GLOBAL STRUCTURE OF POSITIVE SOLUTIONS FOR SEMIPOSITONE THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We are concerned with the global behavior of positive solutions for some classes of semipositone third-order nonlinear boundary value problems

$$u''' = \lambda f(t,u),\ t \in (0,1),$$
$$u(0) = u'(\eta) = 0,\ u''(1) + g(u(1))u(1) = 0,$$

where $\eta \in (\frac{1}{2}, 1)$, λ is a positive parameter, $g \in C([0,\infty),[0,\infty))$ and $f \in C([0,1]\times[0,\infty),\mathbb{R})$ with $f(t,0) < 0$. The proof of our main results are based upon bifurcation theory.

1. Introduction.

In this paper, we investigate the global behavior of positive solutions for third-order nonlinear boundary value problems

$$u''' = \lambda f(t,u),\ t \in (0,1),$$
$$u(0) = u'(\eta) = 0,\ u''(1) + g(u(1))u(1) = 0, \tag{1.1}$$

where $\eta \in (\frac{1}{2}, 1)$ and λ is a positive parameter. We assume that the following assumptions.

(F_1) (semipositone) $f : [0,1]\times[0,\infty) \to \mathbb{R}$ is continuous with $f(t,0) < 0$ for all $t \in [0,1]$.

(G_1) $g : [0,\infty) \to [0,\infty)$ is continuously differentiable and nondecreasing about ξ, that is, $0 \le g(0) \le g(\xi) \le g(\infty) < \infty$.

This is a generalization of the right focal boundary value problems used in **[1, 2, 3, 4, 5, 6]** with $g(u(1)) = 0$ and **[7]** with $g(u(1)) = \gamma$. However, this is not the case with nonlinear boundary conditions as the problem

$$u''' = 1,\ t \in (0,1),$$
$$u(0) = u'(\eta) = 0,\ u''(1) + g(u(1))u(1) = 0,$$

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with $g\left(\frac{2}{15}\right) = 1$ and $g\left(\frac{1}{6}\right) = 2$ has at least two solutions $u(t) = \frac{t^3}{6} - \frac{t^2}{2}m + (m\eta - \frac{\eta^2}{2})t$ where $m \in \{\frac{17}{15}, \frac{4}{3}\}$ and $\eta = \frac{2}{3}$. Meanwhile, we notice that all of the main results in the above works use a key condition that the nonlinearities are nonnegative at $u = 0$, i.e., $f(t, 0) \geq 0$, which is called a positone problem. On the contrary, few works of positive solutions of the boundary value problems with (F_1) are developed in **[8, 9]**. In this case, the maximum principle may fail. In order to overcome this difficulty, as a first step, they translated the semipositone problem into a positone problem using the transformations $f(t, u) + M > 0$ where $M > 0$ is a constant. Naturally, a question is raised that how to deal with the semipositone problems without transformation. Concurrently, we found that they provided no information about the global behavior of the set of positive solutions since the spectrum structure of third-order linear eigenvalue problems has not been established so far. The purpose of this paper is to fill, at least partially, this gap.

To our knowledge, existence, uniqueness and multiplicity of positive solutions of semipositone second-order boundary value problems have been studied, see **[10, 11, 12, 13]** with linear boundary conditions, and **[14, 15, 16]** with nonlinear boundary conditions. It is worth noting that based upon the bifurcation method, with the exception of Ambrosetti et al. **[10]** that deals with semipositone elliptic problems with linear boundary conditions, recently, Ma **[15, 16]** obtained the existence of positive solutions for second-order semipositone problems with nonlinear boundary conditions

$$-u'' = \lambda \tilde{f}(t, u), \ t \in (0, 1),$$

$$u(0) = 0, \ u'(1) + \tilde{c}(u(1))u(1) = 0.$$

Comparing with that, the difference in the study of third-order boundary value problems is that the third-order differential operator is non-self adjoint.

Motivated by the above works, in this paper, we investigate the global behavior of positive solutions for the semipositone third-order problems with nonlinear boundary conditions. Depending on the behavior of $f(t, u)$ as $u \to \infty$, the same abstract setting is employed to deal with both asymptotically linear, superlinear as well as sublinear problems. All results are obtained by showing that there exists a global branch of solutions of (1.1) which bifurcates from infinity and proving that for λ near the bifurcation value, solutions of large norms are indeed positive to apply bifurcation theory or topological methods. Since there are some differences between second- and third-order cases, we have to overcome several new difficulties in the proof of our main results.

The rest of the paper is arranged as follows. Some notation and preliminaries are listed in Section 2. In Section 3, we deal with asymptotically linear problems. In Section 4, we prove that (1.1) has at least one solution for $\lambda \in (0, \lambda^*]$ in the case that f satisfies the superlinear condition. Similar arguments can be used in the sublinear case, we show that (1.1) possesses positive solutions provided λ is large enough in Section 5.

