On Positive Definiteness of some Linear Functionals. An Inverse Problem.

Wathek Chammam, [a,b,d,*] and Mongia Khlifi [b,c]

[a] *Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia*

[b] *Research Laboratory Mathematics and Applications LR17ES11, Gabes University, Erriadh City, 6072 Zrig, Gabes, Tunisia*

[c] *Department of Mathematics,Faculty of Sciences of Sfax, Sfax University, Sfax Tunisia*

[d] *University of Carthage, National Institute of Applied Sciences and Technology, Tunisia*

Abstract

In this paper, we analyze a positivity problem of a linear functional $\mathscr{U}$ in the space $\mathbb{P}$ of polynomials in one variable with complex coefficients. Some new results connection relations for generalized Hermite polynomials and generalized Gegenbauer polynomials are established.

Key words: Positive-definite linear functional; Inverse Problem; generalized Hermite polynomials; generalized Gegenbauer polynomials; Orthogonal polynomials
MSC: 33C45; 42C05

1 Introduction

The study of the orthogonal polynomials is important in various mathematical areas, such as approximation theory, interpolation, probability, special functions, because the solution of a large number of problems depends on whether a characterization and connections between orthogonal polynomials. For example, study classical orthogonal polynomials via a linear differential operators

* Corresponding author.
Email address: w.chammam@mu.edu.sa (Wathek Chammam,).

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and results on the relations between classical polynomials have been studied in Aloui and his co-authors [1, 2, 3, 4, 5, 6]. Moreover, reference should be made to the work of Atia [7, 8] on the propriety and relations of semi-classical orthogonal polynomials.
A linear functional $\mathscr{U}$ is said to be positive-definite if and only if $\langle \mathscr{U}, p^2\rangle > 0$, for every non-zero real polynomial p (see [18]). To construct positive-definite linear functionals from positive-definite linear data, Christoffel proved that the product of a positive-definite linear functional and a positive polynomial yields a positive-definite linear form [14, 27].
The paper is organized as follows. In Section 2, the basic background and notations are introduced. In Section 3, the main results are presented. In particular, it is proved that for any pair $(\mu, c) \in \mathbb{R}_+^* \times \mathbb{R}$ and any positive-definite linear functional $\mu\mathscr{U}$, the linear functional $\mathscr{U}(\mu, c)$ given by $(\mu-1)\mathscr{U}(\mu, c) - (x-c)\Big(\mathscr{U}(\mu, c)\Big)' = \mu\mathscr{U}$ is also positive-definite. Finally, new results on the relations between the corresponding sequences of monic generalized Hermite polynomials and generalized Gegenbauer polynomials are presented.

2 Orthogonality and positive-definiteness of linear functionals

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its algebraic dual space. $\langle \mathscr{U}, p\rangle$ denotes the action of $\mathscr{U} \in \mathbb{P}'$ on $p \in \mathbb{P}$, and $(\mathscr{U})_n := \langle \mathscr{U}, x^n\rangle$, $n \geq 0$, the sequence of moments of $\mathscr{U}$ with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$. Let the following operations in $\mathbb{P}'$ be defined. For any linear functionals $\mathscr{U}$ and $\mathscr{V}$, any polynomial q, and any $(a, b, c) \in \mathbb{C}^* \times \mathbb{C}^2$, let $D\mathscr{U} = \mathscr{U}'$, $q\mathscr{U}$, $(x-c)^{-1}\mathscr{U}$, $\tau_{-b}\mathscr{U}$, $h_a\mathscr{U}$, and $\mathscr{U}\mathscr{V}$ be the linear functionals defined by duality as follows [18]:

$$\begin{aligned}
\langle \mathscr{U}', p\rangle &:= -\langle \mathscr{U}, p'\rangle,\\
\langle q\mathscr{U}, p\rangle &:= \ \langle \mathscr{U}, qp\rangle,\\
\langle (x-c)^{-1}\mathscr{U},\ p\rangle &:= \langle \mathscr{U},\ \theta_c p\rangle = \Big\langle \mathscr{U},\ \frac{p(x)-p(c)}{x-c}\Big\rangle,\\
\langle \tau_{-b}\mathscr{U},\ p\rangle &:= \ \langle \mathscr{U},\ \tau_b p\rangle = \ \langle \mathscr{U},\ p(x-b)\rangle,\\
\langle h_a\mathscr{U},\ p\rangle &:= \langle \mathscr{U},\ h_a p\rangle = \ \langle \mathscr{U},\ p(ax)\rangle,\\
\langle \mathscr{U}\mathscr{V}, p\rangle &:= \langle \mathscr{U}, \mathscr{V}p\rangle,\ p \in \mathbb{P},
\end{aligned}$$

where the right-multiplication of $\mathscr{V}$ by p yields a polynomial given by

$$(\mathscr{V}p)(x) := \langle \mathscr{V}_y, \frac{xp(x)-yp(y)}{x-y}\rangle, \quad p \in \mathbb{P}.$$

For any $c \in \mathbb{C}$, $(x-c)^{-1}(x-c)\mathscr{U} = \mathscr{U} - (\mathscr{U})_0\delta_c$, where δ_c is the Dirac mass defined by $\langle \delta_c, p\rangle = \ p(c),\ p \in \mathbb{P}$.
It should be noticed that if $\mathscr{U} \in \mathbb{P}'$ is such that $\mathscr{U}' = 0$, then $\mathscr{U} = 0$.

