On a Nonlinear Fractional Boundary Value Problem including Hadamard fractional derivative with p-Laplacian

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Abstract
This paper presents the existence results of the p-Laplacian Hadamard fractional boundary value problem. We analyze the existence of multiple positive solutions for the fractional boundary value problem with the nonlinear term which involves the derivative term with the help of a fixed point theorem. An example is considered to describe the validity of the main result.

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1 Introduction

Fractional calculus has been a widely studied area of research in recent decades, especially because of its ability to describe complex issues in various areas of science and engineering, by the help of arbitrary order approach. Also, with respect to integer order derivatives, fractional order derivatives are more suitable to describe the memory and inherent properties of many kinds of materials and processes. Fractional order derivative is a global operator, that can serve as a tool for various applied phenomena such as mathematical biology [27], control theory [30], dynamical process [31], *etc.* For the newest studies on fractional calculus, we refer to some books [1, 17, 18, 19, 23, 25, 28]. Moreover, many scientists have introduced the existence results of fractional boundary value problems whereby many various techniques, such as the mixed monotone operator theory, Banach's contraction mapping principle, see [5, 7, 10, 12, 14, 22, 26, 34, 36, 38, 39].

Most common fractional derivative types that have been researched in fractional calculus field are Riemann-Liouville and Caputo fractional derivatives. Moreover, Hadamard fractional derivative which was demonstrated by Hadamard [13] has recently received attention from researchers. Hadamard fractional derivative and integral are different because their definitions include logarithmic function of arbitrary order. Within this field, discovering new generalizations for the existing fractional derivatives has always been a major focus of research. Using these generalized operators, we will provide new opportunities to develop existing results from theoretical and applied point of views. In [8], the authors introduced some operators that include Hadamard derivatives and presented the advantage of their approach through some examples. [8] showed a new approach based on linear integro-differential operators with logarithmic kernel via Hadamard fractional calculus for a generalization of the Lomnitz logarithmic creep law. Hadamard fractional calculus can show the mathematical essential of this creep law more exactly in this subject [8]. In addition, the works in [1, 4, 18] contributed significantly to the development of Hadamard derivatives within fractional calculus. At the same time, the subject of the fractional differential equations with Hadamard-type derivative has gained a lot of popularity due to their capability to modeling real world phenomena.

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To examine some of the recent developments on the Hadamard fractional boundary value problem (HFBVP for short), see [2, 3, 24, 29, 32, 35, 37]. Furthermore, the analysis of solutions with p-Laplacian operator for the HFBVP is still an area of further research and many researchers have shown more interest to the HFBVP including p-Laplacian operator, see [11, 15, 16, 21, 33]. Also, the function ϱ which involves Hadamard fractional derivative has been examined by only a few researchers, see [6, 11]. For a detail monograph on Hadamard fractional calculus and integral, see [1].

In [21], Li and Lin studied the following HFBVP involving the p-Laplacian operator

$$\begin{cases} \mathfrak{D}^{\beta}(\phi_p(\mathfrak{D}^{\alpha}u(z))) = \varrho(z,u(z)),\ 2<\alpha\le 3,\ 1<\beta\le 2,\ 1<z<e, \\ u(1)=u'(1)=u'(e)=0,\ \mathfrak{D}^{\alpha}u(1)=\mathfrak{D}^{\alpha}u(e)=0, \end{cases}$$

where $\varrho \in \mathcal{C}([1,e]\times[0,+\infty),[0,+\infty))$. They obtained the existence and uniqueness of the positive solution of the problem.

In [15], Xu *et.al.* examined the following problem

$$\begin{cases} -\mathfrak{D}^{\alpha}(\phi_p(\mathfrak{D}^{\beta}u(z))) = \varrho(z,u(z)),\ z\in(1,T), \\ u(1)=\delta u(1)=\delta u(e)=u(e)=\mathfrak{D}^{\beta}u(1)=0,\ \mathfrak{D}^{\beta}u(e)=b\mathfrak{D}^{\beta}u(\eta), \end{cases}$$

in which $1<\alpha\le 2$, $3<\beta\le 4$, $\varrho\in\mathcal{C}([1,e]\times[0,\infty),[0,\infty))$ and δ means delta derivative. By using fixed point theorems, the existence results for the HFBVP is ensured.