2. Notation and Preliminaries. We will work in $X = C[0,1]$ with the norm $\|u\| := \max_{t\in[0,1]} |u(t)|$ and the inner product in $L^2(0,1)$ by $\langle\cdot,\cdot\rangle$. Also we set $B_r = \{u \in X : \|u\| < r\}$.

Denote that $G_\gamma(t,s)$ is the Green's function of the linear problem ($e \in X$)

$$\begin{aligned} &u'''(t) = e(t),\ t \in (0,1), \\ &u(0) = u'(\eta) = 0,\ u''(1) + \gamma u(1) = 0, \end{aligned} \tag{2.1}$$

which is explicitly given by

$$G_\gamma(t,s) = \begin{cases} s \in [0,\eta] : \begin{cases} \frac{t(2s-t)}{2} + \frac{\gamma t s^2(2\eta-t)}{2(2+\gamma(1-2\eta))},\ t \le s, \\ \frac{s^2}{2} + \frac{\gamma t s^2(2\eta-t)}{2(2+\gamma(1-2\eta))},\ t \ge s, \end{cases} \\ s \in [\eta,1] : \begin{cases} \frac{t(2\eta-t)(2+\gamma(1-s)^2)}{2(2+\gamma(1-2\eta))},\ t \le s, \\ \frac{t(2\eta-t)(2+\gamma(1-s)^2)}{2(2+\gamma(1-2\eta))} + \frac{(t-s)^2}{2},\ t \ge s. \end{cases} \end{cases}$$

Notice that in the special case $\gamma = 0$, (2.1) can be reduced to a class of three-point boundary value problems coupled with $u(0) = u'(\eta) = u''(1) = 0$. This result just coincides with **[1]**.

Lemma 2.1. (**[7,** Lemma 2.1]) Let $\gamma \in [0, \frac{2}{2\eta-1})$ with $\eta \in (\frac{1}{2}, 1)$. Then

(i) $G_\gamma(t,s)$ is nonnegative and increasing in γ for fixed $(t,s) \in [0,1]\times[0,1]$;

(ii) $l(t)G_\gamma(\eta,s) \le G_\gamma(t,s) \le G_\gamma(\eta,s)$, where $l(t) := \frac{t(2\eta-t)}{\eta^2}$.

Next, we consider the principal eigenvalue of the following linear eigenvalue problem

$$\begin{aligned} &u''' = \lambda a(t)u,\ t \in (0,1), \\ &u(0) = u'(\eta) = 0,\ u''(1) + \gamma u(1) = 0. \end{aligned} \tag{2.2}$$

Let $\theta \in (0, 1-\eta)$. Define the $P \subset X$ by

$$P := \left\{ u \in X | \min_{t\in[\eta-\theta,\eta+\theta]} u(t) \ge l(\eta+\theta)\|u\| \right\}.$$

Let $A : P \to X$ be the map defined by

$$Au(t) := \lambda \int_0^1 G_\gamma(t,s)a(s)u(s)ds,\ t \in [0,1].$$

Lemma 2.2. For $\gamma \in [0, \frac{2}{2\eta-1})$ with $\eta \in (\frac{1}{2}, 1)$, (2.2) has a principal eigenvalue $\lambda_1[a(\cdot),\gamma]$, which is positive and simple, and the corresponding eigenfunction $\phi_1(t)$ is positive on $[0,1]$.

Proof. From the definition of P, we know that P is normal and has nonempty interior. Obviously, $X = \overline{P-P}$. Since $G_\gamma(t,s) > 0$, then A is a strong positive operator, that is, $A \in \operatorname{int} P$. By Krein-Rutman Theorem **[18,** Theorem 19.3(a)], the spectral radius $r(A)$ is positive, and there exists $\phi_1 \in X$ such that $\phi_1 > 0$ on $[0,1]$ and $A\phi_1 = r(A)\phi_1$. Thus, $\lambda_1[a(\cdot),\gamma] = (r(A))^{-1} > 0$. Let A^* be conjugate operator of A, then $A^*\psi_1 = r(A^*)\psi_1$, where $\psi_1 \in X$ such that $\psi_1 > 0$ on $[0,1]$ corresponding to $\lambda_1[a(\cdot),\gamma]$. Since

$$\langle A\phi_1, \psi_1\rangle = \lambda_1[a(\cdot),\gamma]\langle \phi_1, \psi_1\rangle = \langle \phi_1, A^*\psi_1\rangle,$$

then the algebraic multiplicity of $\lambda_1[a(\cdot),\gamma]$ is 1. Thus, $\lambda_1[a(\cdot),\gamma]$ is the principal eigenvalue of (2.2). □

Lemma 2.3. Assume that (G_1) holds. Then for $e \in X$, the problem

$$\begin{aligned} &v'''(t) = e(t),\ t \in (0,1), \\ &v(0) = v'(\eta) = 0,\ v''(1) + g(v(1))v(1) = 0, \end{aligned} \tag{2.3}$$

has a unique solution $v \in C^3[0,1]$.