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A linear functional $\mathscr{U}$ is said to be quasi-definite (regular) if a monic polynomial sequence (MPS) $\{B_n\}_{n\geq 0}$ with $\deg B_n = n,\ n \geq 0$, can be associated with it, so that

$$\langle \mathscr{U}, B_n B_m \rangle = r_n \delta_{n,m},\ m,\ n \geq 0,\ \text{and where } r_n \neq 0,\ n \geq 0, \tag{2.1}$$

where $\delta_{n,m}$ is the Kronecker delta.
In such a situation, $\{B_n\}_{n\geq 0}$ is said to be the monic orthogonal polynomial sequence (MOPS) with respect to $\mathscr{U}$. The MOPS $\{B_n\}_{n\geq 0}$ satisfies the following three-term recurrence relation (TTRR):

$$\begin{cases} B_{-1}(x) & = 0, \quad B_0(x) = 1, \\ B_{n+1}(x) & = (x - \beta_n) B_n(x) - \gamma_n B_{n-1}(x),\ n \geq 0, \end{cases} \tag{2.2}$$

where $\beta_n \in \mathbb{C}$, $\gamma_n \in \mathbb{C}^*$ for every integer $n \geq 0$, and $\gamma_0 = (\mathscr{U})_0$. Furthermore, the following formula will be used:

$$\beta_n = \frac{\langle \mathscr{U}, x B_n^2 \rangle}{\langle \mathscr{U}, B_n^2 \rangle},\ n \geq 0, \quad \gamma_{n+1} = \frac{\langle \mathscr{U}, B_{n+1}^2 \rangle}{\langle \mathscr{U}, B_n^2 \rangle},\ n \geq 0, \tag{2.3}$$

$$\langle \mathscr{U}, B_n^2 \rangle = \prod_{\nu=0}^{n} \gamma_\nu,\ n \geq 0, \qquad (\gamma_0 = (\mathscr{U})_0). \tag{2.4}$$

When $\mathscr{U}$ is quasi-definite and $A \in \mathbb{P}$ is such that $A\mathscr{U} = 0$, then $A = 0$.
A linear functional $\mathscr{U}$ is said to be positive-definite if it is quasi-definite, i.e., it satisfies (2.1) and is such that $r_n > 0$ for every integer $n \geq 0$.
The quasi-definiteness of a linear functional $\mathscr{U}$ is equivalent to (see [13, 22])

$$\Delta_n(\mathscr{U}) = \begin{vmatrix} (\mathscr{U})_0 & (\mathscr{U})_1 & (\mathscr{U})_2 & \dots & (\mathscr{U})_n \\ (\mathscr{U})_1 & (\mathscr{U})_2 & (\mathscr{U})_3 & \dots & (\mathscr{U})_{n+1} \\ (\mathscr{U})_2 & (\mathscr{U})_3 & (\mathscr{U})_4 & \dots & (\mathscr{U})_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ (\mathscr{U})_n & (\mathscr{U})_{n+1} & (\mathscr{U})_{n+2} & \dots & (\mathscr{U})_{2n} \end{vmatrix} \neq 0, \tag{2.5}$$

where $\Delta_n(\mathscr{U})$ is the Hankel determinant of order n of $\mathscr{U}$.
We thereby have the following formula [13]:

$$\langle \mathscr{U}, B_n^2 \rangle = \frac{\Delta_n(\mathscr{U})}{\Delta_{n-1}(\mathscr{U})},\ n \geq 0, \quad (\Delta_{-1}(\mathscr{U}) = 1). \tag{2.6}$$

By using the first definition of positive-definite linear functional already mentioned in the introduction and by using the last formula we find in an equivalent, $\mathscr{U}$ is positive-definite if and only if $\Delta_n(\mathscr{U}) > 0$ for every integer $n \geq 0$. Also, we can characterize the criterion of positive-definite linear functional by this following definition:

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Definition 2.1 *[13] A linear functional $\mathscr{U}$ is said to be positive-definite if $\langle \mathscr{U}, p\rangle > 0$, for every real polynomial p that is not identically zero and is non-negative.*

Lemma 1 *[Christoffel] Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional $\mathscr{U}$ and let A be a monic polynomial with $\deg A = t$. The following statements are equivalent:*

(i) *The linear functional $A\mathscr{U}$ is quasi-definite.*
(ii) *There exists a MPS $\{Q_n\}_{n\geq 0}$ satisfying*

$$AQ_n(x) = \sum_{\nu=n}^{n+t} \lambda_{n,\nu} B_\nu(x), \quad n \geq 0,$$
$$\lambda_{n,n} \neq 0, \; n \geq 0.$$

In this case, $\{Q_n\}_{n\geq 0}$ is the MOPS with respect to $A\mathscr{U}$.