In light of the papers mentioned above, we generate some new results about the following problem involving Hadamard fractional derivative

$$\begin{cases} {}^{H}\mathfrak{D}^{\sigma_1}_{1+}(\phi_p({}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(z))) = \varrho(z,\kappa(z),{}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(z)),\quad 2<\sigma_1,\sigma_2\le 3,\quad z\in(1,e), \\ \kappa(1)=\kappa'(1)={}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(1)=(\phi_p({}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(1)))'=0, \\ \phi_p({}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(e)) = \sum_{i=1}^{k} a_i\phi_p({}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(\eta_i)) + \int_1^e \phi_p({}^{H}\mathfrak{D}^{\sigma_2}_{1+}\kappa(z))g(z)\frac{dz}{z}, \\ \kappa(e) = \sum_{j=1}^{q} b_j\kappa(\xi_j) + \int_1^e \kappa(z)h(z)\frac{dz}{z}, \end{cases} \tag{1.1}$$

in which $5<\sigma_1+\sigma_2\le 6$, ${}^{H}\mathfrak{D}^{\sigma_1}_{1+}$ and ${}^{H}\mathfrak{D}^{\sigma_2}_{1+}$ are the Hadamard fractional derivative, ϕ_p is the p-Laplacian, *i.e.*, $\phi_p(s)=|s|^{p-2}s$, with $s\in\mathbb{R}$, $p>1$, $\phi_p^{-1}=\phi_q$, and $\frac{1}{p}+\frac{1}{q}=1$. $a_i\ge 0$, $b_j\ge 0$ $(i=1,2,...,k)$, $(j=1,2,...,q)$, $1<\eta_1<\eta_2<...<\eta_k<e$ and $1<\xi_1<\xi_2<...<\xi_q<e$.

Let us give the necessary assumptions below:

(C1) $\varrho\in\mathcal{C}([1,e]\times[0,\infty)\times(-\infty,\infty),(0,\infty))$, $g,h\in\mathcal{C}([1,e],[0,\infty))$,

(C2) $\sum_{i=1}^{k} a_i(\log\eta_i)^{\sigma_1-1}+\int_1^e g(z)(\log z)^{\sigma_1-1}\frac{dz}{z}<1$, $\sum_{j=1}^{q} b_j(\log\xi_j)^{\sigma_2-1}+\int_1^e h(z)(\log z)^{\sigma_2-1}\frac{dz}{z}<1$.

In this study, by using the theory of fixed point under appropriate conditions, we aim to ensure the multiple positive solutions for a HFBVP involving p-Laplacian operator. The function ϱ which involves of derivative operator ${}^{H}\mathfrak{D}^{\sigma_2}_{1+}$ is studied by only a few scientists. In comparison to [15],[21], our problem includes the Hadamard fractional derivative in the nonlinear term and the boundary conditions are more complicated because of that the boundary conditions consist of the linear combination of the unknown function and integral boundary condition. Also, Hadamard derivative in nonlinear term is complicated, so it requires a more careful study and this type of derivative is a growing area of research.

2 Preliminaries

To meet the requirements in the next part, the fundamental principles of Hadamard fractional calculus are shown here.

Definition 2.1 *[25]* The Hadamard fractional derivative of fractional order ϖ for a function $c : [1,\infty) \to \mathbb{R}$ is expressed as

$$ {}^{H}\mathfrak{D}_{1^+}^{\varpi} c(z) = \frac{1}{\Gamma(m-\varpi)} \Big(z\frac{d}{dz} \Big)^m \int_1^t \Big(\log\frac{z}{s} \Big)^{m-\varpi-1} c(s)\frac{ds}{s}, \quad m-1<\varpi<m,\ m=[\varpi]+1, $$

where $[\varpi]$ shows the integer part of the real number ϖ and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 *[25]* The Hadamard fractional integral of order ϖ for a function $c : [1,\infty) \to \mathbb{R}$ is expressed as

$$ {}^{H}I_{1^+}^{\varpi} c(z) = \frac{1}{\Gamma(\varpi)} \int_1^z \big(\log\frac{z}{s} \big)^{\varpi-1} c(s)\frac{ds}{s}, \quad \varpi>0, $$

in the event that the associated integral exists.