Proof. First, we show that (2.3) exists at least one solution, which is equivalent to the fixed point of the following equation

$$v(t) = \tilde{\mathcal{T}}v(t) := \int_0^1 G_{g(v(1))}(t,s)e(s)ds.$$

It is not hard to show that $\tilde{\mathcal{T}}$ is completely continuous and

$$v \le \max_{0\le t,s\le 1} G_{g(v(1))} \cdot \|e\| := \rho.$$

By Schauder fixed point theorem, $\tilde{\mathcal{T}}$ has a fixed point in B_ρ. So (2.3) has a solution.

Next, we show that (2.3) has a unique solution in $C^3[0,1]$.

Suppose the contrary that v_1 and v_2 are two solutions of (2.3) with $v_1 \neq v_2$. Then we can obtain that

$$\begin{aligned} &(v_2 - v_1)''' = 0,\ t \in (0,1), \\ &(v_2 - v_1)(0) = (v_2' - v_1')(\eta) = 0, \\ &(v_2'' - v_1'')(1) + [g(v_2(1))v_2(1) - g(v_1(1))v_1(1)] = 0. \end{aligned}$$

Since

$$(g(x)x)' = g'(x)x + g(x) \ge g(0) \ge 0 \tag{2.4}$$

and

$$g(v_2(1))v_2(1) - g(v_1(1))v_1(1) = [g'(\xi)\xi + g(\xi)](v_2 - v_1),$$

for a $\xi \in [\min\{u(1), v(1)\}, \max\{u(1), v(1)\}]$. Thus

$$(v_2 - v_1)''' = 0, \ t \in (0, 1),$$
$$(v_2 - v_1)(0) = (v_2' - v_1')(\eta) = 0,$$
$$(v_2'' - v_1'')(1) + [g'(\xi)\xi + g(\xi)](v_2 - v_1) = 0,$$

which contradicts with (2.4). Therefore,

$$v_2(t) - v_1(t) \equiv 0, \ t \in [0, 1].$$

□

In view of Lemma 2.3, we can define a nonlinear operator $\mathcal{K} : X \to C^3[0, 1]$ by

$$u := \mathcal{K}e,$$

where $u \in C^3[0, 1]$ is the unique solution of (2.3). It is easy to check that $\mathcal{K}$ is completely continuous. From the above notation, it follows that (1.1) is equivalent to

$$u - \lambda\mathcal{K}f(\cdot, u) = 0, \ u \in X. \tag{2.5}$$

Hereafter we will use the same symbol to denote both the function and the associated Nemytskii operator.

We denote that if there exists a sequence (μ_n, u_n) with $\mu_n \to \lambda_\infty$ and $u_n \in X$, such that $u_n - \mu_n\mathcal{K}f(u_n) = 0$ and $\|u_n\| \to \infty$, then λ_∞ is a bifurcation from infinity for (2.5).

In some situations, like the specific ones we will discuss later, an appropriate rescaling allows us to find bifurcation from infinity by means of the Leray-Schauder topological degree, denoted by $\deg(\cdot, \cdot, \cdot)$. Recall that $\mathcal{K} : X \to X$ is continuous and compact, and hence it makes sense to consider the topological degree of $I - \lambda\mathcal{K}f$, where I is the identity map.

3. Asymptotically linear problems.

Theorem 3.1. Assume that (F_1) and (G_1) are satisfied. f satisfies the following assumption.

(F_2) There exists a function $c \in X$ with $c(t) > 0$ for $t \in [0, 1]$ such that

$$\lim_{u \to \infty} \frac{f(t, u)}{u} = c.$$

Then there exists $\varepsilon > 0$ such that (1.1) has positive solutions provided either

(i) $\alpha > 0$ (possibly ∞) in $[0, 1]$ and $\lambda \in [\lambda_\infty - \varepsilon, \lambda_\infty)$; or

(ii) $\beta < 0$ (possibly $-\infty$) in $[0,1]$ and $\lambda \in (\lambda_\infty, \lambda_\infty + \varepsilon]$,

where $\lambda_\infty = \frac{\lambda_1[c(\cdot),g(u(1))]}{c}$ and

$$\alpha(t) = \liminf_{u\to\infty}(f(t,u) - cu),\ \beta(t) = \limsup_{u\to\infty}(f(t,u) - cu).$$

In order to show Theorem 3.1, we firstly extend $f(t,\cdot)$ to all of $\mathbb{R}$ by setting

$$F(t,u) = f(t,|u|). \tag{3.1}$$

Set

$$\Phi(\lambda,u) := u - \lambda\mathcal{K}F(t,u),\ u \in X. \tag{3.2}$$

Obviously, the solution $u > 0$ of $\Phi(\lambda,u) = 0$ is a positive solution of (1.1).