Remark 2.2 *The last Lemma expressions is known in the literature of orthogonal polynomials as Christoffel's formula [14, 27] and indicates a finite-type relations between two monic polynomial sequences, for more details for the readers we can cite the work of Maroni [18, pp 21],[19, pp 302-303] and Petronilho [20], also we can cite the work of Marcellán and Sfaxi [16] and recently in the paper of Salah [23].*

3 Main results

For any $(\mu, c) \in \mathbb{C}^2$ with $\mu \neq -n, \; n \geq 0$, and $\mathscr{U} \in \mathbb{P}'$, let $\mathscr{U}(\mu, c)$ be the linear functional whose moments with respect to the MPS $\{(x-c)^n\}_{n\geq 0}$, which are denoted by $\Big(\mathscr{U}(\mu,c)\Big)_{n,\,c}$, $n \geq 0$, are given by

$$\Big(\mathscr{U}(\mu,c)\Big)_{n,c} = \frac{\mu}{n+\mu}(\mathscr{U})_{n,\,c}, \; n \geq 0. \tag{3.1}$$

Lemma 2 *For any $(\mu, c) \in \mathbb{C}^2$ with $\mu \neq -n, \; n \geq 0$, and $\mathscr{U} \in \mathbb{P}'$, the linear functional $\mathscr{U}(\mu, c)$ satisfies the distributional equation*

$$(\mu - 1)\mathscr{U}(\mu, c) - (x-c)\Big(\mathscr{U}(\mu,c)\Big)' = \mu\mathscr{U}. \tag{3.2}$$

Proof. For any $(\mu, c) \in \mathbb{C}^2$ with $\mu \neq -n, \; n \geq 0$, we have

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$$
\begin{aligned}
(\mu-1)\langle \mathscr{U}(\mu,c),(x-c)^n\rangle - \langle (x-c)\Big(\mathscr{U}(\mu,c)\Big)',(x-c)^n\rangle &= \mu\langle \mathscr{U},(x-c)^n\rangle\\
(\mu-1)\langle \mathscr{U}(\mu,c),(x-c)^n\rangle - \langle \Big(\mathscr{U}(\mu,c)\Big)',(x-c)^{n+1}\rangle &= \mu\langle \mathscr{U},(x-c)^n\rangle\\
(\mu-1)\langle \mathscr{U}(\mu,c),(x-c)^n\rangle + (n+1)\langle \mathscr{U}(\mu,c),(x-c)^n\rangle &= \mu\langle \mathscr{U},(x-c)^n\rangle\\
(n+\mu)\langle \mathscr{U}(\mu,c),(x-c)^n\rangle &= \mu\langle \mathscr{U},(x-c)^n\rangle\\
(n+\mu)\Big(\mathscr{U}(\mu,c)\Big)_{n,\,c} &= \mu(\mathscr{U})_{n,\,c}
\end{aligned}
$$

■

The following operational formulas are immediate.

Lemma 3 *For any integer $m \geq 0$, any $(\mu,a,b,c,\alpha,\beta) \in \mathbb{C}^6$, $a \neq 0$, $\mu \neq -n$, $n \geq 0$, and any $\mathscr{U} \in \mathbb{P}'$, the following formulas hold:*

$$\Big(\alpha\mathscr{U}+\beta\mathscr{V}\Big)(\mu,c) = \alpha\mathscr{U}(\mu,c)+\beta\mathscr{V}(\mu,c), \tag{3.3}$$

$$\Big(\tau_b \circ h_a\mathscr{U}\Big)(\mu,b+ac) = \tau_b \circ h_a\Big(\mathscr{U}(\mu,c)\Big), \tag{3.4}$$

$$\Big((x-c)^m\mathscr{U}\Big)(\mu+m,c) = \frac{\mu+m}{\mu}(x-c)^m\Big(\mathscr{U}(\mu,c)\Big), \tag{3.5}$$

$$\delta_c^{(m)}(\mu,c) = \frac{\mu}{\mu+m}\delta_c^{(m)}, \tag{3.6}$$

$$\Big(\mathscr{U}+\sum_{\nu=0}^{m}\lambda_\nu\delta_c^{(\nu)}\Big)(\mu,c) = \mathscr{U}(\mu,c)+\sum_{\nu=0}^{m}\frac{\mu}{\mu+\nu}\lambda_\nu\delta_c^{(\nu)}. \tag{3.7}$$

The analysis of the quasi-definiteness of $\mathscr{U}(\mu,c)$ is very difficult in the general case. But when $\mathscr{U}$ is positive-definite, we can prove that $\mathscr{U}(\mu,c)$ is also positive-definite. It is the aim of this section.

3.1 Preservation of positive-definiteness

Assuming that $\mathscr{U}$ is positive-definite, a necessary and sufficient condition on the real parameter μ will be obtained whereby the linear functional $\mathscr{U}(\mu,c)$ is also positive-definite.

To this end, the following Lemma is required.

Lemma 4 *Let μ be a real number with $\mu \neq -n$, $n \geq 0$, and let $c \in \mathbb{R}$. The following statements are equivalent:*

(i) $\mu \in]0,+\infty[$.