Lemma 2.1 *[25]* Let $\varpi > 0$ and $a \in \mathcal{C}[1,\infty) \cap L^1[1,\infty)$, then the solution of Hadamard fractional differential equation ${}^{H}\mathfrak{D}_{1^+}^{\varpi} a(z) = 0$ is expressed as

$$ a(z) = \sum_{j=1}^{m} t_j (\log z)^{\varpi-j}, $$

then,

$$ {}^{H}I_{1^+}^{\varpi}\, {}^{H}\mathfrak{D}_{1^+}^{\varpi} a(z) = a(z) + \sum_{j=1}^{m} t_j (\log z)^{\varpi-j}, $$

in which $t_j \in \mathbb{R}, j = 1,2,...,m, m-1<\varpi<m, m=[\varpi]+1$.

Here, we introduce $\mathfrak{X} = \big\{ \kappa \in \mathcal{C}[1,e] : {}^{H}\mathfrak{D}_{1^+}^{\sigma_2}\kappa \in \mathcal{C}[1,e] \big\}$ by the norm

$$ \|\kappa\| = \max \left\{ \max_{z\in[1,e]} |\kappa(z)|,\ \max_{z\in[1,e]} |{}^{H}\mathfrak{D}_{1^+}^{\sigma_2}\kappa(z)| \right\}. $$

Hence, $\mathfrak{X}$ is a Banach space. Demonstrate a cone M by

$$ M = \{ \kappa \in \mathfrak{X} : \kappa(z) \geq 0, \forall z \in [1,e] \}. $$

Lemma 2.2 *Let $r \in \mathcal{C}[1,e]$ be given, then $c \in \mathfrak{X}$ is a solution of*

$$ \begin{cases} {}^{H}\mathfrak{D}_{1^+}^{\sigma_1} c(z) = r(z), \quad 2<\sigma_1\leq 3, \quad z\in(1,e), \\ c(1) = c'(1) = 0, \qquad c(e) = \sum_{i=1}^{k} a_i c(\eta_i) + \int_1^e g(z)c(z)\frac{dz}{z}, \end{cases} \tag{2.1} $$

provided that c satisfies the integral equation

$$ c(z) = -\int_1^e H(z,s) r(s) \frac{ds}{s}, \qquad z\in[1,e], $$

where

$$ H(z,s) = H_1(z,s) + H_2(z,s), \tag{2.2} $$

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$$H_1(z,s) = h(z,s) + \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}}{\Delta} h(\eta_i, s),$$

$$H_2(z,s) = \frac{(\log z)^{\sigma_1-1}}{\Delta_1} \int_1^e H_1(z,s) g(z) \frac{dz}{z},$$

with

$$h(z,s) = \frac{1}{\Gamma(\sigma_1)} \begin{cases} (\log z)^{\sigma_1-1}(1-\log s)^{\sigma_1-1} - (\log z - \log s)^{\sigma_1-1}, & 1 \le s \le z \le e, \\ (\log z)^{\sigma_1-1}(1-\log s)^{\sigma_1-1}, & 1 \le z \le s \le e, \end{cases} \tag{2.3}$$

and $\Delta = 1 - \sum_{i=1}^{k} a_i (\log \eta_i)^{\sigma_1-1}$, $\Delta_1 = \Delta - \int_1^e g(z)(\log z)^{\sigma_1-1} \frac{dz}{z}$.

Proof. From Lemma 2.1, the problem (2.1) can be shown by

$$c(z) = \frac{1}{\Gamma(\sigma_1)} \int_1^z (\log \frac{z}{s})^{\sigma_1-1} r(s) \frac{ds}{s} + t_1(\log z)^{\sigma_1-1} + t_2(\log z)^{\sigma_1-2} + t_3(\log z)^{\sigma_1-3},$$

for $t_1, t_2, t_3 \in \mathbb{R}$. Using $c(1) = c'(1) = 0$, we get $t_2 = 0$, $t_3 = 0$ and

$$c(z) = \frac{1}{\Gamma(\sigma_1)} \int_1^z (\log \frac{z}{s})^{\sigma_1-1} r(s) \frac{ds}{s} + t_1(\log z)^{\sigma_1-1}. \tag{2.4}$$

If we apply the boundary condition $c(e) = \sum_{i=1}^{k} a_i c(\eta_i) + \int_1^e g(z)c(z)\frac{dz}{z}$, from (2.4), we have

$$t_1 = \frac{1}{\Delta}\Big(- \frac{1}{\Gamma(\sigma_1)} \int_1^e (\log \frac{e}{s})^{\sigma_1-1} r(s) \frac{ds}{s} + \sum_{i=1}^{k} \frac{a_i}{\Gamma(\sigma_1)} \int_1^{\eta_i} (\log \frac{\eta_i}{s})^{\sigma_1-1} r(s) \frac{ds}{s} + \int_1^e g(z)c(z)\frac{dz}{z} \Big). \tag{2.5}$$