Lemma 3.1. For every compact interval $\Lambda \subset [0,+\infty)\backslash\{\lambda_\infty\}$, there exists $r > 0$ such that $\Phi(\lambda,u) \neq 0$, for all $\lambda \in \Lambda$, $\|u\| \geq r$. Moreover,

(i) if $\alpha > 0$, then $\Lambda = [\lambda_\infty, \lambda]$, for $\lambda > \lambda_\infty$;

(ii) if $\beta < 0$, then $\Lambda = [0, \lambda_\infty]$.

Proof. Let $\mu_n \to \mu \geq 0 \in \Lambda$, that is, $\mu \neq \lambda_\infty$ and $\|u_n\| \to \infty$ be such that

$$u_n = \mu_n\mathcal{K}F(t,u_n).$$

Setting $w_n = \frac{u_n}{\|u_n\|}$, it follows that

$$w_n = \mu_n\|u_n\|^{-1}\mathcal{K}F(t,u_n).$$

It follows from (F_2) and (3.1) that, up to a subsequence, $w_n \to w$ in X, where w satisfies the problem

$$w''' = \mu c|w|,\ t \in (0,1),$$
$$w(0) = w'(\eta) = 0,\ w''(1) + g(w(1))w(1) = 0,$$

and $\|w\| = 1$. By the maximum principle, we know that $w \geq 0$. Since $\|w\| = 1$, then $\mu c = \lambda_1[c(\cdot), g(w(1))]$, namely $\mu = \lambda_\infty$, which is a contradiction.

Next, we give a short illustration of Lemma 3.1 (i). And (ii) follows similarly. Now, suppose that there exists a sequence $(\mu_n, u_n) \in (0,\infty) \times X$ with $\mu_n \to \lambda_\infty$, $\|u_n\| \to \infty$ and $\mu_n > \lambda_\infty$ such that

$$\Phi(\mu_n, u_n) = 0. \tag{3.3}$$

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Note that $u_n \in X$ has a unique decomposition

$$u_n = v_n + s_n\phi_1, \tag{3.4}$$

where $s_n \in \mathbb{R}$, since $u_n > 0$, $\psi_1 > 0$ and $\int_0^1 v_n(t)\psi_1(t)dt = 0$, by (3.4), we have

$$s_n = \left(\int_0^1 u_n(t)\psi_1(t)dt\right)\left(\int_0^1 \phi_1(t)\psi_1(t)dt\right)^{-1} > 0,\ n \in \mathbb{N}. \tag{3.5}$$

Then from (3.3), it follows that

$$\int_0^1 \psi_1(t)u_n(t)dt = \mu_n \int_0^1 \psi_1(t)\mathcal{K}F(u_n(t))dt.$$

And since $\int_0^1 \psi_1(t)\mathcal{K}F(u_n(t))dt = -\lambda_1[c(\cdot), g(u(1))]\int_0^1 \psi_1(t)u_n(t)dt$, we obtain

$$\begin{aligned}
-\lambda_1[c(\cdot), g(u(1))]\int_0^1 \psi_1(t)u_n(t)dt &= \int_0^1 \psi_1'''(t)u_n(t)dt \\
&= -\int_0^1 u_n'''(t)\psi_1(t)dt \\
&= -\int_0^1 \mu_n f(t, u_n(t))\psi_1(t)dt \\
&= -\int_0^1 \mu_n(f(t, u_n(t)) - cu_n(t))\psi_1(t)dt - \int_0^1 \mu_n u_n(t)c\psi_1(t)dt,
\end{aligned}$$

then

$$(\mu_n c - \lambda_1[c(\cdot), g(u(1))])\int_0^1 \psi_1(t)u_n(t)dt = -\int_0^1 \mu_n(f(t, u_n(t)) - cu_n(t))\psi_1(t)dt.$$

Since $\mu_n > \lambda_\infty$ and $\int_0^1 u_n(t)\psi_1(t)dt > 0$ for n large enough, we infer that $\langle f(u_n) - cu_n, \psi_1\rangle < 0$ and from Fatou lemma, we yields

$$0 \geq \liminf_{n\to\infty}\langle f(u_n) - cu_n, \psi_1\rangle \geq \langle \alpha, \psi_1\rangle,$$

which contradicts with $\alpha > 0$. $\square$

Lemma 3.2. For $\lambda \in (\lambda_\infty, \infty)$, there exists $r > 0$ such that

$$\Phi(\lambda, u) \neq \tau\phi_1,\ \text{for all } \tau \geq 0,\ \|u\| \geq r.$$

Proof. Suppose that there exist a sequence $\{u_n\}$ in X with $\|u_n\| \to \infty$ and numbers $\tau_n \geq 0$ such that $\Phi(\lambda, u_n) = \tau_n\phi_1$. Then

$$u_n''' = \lambda F(t, u_n) + \tau_n\lambda_1[c(\cdot), g(u(1))]\phi_1.$$

Since $F(t, u) \approx c|u| \to \infty$ and $\tau_n\lambda_1[c(\cdot), g(u(1))]\phi_1 \geq 0$, by (2.3), we know that $u_n > 0$ for all $t \in [0, 1]$.