(ii) *For every real non-negative polynomial $p \geq 0$, the first-order differential equation $\mu y(x)+(x-c)y'(x) = (\mu+\deg p)p(x)$ has a unique polynomial*

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solution $y = q(.;\mu, c) \in \mathbb{P}$ *such that* $q(x; c, \mu) \geq 0$ *for all* $x \in \mathbb{R}$ *and* $\deg q(.; c, \mu) = \deg p$.

Proof. It is first assumed that (i) holds. By the uniqueness theorem and direct computation,

$$y(x) = \sum_{\nu=0}^{\deg p} \Big(\frac{\mu + \deg p}{\mu + \nu}\Big) \frac{p^{(\nu)}(c)}{\nu!}(x - c)^{\nu} \ \sim p(x),\ x \to \pm\infty, \tag{3.8}$$

is the expression of the polynomial solution $q(.; c, \mu)$. Let α be an absolute minimum of $y = q(.; c, \mu)$ on $\mathbb{R}$, which clearly exists because $y(x) = q(x; c, \mu)$ tends to infinity as $x \to \pm\infty$. It should be noticed that $q'(\alpha; c, \mu) = 0$, as α is a critical point of $q(.; c, \mu)$. Accordingly, $\mu q(\alpha; c, \mu) = (\mu + \deg p)p(\alpha) \geq 0$, as $p(\alpha) \geq 0$ and $\mu > 0$. Thus, $q(x; c, \mu) \geq q(\alpha; c, \mu) \geq 0$ for all $x \in \mathbb{R}$.
Hence, (ii) holds.

It is now assumed that (ii) holds. For any integer $n \geq 1$, let the real non-negative polynomial $p_n(x) = (x - c)^{2n} + 1$ be considered. By the assumption (ii) with $p = p_n$, the first-order differential equation $\mu y(x) + (x - c)y'(x) = (\mu + \deg p)p_n(x)$ has a unique polynomial solution $y = q_n(.; c, \mu)$ such that $q_n(x; c, \mu) \geq 0$ for all $x \in \mathbb{R}$ and $\deg q_n(.; c, \mu) = \deg p_n$. However, it follows from (3.8) that $q_n(x; c, \mu) = (x - c)^{2n} + \frac{\mu+2n}{\mu}$ for all $x \in \mathbb{R}$. In particular, we must have $q_n(c; c, \mu) \geq 0$ for all $n \geq 1$, i.e., $\frac{\mu+2n}{\mu} \geq 0$ for all $n \geq 1$. This yields (i). ■

The following result can now be established.

Theorem 5 *Let* μ *be a real number with* $\mu \neq -n,\ n \geq 0$, *let* $c \in \mathbb{R}$, *and let* $\mathscr{U}$ *be a positive-definite linear functional. The following statements are equivalent:*

(i) $\mu \in]0, +\infty[$.
(ii) *The unique linear functional* $\mathscr{U}(\mu, c)$ *satisfying*

$$(\mu - 1)\mathscr{U}(\mu, c) - (x - c)\Big(\mathscr{U}(\mu, c)\Big)' = \mu\mathscr{U} \quad \textit{is positive-definite.}$$

Proof. It is first assumed that (i) holds. Let p be a real non-negative polynomial $p \geq 0$. By the previous Lemma, the unique solution $q(.; c, \mu)$ of the first-order differential equation $\mu y(x) + (x - c)y'(x) = (\mu + \deg p)p(x)$ is a real non-negative polynomial $q(.; c, \mu) \geq 0$. As $\mathscr{U}$ is positive-definite, it holds that

$$\begin{aligned}\langle \mathscr{U}(\mu, c), p\rangle &= (\mu + \deg p)^{-1}\langle \mathscr{U}(\mu, c), \mu q(x; c, \mu) + (x - c)q'(x; c, \mu)\rangle \\ &= (\mu + \deg p)^{-1}\langle(\mu - 1)\mathscr{U}(\mu, c) - (x - c)\Big(\mathscr{U}(\mu, c)\Big)', q(x; c, \mu)\rangle \\ &= \mu(\mu + \deg p)^{-1}\langle \mathscr{U}, q(x; c, \mu)\rangle > 0.\end{aligned}$$

It is now assumed that (ii) holds. From (3.1), it follows that $\Big(\mathscr{U}(\mu, c)\Big)_{2n,\, c} =$

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$\frac{\mu}{2n+\mu}(\mathscr{U})_{2n,\,c}, n \geq 1$. As $\mathscr{U}(\mu,c)$ and $\mathscr{U}$ are positive-definite, $\Big(\mathscr{U}(\mu,c)\Big)_{2n,\,c} > 0$ and $(\mathscr{U})_{2n,\,c} > 0$. Therefore, $\frac{\mu}{2n+\mu} > 0$ for all $n \geq 1$. This implies (i). ■

Remark 3.1 *The case where $\mu = 1$ and $c \in \mathbb{R}$, is totally study in [24].*

Corollary 3.2 *Let $c \in \mathbb{R}$, and let $\mathscr{U}$ be a positive-definite linear functional. For any integer $m \geq 1$, any $\lambda \geq 0$, any positive real numbers $\alpha_i,\ i = 1, ..., m$, and $\mu_i,\ i = 1, ..., m$, the following hold:*