Inserting (2.5) into (2.4),

$$\begin{aligned} c(z) =& \frac{1}{\Gamma(\sigma_1)} \int_1^z (\log \frac{z}{s})^{\sigma_1-1} r(s) \frac{ds}{s} - \frac{(\log z)^{\sigma_1-1}}{\Delta\Gamma(\sigma_1)} \int_1^e (\log \frac{e}{s})^{\sigma_1-1} r(s) \frac{ds}{s} \\ &+ \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}}{\Delta\Gamma(\sigma_1)} \int_1^{\eta_i} (\log \frac{\eta_i}{s})^{\sigma_1-1} r(s) \frac{ds}{s} + \frac{(\log z)^{\sigma_1-1}}{\Delta} \int_1^e g(z)c(z)\frac{dz}{z} \\ =& \frac{1}{\Gamma(\sigma_1)} \int_1^z (\log \frac{z}{s})^{\sigma_1-1} r(s) \frac{ds}{s} - \frac{(\log z)^{\sigma_1-1}}{\Gamma(\sigma_1)} \int_1^e (\log \frac{e}{s})^{\sigma_1-1} r(s) \frac{ds}{s} \\ &- \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}(\log \eta_i)^{\sigma_1-1}}{\Delta\Gamma(\sigma_1)} \int_1^e (\log \frac{e}{s})^{\sigma_1-1} r(s) \frac{ds}{s} \\ &+ \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}}{\Delta\Gamma(\sigma_1)} \int_1^{\eta_i} (\log \frac{\eta_i}{s})^{\sigma_1-1} r(s) \frac{ds}{s} + \frac{(\log z)^{\sigma_1-1}}{\Delta} \int_1^e g(z)c(z)\frac{dz}{z} \\ =& -\int_1^e h(z,s) r(s) \frac{ds}{s} - \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}}{\Delta} \int_1^e h(\eta_i, s) r(s) \frac{ds}{s} + \frac{(\log z)^{\sigma_1-1}}{\Delta} \int_1^e g(z)c(z)\frac{dz}{z} \\ =& -\int_1^e H_1(z,s) r(s) \frac{ds}{s} + \frac{(\log z)^{\sigma_1-1}}{\Delta} \int_1^e g(z)c(z)\frac{dz}{z}. \end{aligned}$$

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Furthermore,

$$\int_1^e g(z)c(z)\frac{dz}{z} = -\int_1^e g(z)\int_1^e H_1(z,s)r(s)\frac{ds}{s}\frac{dz}{z} + \frac{1}{\Delta}\int_1^e g(z)(\log z)^{\sigma_1-1}\frac{dz}{z}\int_1^e g(z)c(z)\frac{dz}{z},$$

which ensure

$$\int_1^e g(z)c(z)\frac{dz}{z} = -\frac{\Delta}{\Delta_1}\int_1^e g(z)\int_1^e H_1(z,s)r(s)\frac{ds}{s}\frac{dz}{z},$$

Hence,

$$\begin{aligned} c(z) &= -\int_1^e H_1(z,s)r(s)\frac{ds}{s} - \int_1^e H_2(z,s)r(s)\frac{ds}{s} \\ &= -\int_1^e H(z,s)r(s)\frac{ds}{s}. \end{aligned}$$

□

Lemma 2.3 *For $c \in \mathcal{C}[1,e]$ and $\kappa \in \mathfrak{X}$, the following problem*

$$\begin{cases} {}^{H}\mathfrak{D}^{\sigma_2}_{1^+}\kappa(z) = \phi_q(c(z)), \quad 2 < \sigma_2 \leq 3, \quad z \in (1,e), \\ \kappa(1) = \kappa'(1) = 0, \quad \kappa(e) = \sum_{j=1}^{q} b_j\kappa(\xi_j) + \int_1^e h(z)\kappa(z)\frac{dz}{z} \end{cases} \tag{2.6}$$