Choose $\epsilon > 0$ such that

$$\lambda_\infty < \lambda(1 - \epsilon).$$

From condition (F_2), there exists a positive constant R_0 such that

$$f(t,u) \geq (1-\epsilon)cu,\ \forall\ u > R_0,\ t \in [0,1].$$

It follows from $\|u_n\| \to \infty$ that there exists a positive constant N^* such that

$$u_n > R_0,\ \forall\ n \geq N^*,$$

and

$$f(t,u_n) \geq (1-\epsilon)cu_n. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\begin{aligned} s_n\lambda_1[c(\cdot),g(u(1))]\int_0^1 \phi_1(t)\psi_1(t)dt =& -\int_0^1 \psi_1'''(t)u_n(t)dt \\ =& \int_0^1 u_n'''(t)\psi_1(t)dt \\ =&\lambda\int_0^1 F(t,u_n)\psi_1(t)dt + \tau\lambda_1[c(\cdot),g(u(1))]\int_0^1 \phi_1(t)\psi_1(t)dt \\ \geq&\lambda\int_0^1 F(t,u_n)\psi_1(t)dt \\ \geq&\lambda\int_0^1 (1-\epsilon)cu_n(t)\psi_1(t)dt \\ =&\lambda(1-\epsilon)cs_n\int_0^1 \phi_1(t)\psi_1(t)dt. \end{aligned}$$

Thus,

$$\lambda_\infty \geq \lambda(1-\epsilon),$$

which is a contradiction. □

For $u \neq 0$, we set $z = \frac{u}{\|u\|^2}$. Letting

$$\Psi(\lambda,z) := \frac{\Phi(\lambda,u)}{\|u\|^2} = \frac{u - \lambda\mathcal{K}F(t,u)}{\|u\|^2} = z - \lambda\|z\|^2\mathcal{K}F\left(t, \frac{z}{\|z\|^2}\right).$$

λ_∞ is a bifurcation from infinity for (3.2) if and only if it is a bifurcation from the trivial $z=0$ for $\Psi = 0$. From Lemma 3.1, for all $\lambda < \lambda_\infty$, it follows by homotopy that

$$\begin{aligned} \deg(\Psi(\lambda,\cdot), B_{\frac{1}{r}}, 0) &= \deg(\Psi(0,\cdot), B_{\frac{1}{r}}, 0) \\ &= \deg(I, B_{\frac{1}{r}}, 0) = 1. \end{aligned} \tag{3.7}$$

Similarly, from Lemma 3.1, for all $\tau \in [0,1]$ and $\lambda > \lambda_\infty$,

$$\begin{aligned} \deg(\Psi(\lambda,\cdot), B_{\frac{1}{r}}, 0) &= \deg(\Psi(0,\cdot) - \tau\varphi_1, B_{\frac{1}{r}}, 0) \\ &= \deg(\Psi(0,\cdot) - \phi_1, B_{\frac{1}{r}}, 0) = 0. \end{aligned} \tag{3.8}$$

Set

$$\Sigma := \{(\lambda,u) \in [0,+\infty) \times X : u \neq 0, \Phi(\lambda,u) = 0\}.$$

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From (3.7) and (3.8) and the above discussion, it follows that

Lemma 3.3. λ_∞ is a bifurcation from infinity for (3.2). More precisely, there exists an unbounded closed connected set $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity. Moreover, Σ_∞ bifurcates to the left (to the right) provided that $\alpha > 0$ (respectively $\beta < 0$).

Proof of Theorem 3.1 By the previous lemmas, it is enough to show that if $\mu_n \to \lambda_\infty$ and $\|u_n\| \to \infty$, then for all $t \in [0,1]$ and n large enough, $u_n > 0$. Let $w_n = \frac{u_n}{\|u_n\|}$ and using the preceding arguments, we obtain that, up to subsequence, $w_n \to w$ in X and $w = \vartheta\phi_1$, $\vartheta > 0$. Then, it follows that $u_n > 0$ for n large enough.

4. Superlinear problems.

Theorem 4.1. Assume that (F_1) and (G_1) are satisfied. f satisfies the following condition.

(F_3) There exists a function $c \in X$ with $c(t) > 0$ for $t \in [0,1]$ such that

$$\lim_{u\to\infty} \frac{f(t,u)}{u^p} = c, \ p \in (1,\infty).$$

Then there exists $\lambda^* > 0$ such that (1.1) has positive solutions for $\lambda \in (0, \lambda^*]$. More precisely, there exists a connected set of positive solution of (1.1) bifurcates from infinity at $\lambda_\infty = 0$.

Set

$$\mathcal{G}(t,u) := F(t,u) - c|u|^p,$$

where the definition of $F(t,u)$ is the same as (3.1).