(i) *The linear functional $\mathscr{R}$ given by the following expressions of its moments with respect to the polynomial sequence $\{(x-c)^n\}_{n\geq 0}$*

$$(\mathscr{R})_{n,\,c} = \Big(\lambda + \sum_{\nu=1}^{m} \alpha_\nu \frac{\mu_\nu}{n+\mu_\nu}\Big)(\mathscr{U})_{n,\,c},\ n \geq 0, \textit{is positive-definite.}$$

(ii) *The linear functional $\mathscr{S}$ given by the following expressions of its moments with respect to the polynomial sequence $\{(x-c)^n\}_{n\geq 0}$*

$$(\mathscr{S})_{n,\,c} = \Big(\lambda + \prod_{\nu=1}^{m} \frac{\mu_\nu}{n+\mu_\nu}\Big)(\mathscr{U})_{n,\,c},\ n \geq 0, \textit{is positive-definite.}$$

Proof. (i) Just note that the finite sum of the positive-definite linear functional is always positive-definite and the multiplication of a positive-definite linear functional by a positive real number is also always positive-definite. Now the proof is immediate by using Theorem 5.
(ii) The proof is similar to the last proof, using Theorem 5 and applying the same process as the relation (3.1). ■

3.2 ***Semiclassical case***

A quasi-definite linear functional $\mathscr{W}$ is said to be *semiclassical* if it satisfies the following functional equation (Pearson equation):

$$(\phi\mathscr{W})' + \psi\mathscr{W} = 0, \tag{3.9}$$

where ϕ and ψ are polynomials such that ϕ is monic and $\deg(\psi) \geq 1$.

The corresponding MOPS $\{B_n\}_{n\geq 0}$ is said to be *semiclassical* (for more details, see [18] and the references therein).

If for each zero c of ϕ, we have

$$\mid \phi'(c) + \psi(c) \mid + \mid \langle \mathscr{W}, \theta_c^2\phi + \theta_c\psi \rangle \mid > 0, \tag{3.10}$$

then the nonnegative integer $s := \max\{\deg(\phi) - 2, \deg(\psi) - 1\}$ is said to be either the *class* of $\mathscr{W}$ or the class of $\{B_n\}_{n\geq 0}$.

The family of semiclassical linear functionals is invariant under shifting. Indeed, if $\mathscr{W}$ is a semiclassical linear functional of class s satisfying (3.9) and (3.10), then for any pair $(a,b)\in\mathbb{C}^2$ with $a\neq 0$, the shifted linear functional $\tilde{\mathscr{W}}=(h_{a^{-1}}\circ\tau_{-b})\mathscr{W}$ is also semiclassical of class s and satisfies $(\tilde{\phi}\tilde{\mathscr{W}})'+\tilde{\psi}\tilde{\mathscr{W}}=0$ with $\tilde{\phi}(x)=a^{-t}\phi(ax+b)$, $\tilde{\psi}(x)=a^{1-t}\psi(ax+b)$, where $t=\deg\phi$. A description of the class $s=1$ may be found in [7, 9, 10, 11, 12, 17, 25]. This corresponds to semiclassical linear functionals (*generalized Hermite* and *symmetric generalized Gegenbauer* functionals).
In Rainville's book [21], we find properties and connection relations of classical orthogonal polynomials (Hermite, Laguerre, Bessel, Jacobi), in this part we will establish some new results connection relations for generalized Hermite polynomials and symmetric generalized Gegenbauer polynomials.

Proposition 3.3 *If $\mathscr{W}$ is a semiclassical linear functional of class s ($s\in\mathbb{N}$) with $\deg\phi=1$, then*

$$\Big(\mu^{-1}(\psi+\mu)\mathscr{W}\Big)(\mu,c)=\mathscr{W}. \tag{3.11}$$

Proof. As $\deg\phi=1$, we can write $\phi(x)=x-c$. Let $\mu\in\mathbb{C}$ with $\mu\neq -n,\ n\in\mathbb{N}$. From (3.9), it follows that

$$-(x-c)\mathscr{W}'+(\mu-1)\mathscr{W}=(\psi+\mu)\mathscr{W}. \tag{3.12}$$

Clearly, we have $\Big((\psi+\mu)\mathscr{W}\Big)_{0,c}=\mu(\mathscr{W}\Big)_{0,c}=\mu\neq 0$. Indeed,

$$\begin{aligned}\Big(\mu^{-1}(\psi+\mu)\mathscr{W}\Big)_{n,c}&=\mu^{-1}\Big(-(x-c)\mathscr{W}'+(\mu-1)\mathscr{W}\Big)_{n,c}\\&=\mu^{-1}\Big[(\mu-1)\Big(\mathscr{W}\Big)_{n,c}-\Big(\mathscr{W}'\Big)_{n+1,c}\Big]\\&=\mu^{-1}\Big[(\mu-1)\Big(\mathscr{W}\Big)_{n,c}+(n+1)\Big(\mathscr{W}\Big)_{n,c}\Big]\\&=\frac{n+\mu}{\mu}\Big(\mathscr{W}\Big)_{n,c}.\end{aligned}$$

