(\mathbf{a}-\mathbf{t})\mid\mathbf{t}\in\Gamma(\mathbf{a},t)\}.$$

The conclusion follows. □

We now compute $\min\{g(\mathbf{a}-\mathbf{t})\mid\mathbf{t}\in\Gamma(\mathbf{a},t)\}$ in terms of $\mathbf{a}$ and t. We may assume that $a_j\equiv 2 \pmod 3$ for $j=1,\dots,\alpha_2$, $a_j\equiv 0\pmod 3$ for $j=\alpha_2+1,\dots,\alpha_2+\alpha_0$ and $a_j\equiv 1\pmod 3$ for $j=\alpha_0+\alpha_2+1,\dots,k$. First, we note some further properties of g.

Lemma 4.5. *Let $\mathbf{b}=(a_3,\dots,a_k)$. We have*

(1) $g(3k+1,3l+3,\mathbf{b})\le g(3k+2,3l+2,\mathbf{b})$.
(2) $g(3k+1,3l+1,\mathbf{b})\le g(3k,3l+2,\mathbf{b})$.
(3) $g(3k-2,3l+2,\mathbf{b})\le g(3k,3l,\mathbf{b})$.

Proof. These properties follow easily from the definition of g. We prove one of them for completeness. For (1), we have that

$$g(3k+1,3l+3,\mathbf{b})=1+k+l+1+\sum_{i=3}^{k}\left\lceil\frac{a_i-1}{3}\right\rceil,$$

$$g(3k+2,3l+2,\mathbf{b})=\varepsilon+k+1+l+1+\sum_{i=3}^{k}\left\lceil\frac{a_i-1}{3}\right\rceil,$$

where $\varepsilon=1$ if $a_j=1\pmod 3$ for some $j\ge 3$ and 0 otherwise. The conclusion follows. □

Lemma 4.6. *Assume that $\alpha_2(\mathbf{a})=0$. Then*

$$g(\mathbf{a}-\mathbf{t})=g(\mathbf{a}),$$

for all $\mathbf{t}\in\Gamma(\mathbf{a},2)$.

Proof. By the assumption, $a_j\equiv 0\pmod 3$ for $j=1,\dots,\alpha_0$ and $a_j\equiv 1\pmod 3$ for $j=\alpha_0+1,\dots,k$. In particular,

$$\left\lceil\frac{a_j-1}{3}\right\rceil=\left\lceil\frac{a_j-2}{3}\right\rceil=\left\lceil\frac{a_j-t_j-1}{3}\right\rceil,$$

for all $j=1,\dots,k$. Since $|\mathbf{t}|=1$, we have that $\mathbf{t}=\mathbf{e}_j$ for some $j=1,\dots,k$. Now $\alpha_2(\mathbf{a}-\mathbf{t})\ne 0$ and $\alpha_1(\mathbf{a}-\mathbf{t})=0$ if and only if $j=k=1$, which is a contradiction. Thus, we have

$$g(\mathbf{a}-\mathbf{t})=1+\sum_{i=1}^{k}\left\lceil\frac{a_i-t_i-1}{3}\right\rceil=g(\mathbf{a}).$$

The conclusion follows. □

Lemma 4.7. *Assume that* $\alpha_2(\mathbf{a}) = \alpha_0(\mathbf{a}) = 0$. *Then*

$$\min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a},t)\} = g(\mathbf{a}) - \left\lfloor \frac{t-1}{3} \right\rfloor.$$

Proof. By assumption, we have $a_j \equiv 1 \pmod 3$ for all $j = 1,\dots,k$. By Lemma 4.5, if there exists i, j such that $t_i, t_j \not\equiv 0 \pmod 3$ then we can choose an $\mathbf{u} \in \Gamma(\mathbf{a},t)$ such that $g(\mathbf{a}-\mathbf{u}) \le g(\mathbf{a}-\mathbf{t})$. Hence, there can be at most one j such that $t_j \not\equiv 0 \pmod 3$. By Lemma 4.1 and the fact that if $a_j \ge 4$ and $a_j \equiv 1 \pmod 3$ then

$$\left\lceil \frac{a_j-1}{3} \right\rceil = \left\lceil \frac{a_j-2}{3} \right\rceil = \left\lceil \frac{a_j-3}{3} \right\rceil,$$

the conclusion follows. □

We use the following notations in the next lemma. Let

$$\begin{aligned}
\beta_1 &= \min\{\alpha_2(\mathbf{a}), t-1\},\\
\beta_2 &= \min\left\{\alpha_0(\mathbf{a}), \left\lfloor \frac{\max\{t-1-\alpha_2(\mathbf{a}),0\}}{2} \right\rfloor\right\},\\
\beta_3 &= \left\lfloor \frac{\max\{t-1-\beta_1-2\beta_2,0\}}{3} \right\rfloor.
\end{aligned} \tag{4.10}$$

We then define $\mathbf{b} \in \mathbb{N}^k$ as follows.