is equivalent to,

$$\kappa(z) = -\int_1^e K(z,s)\phi_q(c(s))\frac{ds}{s}, \qquad z \in [1,e],$$

where

$$K(z,s) = K_1(z,s) + K_2(z,s), \tag{2.7}$$

$$K_1(z,s) = k(z,s) + \sum_{j=1}^{q}\frac{b_j(\log z)^{\sigma_2-1}}{\Delta^*}k(\xi_j,s),$$

$$K_2(z,s) = \frac{(\log z)^{\sigma_2-1}}{\Delta_1^*}\int_1^e K_1(z,s)h(z)\frac{dz}{z},$$

with

$$k(z,s) = \frac{1}{\Gamma(\sigma_2)}\begin{cases} (\log z)^{\sigma_2-1}(1-\log s)^{\sigma_2-1} - (\log z - \log s)^{\sigma_2-1}, & 1 \leq s \leq z \leq e, \\ (\log z)^{\sigma_2-1}(1-\log s)^{\sigma_2-1}, & 1 \leq z \leq s \leq e, \end{cases} \tag{2.8}$$

and $\Delta^* = 1 - \sum_{j=1}^{q} b_j(\log\xi_j)^{\sigma_2-1}$, $\Delta_1^* = \Delta^* - \int_1^e h(z)(\log z)^{\sigma_2-1}\frac{dz}{z}$.

Set $n_{\sigma_i}(z) = (\log z)^{\sigma_i-1}(1-\log z), m_{\sigma_i}(z) = (1-\log z)^{\sigma_i-1}\log z$, for $\sigma_i > 2$, $z \in [1,e]$, $i = 1,2$.

Lemma 2.4 *[35] $h(z,s), k(z,s)$ introduced by (2.3) and (2.8) have the below features:*

(i) $h(z,s)$ and $k(z,s)$ are continuous functions and $h(z,s) \geq 0$, $k(z,s) \geq 0$ for any $z,s \in [1,e]$;

(ii) $n_{\sigma_1}(z)m_{\sigma_1}(s) \leq \Gamma(\sigma_1)h(z,s) \leq (\sigma_1 - 1)m_{\sigma_1}(s)$, for any $z,s \in [1,e]$,

(iii) $n_{\sigma_2}(z)m_{\sigma_2}(s) \leq \Gamma(\sigma_2)k(z,s) \leq (\sigma_2 - 1)m_{\sigma_2}(s)$, for any $z,s \in [1,e]$.

Lemma 2.5 *Let $\mu_1 = \frac{\sigma_1-1}{\Gamma(\sigma_1)} + \sum_{i=1}^{k} \frac{a_i(\sigma_1-1)}{\Delta\Gamma(\sigma_1)}$, $\mu_2 = \frac{\sigma_2-1}{\Gamma(\sigma_2)} + \sum_{j=1}^{q} \frac{b_j(\sigma_2-1)}{\Delta^*\Gamma(\sigma_2)}$, $\mu_1^* = 1 + \frac{1}{\Delta_1}\int_1^e g(z)\frac{dz}{z}$, $\mu_2^* = 1 + \frac{1}{\Delta_1^*}\int_1^e h(z)\frac{dz}{z}$ and $V_1 = \sum_{i=1}^{k} \frac{a_i n_{\sigma_1}(\eta_i)}{\Delta\Gamma(\sigma_1)}$, $V_2 = \sum_{j=1}^{q} \frac{b_j n_{\sigma_2}(\xi_j)}{\Delta^*\Gamma(\sigma_2)}$. Then, the functions $H(z,s)$ and $K(z,s)$ introduced by (2.2) and (2.7) ensure the following features:*

(i) $H(z,s)$, $K(z,s)$ are continuous and $H(z,s) \geq 0$, $K(z,s) \geq 0$, for $(z,s) \in [1,e] \times [1,e]$;

(ii)$V_1(\log z)^{\sigma_1-1}m_{\sigma_1}(s) \leq H(z,s) \leq \mu_1\mu_1^* m_{\sigma_1}(s)$, for $(z,s) \in [1,e] \times [1,e]$;

(iii)$V_2(\log z)^{\sigma_2-1}m_{\sigma_2}(s) \leq K(z,s) \leq \mu_2\mu_2^* m_{\sigma_2}(s)$, for $(z,s) \in [1,e] \times [1,e]$.