Taking into account the equation (3.1) we obtain

$$\Big(\mu^{-1}(\psi+\mu)\mathscr{W}\Big)(\mu,c)=\mathscr{W}.$$

■

A particular case will now be studied.
Namely, $c=0$, $\mu=2\zeta+1$, $\psi(x)=2x^2-2\zeta-1$, and $\mathscr{W}=\mathcal{H}^\zeta$, i.e., the generalized Hermite linear functional with parameter $\zeta\neq 0$ and $\zeta\neq-\frac{2n+1}{2}$, $n\geq 0$, which is the unique linear functional satisfying (see [18])

$$(x\mathcal{H}^\zeta)'+(2x^2-2\zeta-1)\mathcal{H}^\zeta=0,\quad (\mathcal{H}^\zeta)_0=1,\ (\mathcal{H}^\zeta)_1=0.$$

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from which he derived an integral representation and the moments

$$\langle \mathcal{H}^\zeta, p\rangle = \frac{1}{\Gamma(\zeta+\frac{1}{2})}\int_{-\infty}^{+\infty} |x|^{2\zeta}\exp(-x^2)p(x)dx, p\in\mathbb{P}, \Re(\zeta) > \frac{1}{2}.$$

$$(\mathcal{H}^\zeta)_{2n} = \frac{1}{2^{2n}}\frac{\Gamma(\zeta+1)\Gamma(2n+2\zeta+1)}{\Gamma(2\zeta+1)\Gamma(n+\zeta+1)}; (\mathcal{H}^\zeta)_{2n+1} = 0,\ n\geq 0.$$

It is clear from the last identity we have $2(2\zeta+1)^{-1}x^2\mathcal{H}^\zeta = \mathcal{H}^{\zeta+1}$, we obtain that $2\zeta\mathcal{H}^\zeta - x(\mathcal{H}^\zeta)' = (2\zeta+1)\mathcal{H}^{\zeta+1}$; thus,

$$\mathcal{H}^{\zeta+1}(2\zeta+1,0) = \mathcal{H}^\zeta. \tag{3.13}$$

The MOPS $\{\hat{H}_n^\zeta(x)\}_{n\geq 0}$ of the generalized Hermite functional was introduced by G. Szegö (see [13, 25]) and satisfies the three-term recurrence relation (TTRR) [11, 13]

$$\begin{cases} \hat{H}_{n+1}^\zeta(x) = x\hat{H}_n^\zeta(x) - \gamma_n(\zeta)\hat{H}_{n-1}^\zeta(x),\ n\geq 0, \\ \hat{H}_0^\zeta(x) = 1,\ \hat{H}_{-1}^\zeta(x) = 0, \end{cases} \tag{3.14}$$

with $\gamma_n(\zeta) = (n-2\zeta\eta_n)/2,\ n\geq 0$, where $\eta_n = \big((-1)^n - 1\big)/2,\ n\geq 0$.
Other characterizations may be found in [17], Table 5.
The first structure relation is

$$x(\hat{H}_n^\zeta)'(x) = 2\zeta\eta_n\hat{H}_n^\zeta(x) + 2\gamma_n(\zeta)x\hat{H}_{n-1}^\zeta(x),\ n\geq 0, \tag{3.15}$$

and the differential equation is

$$x^2(\hat{H}_n^\zeta)''(x) + 2x(-x^2+\zeta)(\hat{H}_n^\zeta)'(x) = -2(nx^2+\zeta\eta_n)\hat{H}_n^\zeta(x),\ n\geq 0. \tag{3.16}$$

Corollary 3.4 *The polynomials $\hat{H}_n^\zeta(x)$ and $\hat{H}_n^{\zeta+1}(x)$ satisfy the following relation:*

$$(\hat{H}_n^\zeta)''(x) - 2x(\hat{H}_n^\zeta)'(x) + 2(n+2\zeta)\hat{H}_n^\zeta(x) = 4\zeta\hat{H}_n^{\zeta+1}(x),\ n\geq 0, \tag{3.17}$$

where $\zeta\neq 0$ and $\zeta\neq -\frac{2n+1}{2},\ n\geq 0$.

Proof. Using (3.15) and (3.14), we have

$$\begin{aligned} x(\hat{H}_n^\zeta)'(x) &= 2(x^2+\zeta\eta_n)\hat{H}_n^\zeta(x) - 2x\hat{H}_{n+1}^\zeta(x) \\ &= 2(x^2+\zeta\eta_n)\hat{H}_n^\zeta(x) - 2\hat{H}_{n+2}^\zeta(x) - 2\gamma_{n+1}(\zeta)\hat{H}_n^\zeta(x) \\ &= \Big(2x^2+2\zeta\eta_n - 2\gamma_{n+1}(\zeta)\Big)\hat{H}_n^\zeta(x) - 2\hat{H}_{n+2}^\zeta(x) \\ &= \Big(2x^2 - n - 1 - 2\zeta\Big)\hat{H}_n^\zeta(x) - 2\hat{H}_{n+2}^\zeta(x),\ n\geq 0. \end{aligned} \tag{3.18}$$