Next, using the rescaling $w = du$ and $\lambda = d^{p-1}$ with $d > 0$ to show that $\lambda_\infty = 0$ is a bifurcation from infinity for

$$u - \lambda\mathcal{K}F(t,u) = 0, \tag{4.1}$$

which is equivalent to (λ, u) is a solution of (4.1) if and only if

$$w - \mathcal{K}\tilde{F}(d,w) = 0, \tag{4.2}$$

where

$$\tilde{F}(d,w) := c|w|^p + d^p\mathcal{G}(d^{-1}w).$$

We extend $\tilde{F}$ to $d = 0$ and set

$$\tilde{F}(0,w) := c|w|^p.$$

By (F_3), we know that $\tilde{F}(d,w)$ is continuous for $(d,w) \in [0,\infty) \times \mathbb{R}$. Let

$$\mathcal{S}(d,w) := w - \mathcal{K}\tilde{F}(d,w),\ d \in (0,\infty).$$

Then $\mathcal{S}(d,\cdot)$ is compact. For $d=0$, solution of $\mathcal{S}(0,w)=0$ are noting but solutions of

$$\begin{aligned} &w''' = c|w|^p,\ t \in (0,1), \\ &w(0) = w'(\eta) = 0,\ w''(1) + g(\infty)w(1) = 0. \end{aligned} \tag{4.3}$$

Now, we claim that there exist two constants R_1, R_2 with $0 < R_1 < R_2$ such that

$$\mathcal{S}(0,w) \neq 0,\ \text{for all } \|w\| \geq R_2, \tag{4.4}$$

$$\mathcal{S}(0,w) \neq 0,\ \text{for all } 0 < \|w\| \leq R_1, \tag{4.5}$$

and

$$\deg(\mathcal{S}(0,w), P_R \backslash \bar{P}_r, 0) = -1,\ \text{for } r \in (0,R_1],\ R \in [R_2,\infty). \tag{4.6}$$

Firstly, we show that there exists a positive constant R such that $\mathcal{S}(0,w) \neq 0$, for all $\|w\| \geq R$.

Suppose that, there exists a sequence $\{w_n\}$ of (4.3) satisfying

$$\lim_{n\to\infty} \|w_n\| = \infty,$$

i.e.,

$$\begin{aligned} &w_n''' = (c|w_n|^{p-1})w_n,\ t \in (0,1), \\ &w_n(0) = w_n'(\eta) = 0,\ w_n''(1) + g(\infty)w_n(1) = 0. \end{aligned}$$

Notice that

$$\lim_{n\to\infty} c|w_n|^{p-1} = \infty,\ t \in [0,1].$$

From the remarks in the final paragraph on P. 56 of **[17]**, w_n must change its sign in $[0,1]$, which contradicts that $w_n(t) > 0$ for all $t \in [0,1]$.

Secondly, we show that there exists $R_1 > 0$ such that $\mathcal{S}(0,w) \neq 0$ for all $0 < \|w\| \leq R_1$.

Suppose to the contrary that (4.5) is not true. Then there exists a sequence w_n of solutions of (4.3), which satisfies

$$\|w_n\| \to 0,\ n \to \infty. \tag{4.7}$$

Let $v_n = \frac{w_n}{\|w_n\|}$. From (4.3), we have

$$\begin{aligned} &v_n''' = c\frac{|w_n|^p}{\|w_n\|},\ t \in (0,1), \\ &v_n(0) = v_n'(\eta) = 0,\ v_n''(1) + g(\infty)v_n(1) = 0. \end{aligned}$$

From (4.7), we have $\lim\limits_{n\to\infty} v_n = 0$ uniformly in $t \in [0,1]$.

By the standard argument, after taking a subsequence and relabeling if necessary, it follows that there exists $v_* \in X$ with $\|v\| = 1$ such that

$$v_n \to v_*, \ n \to \infty$$

and

$$v_*''' = 0, \ t \in (0,1),$$
$$v_*(0) = v_*'(\eta) = 0, \ v_*''(1) + g(\infty)v_*(1) = 0,$$

which implies that $v_* = 0$. However, this is a contradiction. Therefore, (4.5) holds.

In the end, we show (4.6) is valid. Denote

$$P_r := \{u \in P : \|u\| < r\}.$$

Now, from (4.4) and (4.5), we have

$$\mathcal{S}(0,w) \neq 0, \ \forall w \in \partial P_R; \ \mathcal{S}(0,w) \neq 0, \ \forall w \in \partial P_r,$$

which implies

$$\mathcal{S}(0,w) \neq 0, \ \forall w \in \partial(P_R \backslash \bar{P}_r).$$

Thus $\deg(\mathcal{S}(0,w), P_R\backslash\bar{P}_r, 0)$ is well defined.

Note that $\tilde{f}(w) = |w|^p$. Next, we show that $\deg(\mathcal{S}(0,w), P_R\backslash\bar{P}_r, 0) = -1$. It is easy to verity the following conditions

(H_1) $f_0 := \lim\limits_{w\to 0^+} \frac{\tilde{f}(w)}{w} = 0$;

(H_2) $f_\infty := \lim\limits_{w\to +\infty} \frac{\tilde{f}(w)}{w} = \infty$.