Proof. It can be shown that (i) ensures. To show (ii), for $(z,s) \in [1,e] \times [1,e]$, we get,

$$\begin{aligned}
H(z,s) &= H_1(z,s) + H_2(z,s) \\
&= h(z,s) + \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}}{\Delta} h(\eta_i, s) \\
&\quad + \frac{(\log z)^{\sigma_1-1}}{\Delta_1} \int_1^e H_1(z,s)g(z)\frac{dz}{z} \\
&\leq \frac{(\sigma_1-1)m_{\sigma_1}(s)}{\Gamma(\sigma_1)} + \sum_{i=1}^{k} \frac{a_i(\sigma_1-1)m_{\sigma_1}(s)}{\Delta\Gamma(\sigma_1)} \\
&\quad + \frac{1}{\Delta_1}\int_1^e \left(h(z,s) + \sum_{i=1}^{k}\frac{a_i}{\Delta}h(\eta_i,s)\right)g(z)\frac{dz}{z} \\
&\leq \frac{(\sigma_1-1)m_{\sigma_1}(s)}{\Gamma(\sigma_1)} + \sum_{i=1}^{k} \frac{a_i(\sigma_1-1)m_{\sigma_1}(s)}{\Delta\Gamma(\sigma_1)} \\
&\quad + \frac{1}{\Delta_1}\int_1^e \left(\frac{(\sigma_1-1)m_{\sigma_1}(s)}{\Gamma(\sigma_1)} + \sum_{i=1}^{k} \frac{a_i(\sigma_1-1)m_{\sigma_1}(s)}{\Delta\Gamma(\sigma_1)}\right)g(z)\frac{dz}{z} \\
&= \mu_1\mu_1^* m_{\sigma_1}(s),
\end{aligned}$$

and

$$\begin{aligned}
H(z,s) \geq& \sum_{i=1}^{k} \frac{a_i(\log z)^{\sigma_1-1}}{\Delta} h(\eta_i,s) \\
\geq& (\log z)^{\sigma_1-1}\sum_{i=1}^{k}\frac{a_i n_{\sigma_1}(\eta_i)m_{\sigma_1}(s)}{\Delta\Gamma(\sigma_1)} \\
=& V_1(\log z)^{\sigma_1-1}m_{\sigma_1}(s).
\end{aligned}$$

In a similar manner, we observe that (iii) holds.□

Using Lemma 2.2 and Lemma 2.3, let $A : M \to \mathfrak{X}$ be a operator by

$$A\kappa(z) = \int_1^e K(z,s)\phi_q\bigg(\int_1^e H(s,\tau)\varrho(\tau,\kappa(\tau),{}^{H}\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau)\bigg)\frac{d\tau}{\tau}\frac{ds}{s}.$$

Clearly, if κ is a fixed point of the operator A, then κ represents a solution for the problem (1.1).

Lemma 2.6 *Let $(C1)$ and $(C2)$ hold. Then, $A : \mathrm{M} \to \mathrm{M}$ is a completely continuous operator.*

Proof. Let us demonstrate that $A(M) \subset M$. $H(z,s)$, $K(z,s)$, ϱ are continuous functions. Thus, A is continuous. Lemma 2.5 and the selection of ϱ satisfy that $A(z) \geq 0$ for $z \in [1,e]$. Hence, $A : M \to M$. Furthermore, by the Arzela–Ascoli theorem, it is obtained that A is a completely continuous operator. □

Now, the fixed point theorem is demonstrated, which is fundamental in our main result's proof.

Theorem 2.1 *[20] Let $\mathfrak{X}$ be a Banach space, $M \subseteq \mathfrak{X}$ a cone of $\mathfrak{X}$. Set*

$$\begin{aligned} M_y &= \{\kappa \in M : \|\kappa\| < y\}, \\ M(\zeta, j_1, j_2) &= \{\kappa \in M : j_1 \leq \zeta(\kappa), ||\kappa|| \leq j_2\}. \end{aligned}$$

Assume $A : \overline{M_y} \to \overline{M_y}$ be a completely continuous operator and ζ be a nonnegative, continuous, concave functional on M with $\zeta(\kappa) \leq ||\kappa||$ for all $\kappa \in \overline{M_y}$. If there exists $0 < p < n < d \leq y$ such that the following conditions hold:

(B_1) $\{\kappa \in M(\zeta,n,d) : \zeta(\kappa) > n\} \neq \emptyset$ *and* $\zeta(A\kappa) > n$ *for all* $\kappa \in M(\zeta,n,d)$;

(B_2) $||A\kappa|| < p$ *for all* $||\kappa|| \leq p$;

(B_3) $\zeta(A\kappa) > n$ *for* $\kappa \in M(\zeta,n,y)$ *with* $||A\kappa|| > d$.