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For any integer $n \geq 0$, (3.18) and (3.16) imply that

$$\begin{aligned} x^2\Big[(\hat{H}_n^\zeta)''(x) - 2x(\hat{H}_n^\zeta)'(x) + 2n\hat{H}_n^\zeta(x)\Big] &= -2\zeta\Big(x(\hat{H}_n^\zeta)'(x) + \eta_n\hat{H}_n^\zeta(x)\Big) \\ &= 2\zeta(-2x^2 + n + 1 + 2\zeta - \eta_n)\hat{H}_n^\zeta(x) \\ &\quad + 4\zeta\hat{H}_{n+2}^\zeta(x). \end{aligned}$$

Hence,

$$x^2Q_n(x) = \hat{H}_{n+2}^\zeta(x) + \frac{(n+1+2\zeta-\eta_n)}{2}\hat{H}_n^\zeta(x),\ n \geq 0,$$

where $Q_n(x) = (4\zeta)^{-1}\Big[(\hat{H}_n^\zeta)''(x) - 2x(\hat{H}_n^\zeta)'(x) + 2(n+2\zeta)\hat{H}_n^\zeta(x)\Big],\ n \geq 0.$
By Lemma 1, with $A(x) = x^2$, $B_n = \hat{H}_n^\zeta$, and $\lambda_{n,n} = (n+1+2\zeta-\eta_n)/2 \neq 0$, $n \geq 0$, and by the fact that $2(2\zeta+1)^{-1}x^2\mathcal{H}^\zeta = \mathcal{H}^{\zeta+1}$, it can be concluded that $\{Q_n\}_{n\geq 0}$ is a MOPS with respect the linear functional $\mathcal{H}^{\zeta+1}$, and thus $Q_n(x) = \hat{H}_n^{\zeta+1}(x)$, $n \geq 0$. This yields (3.17). ■

The same result was proven by Atia and his co-author ([7]), to establish beautiful proof used successfully at giving a second-order spectral vectorial differential equation (SVDE) and apply it to generalized Hermite polynomial.

Proposition 3.5 *If $\mathscr{W}$ is a semiclassical linear functional of class s ($s \in \mathbb{N}$) with* $\deg\phi \geq 2$, *then*

$$(\mu^{-1}\rho\mathscr{W})(\mu, c) = \theta_c(\phi)\mathscr{W}. \tag{3.19}$$

where $\rho \in \mathbb{P}$ with $\deg\rho \leq s = \deg(\psi) - 1$.

Proof. We have $\deg\phi \geq 2$ and $s = \deg(\phi) - 2 = \deg(\psi) - 1$. Let now c be a zero of ϕ. The Euclidean division of ψ by $\theta_c(\phi)$ yields

$$\psi(x) = -\mu\theta_c(\phi)(x) + \rho(x), \tag{3.20}$$

Clearly, μ is given by

$$\mu = -\frac{\psi^{(s+1)}(0)}{(s+1)!} \neq 0. \tag{3.21}$$

As $\deg\psi = \deg\phi - 1$, μ is the leading coefficient of $-\psi$.
By (3.9) and (3.20), we have

$$(\mu-1)(\theta_c(\phi)\mathscr{W}) - (x-c)(\theta_c(\phi)\mathscr{W})' = \rho\mathscr{W}. \tag{3.22}$$

It should be noticed that $\deg\rho \geq 0$; otherwise, the linear functional $\theta_c(\phi)\mathscr{W}$ would satisfy $(\mu-1)(\theta_c(\phi)\mathscr{W}) - (x-c)(\theta_c(\phi)\mathscr{W})' = 0$ and then its moments with respect to the MPS $\{(x-c)^n\}_{n\geq 0}$ would be such that $(n+\mu)(\theta_c(\phi)\mathscr{W})_{n,c} = 0,\ n \geq 0$. Thus, there exists an integer $n_0 \geq 0$ such that $(\theta_c(\phi)\mathscr{W})_{n,c} = 0,\ n \geq n_0$, i.e. $(x-c)^{n_0}\theta_c(\phi)\mathscr{W} = 0$, which contradicts the quasi-definiteness of $\mathscr{W}$.
It follows immediately that

$$(\mu^{-1}\rho\mathscr{W})(\mu, c) = \theta_c(\phi)\mathscr{W}.$$

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$\blacksquare$

Another particular case will now be studied.
Namely, $c = 0$, $\mu = 2(\alpha+\beta+2)$, and $\mathscr{W} = \mathcal{G}_G(\alpha,\beta)$, i.e. the symmetric generalized Gegenbauer quasi-definite linear functional with parameters α and β where α, β and $\alpha+\beta \neq -n$, $n \geq 1$ and $\beta \neq -\frac{1}{2}$ which is the unique monic and symmetric linear functional satisfying

$$\Big(x(x^2-1)\mathcal{G}_G(\alpha,\beta)\Big)' + \Big(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\Big)\mathcal{G}_G(\alpha,\beta) = 0.$$

It can be easily shown that

$$\mathcal{G}_G(\alpha+1,\beta) = 2(\alpha+\beta+2)(x^2-1)\mathcal{G}_G(\alpha,\beta), \tag{3.23}$$

thus,

$$\Big(2(\alpha+\beta+2)-1\Big)\mathcal{G}_G(\alpha+1,\beta) - x\mathcal{G}_G(\alpha+1,\beta)' = 2(\alpha+\beta+2)\mathcal{G}_G(\alpha,\beta). \tag{3.24}$$