Choose $M_1 > 0$ such that

$$\frac{(\eta+\theta)(\eta-\theta)}{\eta^2} M_1 \int_{\eta-\theta}^{\eta+\theta} G(\eta,s)c(s)ds > 1. \tag{4.8}$$

From (H_2), there exists a constant $R_2 > 0$ such that $f(w) > M_1 w$, $\forall w \geq R_2$. Choose $R > \max\{R_1, R_2\}$, we claim that $\|\mathcal{K}\tilde{F}(0,w)\| > \|w\|$ for $w \in \partial P_R$. In fact, for $w \in \partial P_R$,

$$\begin{aligned}(\mathcal{K}\tilde{F}(0,w))(t) &= \int_0^1 G(t,s)c(s)|w|^p ds \\ &\geq \frac{(\eta+\theta)(\eta-\theta)}{\eta^2} M_1 \|w\| \int_{\eta-\theta}^{\eta+\theta} G(\eta,s)c(s)ds \\ &> \|w\|.\end{aligned}$$

Hence, from the fixed point index theorem of **[18]**, we have

$$i(\mathcal{K}\tilde{F}(0,\cdot), P_R, P) = 0. \tag{4.9}$$

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On the other hand, from (H_1), there exists a constant $\delta > 0$ such that $w \in [0,\delta]$, and

$$\tilde{f}(w) \le \kappa w,$$

where $\kappa > 0$ satisfying

$$\kappa \int_{\eta-\theta}^{\eta+\theta} G(\eta,s)c(s)ds \le 1.$$

Choose $0 < r < \min\{\delta, \frac{R}{2}\}$, for $w \in \partial P_r$,

$$\begin{aligned}\|\mathcal{K}\tilde{F}(0,w)\| =& \max_{t\in[\eta-\theta,\eta+\theta]} \int_0^1 G(t,s)c(s)|w|^p ds\\ \le&\kappa\|w\| \int_{\eta-\theta}^{\eta+\theta} G(\eta,s)c(s)ds\\ \le&\|w\|.\end{aligned}$$

Obviously, $\mathcal{K}\tilde{F}(0,w) \neq w$ for $w \in \partial P_r$. By the fixed point index theorem of **[18]**, then

$$i(\mathcal{K}\tilde{F}(0,\cdot), P_r, P) = 1. \tag{4.10}$$

So from the additivity of the fixed point index, (4.9) and (4.10), we get

$$i(\mathcal{K}\tilde{F}(0,\cdot), P_R\backslash\bar{P}_r, P) = -1. \tag{4.11}$$

From (4.11) and $\mathcal{S}(0,w) : X \to P_R\backslash\bar{P}_r$, it follows that

$$\deg(\mathcal{S}(0,w), P_R\backslash\bar{P}_r, 0) = -1.$$

Lemma 4.1. For $\gamma \in [0, \frac{2}{2\eta-1})$ with $\eta \in (\frac{1}{2}, 1)$, there exists $d_0 > 0$ such that

(i) $\deg(\mathcal{S}(d,\cdot), P_R\backslash\bar{P}_r, 0) = -1$, $\forall d \in [0, d_0]$;

(ii) if $\mathcal{S}(d,w) = 0$, $d \in [0,d_0]$, $\|w\| \in [r,R]$, then $w > 0$.

Proof. Suppose to the contrary that, there exists a sequence (d_n, w_n) with $d_n \to 0$, $\|w_n\| \in \{r,R\}$ and $w_n = \mathcal{K}\tilde{F}(d_n,w_n)$. Since $\mathcal{K}$ is a compact operator, then up to a subsequence, $w_n \to w$ and $\mathcal{S}(0,w) = 0$, $\|w_n\| \in \{r,R\}$, which contradicts with (4.4) and (4.5). Therefore, (i) holds.

In order to prove (ii), we argue again by contradiction. From the preceding argument, we can find a sequence $\{w_n\} \in X$ with $\{t \in [0,1] : w_n \le 0\} \neq \emptyset$, which satisfies $w_n \to w$, $\|w\| \in \{r,R\}$ and $\mathcal{S}(0,w) = 0$; namely, w is a solution of (4.3). From the maximum principle, we have $w > 0$. Therefore, $w_n > 0$ for n large enough, which is a contradiction. □

Proof of Theorem 4.1 From Lemma 4.1, $\forall d \in [0,d_0]$, (4.2) has a positive solution w_d. Recalling, for $d > 0$ and $\lambda = d^{p-1}$, $u = \frac{w}{d}$ gives a solution (λ, u_λ) of (4.1) for all $0 < \lambda < \lambda^* := d_0^{p-1}$.

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Since $w_d > 0$, (λ, u_λ) is a positive solution of (1.1). Finally, $\forall d \in [0, d_0]$, $\|w_d\| \geq d$ implies that $\|u_d\| = \|w_d\|/d \to \infty$ as $d \to 0$.

5. Sublinear problems.

Theorem 5.1. Assume that (F_1) and (G_1) are satisfied. f satisfies the following condition.