Then A has at least three positive solutions κ_j for $j \in \{1,2,3\}$ in $\overline{M_y}$ satisfying

$$||\kappa_1|| < p,\ \zeta(\kappa_2) > n,\ \ p < ||\kappa_3|| \ \textit{with}\ \zeta(\kappa_3) < n.$$

For simplicity, let

$$\begin{aligned} &B = \mu_2\mu_2^*\phi_q(\mu_1\mu_1^*)\int_1^e m_{\sigma_2}(s)\phi_q\bigg(\int_1^e m_{\sigma_1}(\tau)\frac{d\tau}{\tau}\bigg)\frac{ds}{s}, \\ &C = \phi_q(\mu_1\mu_1^*)\phi_q\bigg(\int_1^e m_{\sigma_1}(\tau)\frac{d\tau}{\tau}\bigg), \\ &F = V_2(\log\eta)^{\sigma_2-1}\phi_q(V_1(\log\eta)^{\sigma_1-1})\int_\eta^e m_{\sigma_2}(s)\frac{ds}{s}\phi_q\bigg(\int_\eta^e m_{\sigma_1}(\tau)\frac{d\tau}{\tau}\bigg), \\ &N = V_2(\log\eta)^{\sigma_2-1}\phi_q(V_1(\log\eta)^{\sigma_1-1})\int_\eta^e m_{\sigma_2}(s)\frac{ds}{s}. \end{aligned}$$

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3 Existence Theorem

Here, we ensure the required conditions to get at least three positive solutions for the HFBVP (1.1). Firstly, we introduce the nonnegative, continuous, concave functional $\Lambda : M \to [0, +\infty)$ given by

$$\Lambda(\kappa) = \min_{z\in[\eta,e]} |\kappa(z)|, \quad \forall \kappa \in M,$$

where $\eta \in (1, e)$. It is obvious that $\Lambda(\kappa) \leq ||\kappa||$ for all $\kappa \in M$.

Theorem 3.1 *Consider that* $(C1)$-$(C2)$ *are satisfied and there exist* $0 < h < L < K < r$ *and* $\frac{K(\log \eta)^{\sigma_2}}{\Gamma(\sigma_2+1)} > L$, $K \geq N^{-1} L \max\{1, \mu_2\mu_2^* \int_1^e m_{\sigma_2}(s)\frac{ds}{s}\}\phi_q(\mu_1\mu_1^*)$, $\frac{L}{F} < \min\{\frac{r}{B}, \frac{r}{C}\}$ *such that the following hypotheses on* ϱ *are satisfied.*

(i)$\varrho(z, \kappa, \iota) > \phi_p(\dfrac{L}{F})$ *for* $(z, \kappa, \iota) \in [\eta, e] \times [L, K] \times [-K, 0]$,

(ii)$\varrho(z, \kappa, \iota) \leq \min\{\phi_p(\dfrac{r}{B}), \phi_p(\dfrac{r}{C})\}$ *for* $(z, \kappa, \iota) \in [1, e] \times [0, r] \times [-r, 0]$,

(iii)$\varrho(z, \kappa, \iota) < \min\{\phi_p(\dfrac{h}{B}), \phi_p(\dfrac{h}{C})\}$ *for* $(z, \kappa, \iota) \in [1, e] \times [0, h] \times [-h, 0]$.

Then, the HFBVP (1.1) *has at least three positive solutions* κ_j *for* $j \in \{1, 2, 3\}$ *satisfying*

$$||\kappa_1|| < h, \;\; \Lambda(\kappa_2) > L, \;\;\; h < ||\kappa_3|| \textit{ with } \Lambda(\kappa_3) < L.$$