Let $\{\hat{G}_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ denote the MOPS with respect to the symmetric generalized Gegenbauer linear functional $\mathcal{G}_G(\alpha,\beta)$. Then, $\{\hat{G}_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ satisfies the three-term recurrence relation (TTRR) [15, 18]

$$\begin{cases} \hat{G}_{n+1}^{(\alpha,\beta)}(x) = x\hat{G}_n^{(\alpha,\beta)}(x) - \gamma_n \hat{G}_{n-1}^{(\alpha,\beta)}(x), \ n \geq 0, \\ \hat{G}_0^{(\alpha,\beta)}(x) = 1, \ \hat{G}_{-1}^{(\alpha,\beta)}(x) = 0, \end{cases} \tag{3.25}$$

where for any integer $n \geq 0$,

$$\gamma_{n+1} = \frac{\Big[2n+2+(2\beta+1)(1+(-1)^n)\Big]\Big[2n+2+4\alpha+(2\beta+1)(1+(-1)^n)\Big]}{16(n+\alpha+\beta+1)(n+\alpha+\beta+2)}.$$

Another characterization may be found in [17], Table 6.
The first structure relation of $\{\hat{G}_n^{(\alpha,\beta)}\}_{n\geq 0}$ is given for any integer $n \geq 0$ by

$$x(x^2-1)(\hat{G}_n^{(\alpha,\beta)})'(x) = \Big(nx^2 - (2\beta+1)\eta_n\Big)\hat{G}_n^{(\alpha,\beta)}(x) - 2(n+\alpha+\beta)\gamma_n x\hat{G}_{n-1}^{(\alpha,\beta)}(x), \tag{3.26}$$

where $\eta_n = \frac{(-1)^n-1}{2}$.

Corollary 3.6 *The polynomials $\hat{G}_{n+2}^{(\alpha,\beta)}(x)$ and $\hat{G}_n^{(\alpha+1,\beta)}(x)$ satisfy the following identity:*

$$x(\hat{G}_{n+2}^{(\alpha,\beta)})'(x) - (n+2)\hat{G}_{n+2}^{(\alpha,\beta)}(x) = \varrho_n \hat{G}_n^{(\alpha+1,\beta)}(x), \tag{3.27}$$

where for any integer $n \geq 0$,

$$\varrho_n = n+2-(2\beta+1)\eta_{n+2} - 2(n+\alpha+\beta+2)\gamma_{n+2} \neq 0.$$

Proof.

Using (3.26) and taking (3.25) into account, we obtain for any integer $n \geq 0$

$$(x^2-1)\Big(x(\hat{G}_{n+2}^{(\alpha,\beta)})'(x)-(n+2)\hat{G}_{n+2}^{(\alpha,\beta)}(x)\Big) = -2(n+\alpha+\beta+2)\gamma_{n+2}\gamma_{n+1}\hat{G}_n^{(\alpha,\beta)}(x) + \varrho_n\hat{G}_{n+2}^{(\alpha,\beta)}(x).$$

It should be noticed that $\varrho_n \neq 0,\ n \geq 0$; otherwise, there would exist an integer $j \geq 0$ such that $\varrho_j = 0$. By (3.26) with $x = 1$, and the fact that $(n+\alpha+\beta)\gamma_n \neq 0,\ n \geq 1$, it is easily seen by induction that $\hat{G}_n^{(\alpha,\beta)}(1) = 0,\ 0 \leq n \leq j$. In particular, $\hat{G}_0^{(\alpha,\beta)}(1) = 0$, which is a contradiction, as $\hat{G}_0^{(\alpha,\beta)}(x) = 1$. Accordingly,

$$(x^2-1)Q_n(x) = \hat{G}_{n+2}^{(\alpha,\beta)}(x) - 2\varrho_n^{-1}(n+\alpha+\beta+2)\gamma_{n+2}\gamma_{n+1}\hat{G}_n^{(\alpha,\beta)}(x),\ n \geq 0,$$

where $Q_n(x) = \varrho_n^{-1}\Big(x(\hat{G}_{n+2}^{(\alpha,\beta)})'(x)-(n+2)\hat{G}_{n+2}^{(\alpha,\beta)}(x)\Big),\ n \geq 0$.
By Lemma 1, with $A(x) = x^2-1$, $B_n = \hat{G}_n^{(\alpha,\beta)}$, and $\lambda_{n,n} = -2\varrho_n^{-1}(n+\alpha+\beta+2)\gamma_{n+2}\gamma_{n+1} \neq 0,\ n \geq 0$, and by taking (3.23) into account, the MPS $\{Q_n(x)\}_{n\geq 0}$ is orthogonal with respect the linear functional $\mathcal{G}^{(\alpha+1,\beta)}$. Thus, $Q_n(x) = \hat{G}_n^{(\alpha+1,\beta)}(x)$, $n \geq 0$. This leads to (3.27). ■

Compliance with ethical standards
Research involving human participants and/or animals This research does not contain any studies with human participants and/or animals performed by the author.
Conflict of interest The author declares there is no conflict of interest related to this article.
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