(F_4) There exists a function $c \in X$ with $c(t) > 0$ for $t \in [0,1]$, such that

$$\lim_{u\to\infty} \frac{f(t,u)}{u^q} = c, \text{ uniformly in } t \in [0,1] \text{ with } q \in [0,1).$$

Then there exists $\lambda_* > 0$ such that (1.1) has positive solutions for all $\lambda \in [\lambda_*, \infty)$. More precisely, there exists a connected set of positive solutions of (1.1) bifurcates from infinity for $\lambda_\infty = \infty$.

In this case, we will show that (1.1) exists positive solutions which branch off from ∞ for $\lambda_\infty = \infty$. By the same procedure as the superlinear case, we still use the rescaling $w = du$, $\lambda = d^{q-1}$ and the same notation, with q instead of p. In the case of superlinear problem, (λ, u) is the solution of (4.1) if (d, w) satisfies (4.2). Now, since $q \in [0,1)$, then

$$\lambda \to \infty \Leftrightarrow d \to 0. \tag{5.1}$$

Lemma 5.1. For $\gamma \in [0, \frac{2}{2\eta-1})$ with $\eta \in (\frac{1}{2}, 1)$ and $q \in (0,1)$, the problem

$$\begin{aligned} &w'''(t) = c(t)w^q(t),\ t \in (0,1), \\ &w(0) = w'(\eta) = 0,\ w''(1) + \gamma w(1) = 0, \end{aligned} \tag{5.2}$$

has a unique positive solution w_0.

Proof. Assume that u_1, u_2 are positive solutions of (5.2), i.e.,

$$u_1''' = c(t)u_1^q,\ u_1(0) = u_1'(\eta) = 0,\ u_1''(1) + \gamma u_1(1) = 0,$$

$$u_2''' = c(t)u_2^q,\ u_2(0) = u_2'(\eta) = 0,\ u_2''(1) + \gamma u_2(1) = 0.$$

We will show that $u_1 \geq u_2$ and $u_2 \geq u_1$.

Suppose on the contrary that $u_1 \ngeq u_2$. We consider the element $\bar{u}$ of the form

$$\bar{u} = u_1(t) - \epsilon u_2(t),\ t \in [0,1],$$

where $\epsilon \in (0,1)$. Let there exists $\tau_0 \in (0,1)$ such that

$$\bar{u}(\tau_0) = u_1(\tau_0) - \epsilon_0 u_2(\tau_0) = 0. \tag{5.3}$$

On the other hand,

$$\bar{u}'''(t) = (u_1(t) - \epsilon_0 u_2(t))''' = c(t)[u_1^q(t) - \epsilon_0 u_2^q(t)] \geq c(t)[\epsilon_0^q u_2^q(t) - \epsilon_0 u_2^q(t)] > 0,$$

$$\bar{u}(0) = \bar{u}'(\eta) = 0,\ \ \bar{u}''(1) + \gamma\bar{u}(1) = 0.$$

Thus, $\bar{u}(t) > 0$, which contradicts with (5.3). Therefore, $u_1 \geq u_2$. By the same method, we may prove that $u_1 \leq u_2$. □

As the same treatment as the sperlinear case, we also can obtain that there exist two constants R_3, R_4 with $0 < R_3 < R_4$ such that

$$\mathcal{S}(0, w) \neq 0,\ \text{for all } \|w\| \geq R_3, \tag{5.4}$$

$$\mathcal{S}(0, w) \neq 0,\ \text{for all } 0 < \|w\| \leq R_4, \tag{5.5}$$

and

$$\deg(\mathcal{S}(0, w), P_R \backslash \bar{P}_r, 0) = 1,\ \text{for } r \in (0, R_3],\ \ R \in [R_4, \infty). \tag{5.6}$$

Therefore, the following results can be obtained.

Lemma 5.2. For $\gamma \in [0, \frac{2}{2\eta-1})$ with $\eta \in (\frac{1}{2}, 1)$, there exists $d_0 > 0$ such that

(i) $\deg(\mathcal{S}(d, \cdot), P_R \backslash \bar{P}_r, 0) = 1,\ \forall d \in [0, d_0]$;

(ii) if $\mathcal{S}(d, w) = 0, d \in [0, d_0], \|w\| \in [r, R]$, then $w > 0$.

Proof of Theorem 5.1 By the continuation, there exists a connected subset Γ of solutions of $\mathcal{S}(d, w) = 0$ such that $(0, w_0) \in \Gamma$. From Lemma 5.2, we obtain that there exists $d_0 > 0$ such that these solutions are positive for all $d \in (0, d_0]$. Though the rescaling $\lambda = d^{q-1}$, $u = \frac{w}{d}$, Γ is transformed into a connected subset Σ_∞ of solutions of (1.1). These solutions are positive for all $\lambda > \lambda^* := d_0^{q-1}$ and, by (5.1), Σ_∞ bifurcates from infinity for $\lambda_\infty = +\infty$.

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