$$b_i = \begin{cases} a_i - 1, & \text{for } i = 1,\dots,\beta_1,\\ a_i - 2, & \text{for } i = \alpha_2(\mathbf{a})+1,\dots,\alpha_2(\mathbf{a})+\beta_2,\\ a_i, & \text{otherwise.}\end{cases}$$

Lemma 4.8. *With the above notations, we have that*

$$\min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a},t)\} = g(\mathbf{b}) - \beta_3.$$

Proof. Let $\mathbf{u} \in \Gamma(\mathbf{a},t)$ be such that

$$g(\mathbf{a}-\mathbf{u}) = \min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a},t)\}.$$

We claim that we can choose such an $\mathbf{u}$ with $\alpha_1(\mathbf{a}-\mathbf{u}) = \beta_1 + \beta_2$.

Indeed, by Lemma 4.1, without loss of generality, assume that $u_1 + u_2 \ge 2$ and $a_1 - u_1$ and $a_2 - u_2$ are not congruent to 1 modulo 3. By Lemma 4.5, we can choose a $\mathbf{u}' \in \Gamma(\mathbf{a},t)$ with $\alpha_1(\mathbf{a}-\mathbf{u}') > \alpha_1(\mathbf{a}-\mathbf{u})$ and $g(\mathbf{a}-\mathbf{u}') \le g(a-\mathbf{u})$. By the choice of $\mathbf{u}$ we must have $g(\mathbf{a}-\mathbf{u}') = g(\mathbf{a}-\mathbf{u})$. Replacing $\mathbf{u}$ by $\mathbf{u}'$, we assume that $\gamma(\mathbf{u})$ is as largest as possible. By the definition of βs and Lemma 4.5, we deduce that $\alpha_1(\mathbf{a}-\mathbf{u}) = \beta_1 + \beta_2$.

It remains to prove that

$$g(\mathbf{a}-\mathbf{u}) = g(\mathbf{b}) - \beta_3. \tag{4.11}$$

There are three cases to consider as follows.

Case 1. $t-1 \le \alpha_2(\mathbf{a})$. Then $\beta_2 = \beta_3 = 0$ and $\mathbf{a}-\mathbf{u} = \mathbf{b}$. The equation (4.11) follows immediately.

Case 2. $t-1 > \alpha_2(\mathbf{a})$ and $\left\lfloor \frac{t-1-\alpha_2(\mathbf{a})}{2} \right\rfloor \le \alpha_0(\mathbf{a})$. Then $\beta_3 = 0$. If $t-1-\alpha_2(\mathbf{a}) \equiv 0 \pmod 2$ then $\mathbf{a}-\mathbf{u} = \mathbf{b}$ and the equation (4.11) holds immediately. Thus, we assume that $t-1-\alpha_2(\mathbf{a}) \equiv 1 \pmod 2$. Replacing $\mathbf{a}$ by $\mathbf{b}$ if necessary, we may assume that $\alpha_2(\mathbf{a}) = 0$ and $t = 2$. Eq. (4.11) follows from Lemma 4.6.

Case 3. $t-1 > \alpha_2(\mathbf{a})$ and $\left\lfloor \frac{t-1-\alpha_2(\mathbf{a})}{2} \right\rfloor \ge \alpha_0(\mathbf{a})$. Then we have $\beta_3 = \left\lfloor \frac{t-1-\beta_1-2\beta_2}{3} \right\rfloor$. Replacing $\mathbf{a}$ by $\mathbf{b}$ if necessary, we may assume that $\alpha_2(\mathbf{a}) = 0$ and $\alpha_0(\mathbf{a}) = 0$. Eq. (4.11) follows from Lemma 4.7. □

We are now ready to give an upper bound for the depth of powers of edge ideals of starlike trees.

Lemma 4.9. *Assume that $k \ge 2$. Let $\mathbf{a} = (a_1,\dots,a_k) \in \mathbb{N}^k$ be a vector of positive integers. Assume that $2 \le t < |\mathbf{a}-\mathbf{1}|$. Then*

$$\operatorname{depth}(S/I(T_{\mathbf{a}})^t) \le \min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a},t)\}.$$

Proof. By the proof of Lemma 4.8, we may choose an $\mathbf{u} \in \Gamma(\mathbf{a},t)$ such that

$$g(\mathbf{a}-\mathbf{u}) = \min\{g(\mathbf{a}-\mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a},t)\},$$

and $\alpha_1(\mathbf{a}-\mathbf{u}) = \beta_1+\beta_2$. In particular, if either $t \ge 3$ or $\alpha_1(\mathbf{a}) > 0$ then we can always choose an $\mathbf{u}$ such that $\alpha_1(\mathbf{a}-\mathbf{u}) > 0$. In these cases, we have

$$g(\mathbf{a}-\mathbf{u}) = 1 + \sum_{i=1}^{k} \left\lceil \frac{a_i-u_i-1}{3} \right\rceil.$$

We no longer assume that $a_1 \ge \cdots \ge a_k$. Instead, we assume that $u_i > 0$ for $i = 1,\dots,\ell$ and $u_i = 0$ for $i = \ell+1,\dots,k$. For each $i = 1,\dots,\ell$, we set

$$m_i = \begin{cases} (x_1 x_{p_{i-1}+2}) & \text{if } u_i = 1, \\ (x_1 x_{p_{i-1}+2})(x_{p_{i-1}+2} x_{p_{i-1}+3}) \cdots (x_{p_{i-1}+u_i} x_{p_{i-1}+u_i+1}) & \text{if } u_i > 1. \end{cases}$$

Let $m_{\mathbf{u}} = m_1 \cdots m_\ell$. Furthermore, we set

$$\begin{aligned} U_i &= \{p_{i-1}+2, p_{i-1}+4, \dots, p_{i-1}+2j \mid 2j \le u_i+2\} \\ V_i &= \{p_{i-1}+3, \dots, p_{i-1}+2j+1 \mid 2j+1 \le u_i+2\} \end{aligned} \tag{4.12}$$

By Lemma 2.14, we have that

$$(I(T_{\mathbf{a}})^t : m_{\mathbf{u}}) = I(G)$$

where G is the graph on $V(G) = [n]$ with edge set

$$E(G) = E(B_1) \cup E(H), \tag{4.13}$$

where $B_1 = K_{U,V}$ is a complete bipartite graph on

$$\begin{aligned} U &= U_1 \cup U_2 \cup \cdots \cup U_\ell \cup \{p_{\ell+1}+2,\dots,p_{k-1}+2\}, \\ V &= V_1 \cup V_2 \cup \cdots \cup V_\ell \cup \{1\}, \end{aligned} \tag{4.14}$$

and H is the induced subgraph of $T_{\mathbf{a}}$ on

$$([n] \setminus V(B_1)) \cup \{p_0+t_1+2, p_1+t_2+2, \dots, p_{\ell-1}+t_\ell+2, p_\ell+2, \dots, p_{k-1}+2\}.$$

By Lemma 2.16, we have that

$$\operatorname{depth}(S/I(G)) \leq 1 + \sum_{i=1}^{k} \left\lceil \frac{a_i - u_i - 1}{3} \right\rceil = g(\mathbf{a} - \mathbf{u}).$$

It remains to consider the case $t = 2$ and $\alpha_1(\mathbf{a}) = 0$. If $\alpha_0(\mathbf{a}) = 0$ then for any $\mathbf{t}$ with $|\mathbf{t}| = 1$, we have $\alpha_1(\mathbf{a} - \mathbf{t}) > 0$ and we can proceed as in the previous case. If $\alpha_2(\mathbf{a}) > 0$, then by the definition of g, we see that for any $\mathbf{t}$ with $|\mathbf{t}| = 1$, we have $g(\mathbf{a} - \mathbf{t}) = g(\mathbf{a})$ and the conclusion is clear. Thus, we may assume that $a_i \equiv 0 \pmod 3$ for all $i = 1, \ldots, k$. By Lemma 2.14, we have that

$$(I(T_{\mathbf{a}})^2 : (x_2 x_3)) = I(G),$$

where $E(G) = E(T_{\mathbf{a}}) \cup \{1,4\}$. Let B_0 be the induced subgraph of G on $\{1,2,3,4\}$, B_1 be the induced subgraph of G on $\{5, \ldots, a_1 + 1\}$ and B_j be the induced subgraph of G on $\{p_{j-1} + 2, \ldots, p_j + 1\}$ for $j = 2, \ldots, k$. Then B_1 is isomorphic to $P_{a_1 - 3}$ and $B_j \cong P_{a_j}$ for $j = 2, \ldots, k$. By Lemma 2.16, we have that

$$\operatorname{depth}(S/I(G)) \leq 1 + \left\lceil \frac{a_1 - 3}{3} \right\rceil + \sum_{j=2}^{k} \left\lceil \frac{a_j}{3} \right\rceil = \sum_{j=1}^{k} \left\lceil \frac{a_j - 1}{3} \right\rceil = g(\mathbf{a}) - 1.$$

The conclusion follows. □

We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.4 and Lemma 4.9, we have that

$$\operatorname{depth}(S/I(T_{\mathbf{a}})^t) = \min\{g(\mathbf{a} - \mathbf{t}) \mid \mathbf{t} \in \Gamma(\mathbf{a}, t)\}.$$

The conclusion then follows from Lemma 4.8. □

Example 4.10. Let $\mathbf{a} = (3,4,5)$. Then $\alpha_0 = 1$, $\alpha_1 = 1$ and $\alpha_2 = 1$. By Theorem 1.2, we see that the sequence $\{\operatorname{depth}(S/I(T_{\mathbf{a}})^t) \mid 1 \leq t \leq 10\}$ is $\{5,4,4,3,3,3,2,2,2,1\}$.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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