Proof. We will ensure that the conditions of Theorem 2.1 will be achieved. First, we will demonsrate $A : \overline{M_r} \to \overline{M_r}$. If $\kappa \in \overline{M_r}$, then $||\kappa|| \leq r$, using Lemma 2.5 and assumption (ii), for $z \in [1, e]$, we get

$$\begin{aligned}
|A\kappa(z)| &= \left| \int_1^e K(z, s)\phi_q\left(\int_1^e H(s, \tau)\varrho(\tau, \kappa(\tau), {}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau} \right) \frac{ds}{s} \right| \\
&\leq \mu_2\mu_2^* \int_1^e m_{\sigma_2}(s)\phi_q\left(\int_1^e \mu_1\mu_1^* m_{\sigma_1}(\tau)\varrho(\tau, \kappa(\tau), {}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\leq \frac{r}{B}\mu_2\mu_2^*\phi_q(\mu_1\mu_1^*) \int_1^e m_{\sigma_2}(s)\phi_q\left(\int_1^e m_{\sigma_1}(\tau)\frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&= r,
\end{aligned}$$

and

$$\begin{aligned}
|{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(z)| &= \left| -\phi_q\left(\int_1^e H(s, \tau)\varrho(\tau, \kappa(\tau), {}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau} \right) \right| \\
&\leq \phi_q(\mu_1\mu_1^*)\phi_q\left(\int_1^e m_{\sigma_1}(\tau)\varrho(\tau, \kappa(\tau), {}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau} \right) \\
&\leq \frac{r}{C}\phi_q(\mu_1\mu_1^*)\phi_q\left(\int_1^e m_{\sigma_1}(\tau)\frac{d\tau}{\tau} \right) \\
&= r.
\end{aligned}$$

Thus, $A : \overline{M_r} \to \overline{M_r}$. By (iii) and similarly to the proof above, it can be obtained that $A : \overline{M_h} \to M_h$.

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Now, we ensure that condition $(B1)$ of Theorem 2.1 is obtained. By choosing $\kappa(z) = \frac{K(\log z)^{\sigma_2}}{\Gamma(\sigma_2+1)}$ for $z \in [1,e]$, we obtain, $\max_{z\in[1,e]} |\kappa(z)| = \dfrac{K}{\Gamma(\sigma_2+1)}$, $\max_{z\in[1,e]} |\mathfrak{D}^{\sigma_2}\kappa(z)| = K$ and $\Lambda(\kappa) = \min_{z\in[\eta,e]} |\kappa(z)| = \frac{K(\log \eta)^{\sigma_2}}{\Gamma(\sigma_2+1)} > L$ and $\|\kappa\| = K$. Hence, $\kappa(z) = \frac{K(\log z)^{\sigma_2}}{\Gamma(\sigma_2+1)} \in M(\Lambda, L, K)$, that is,

$$\{\kappa \in M(\Lambda, L, K) : \Lambda(\kappa) > L\} \neq \emptyset.$$

If $\kappa \in M(\Lambda, L, K)$, then $L \leq \kappa(z) \leq K$ and $-K \leq \mathfrak{D}^{\sigma_2}\kappa(z) \leq 0$ for any $z \in [\eta, e]$. Using (i), we ensure

$$\begin{aligned}
\Lambda(A\kappa) &= \min_{z\in[\eta,e]} A\kappa(z) \\
&\geq V_2(\log\eta)^{\sigma_2-1}\int_\eta^e m_{\sigma_2}(s)\phi_q\bigg(\int_\eta^e V_1(\log s)^{\sigma_1-1} m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\gamma-1}\kappa(\tau))\frac{d\tau}{\tau}\bigg)\frac{ds}{s} \\
&\geq V_2(\log\eta)^{\sigma_2-1}\phi_q(V_1(\log\eta)^{\sigma_1-1})\int_\eta^e m_{\sigma_2}(s)\frac{ds}{s}\phi_q\bigg(\int_\eta^e m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\gamma-1}\kappa(\tau))\frac{d\tau}{\tau}\bigg) \\
&> \frac{L}{F}V_2(\log\eta)^{\sigma_2-1}\phi_q(V_1(\log\eta)^{\sigma_1-1})\int_\eta^e m_{\sigma_2}(s)\frac{ds}{s}\phi_q\bigg(\int_\eta^e m_{\sigma_1}(\tau)\frac{d\tau}{\tau}\bigg) \\
&= L.
\end{aligned}$$

This proves condition $(B1)$ of Theorem 2.1.

Lastly, we will prove the condition $(B3)$ of Theorem 2.1. If $\kappa \in M(\Lambda, L, r)$ and $||A\kappa|| > K$, for $z \in [1,e]$,

$$\begin{aligned}
|A\kappa(z)| &= \bigg|\int_1^e K(z,s)\phi_q\bigg(\int_1^e H(s,\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\bigg)\frac{ds}{s}\bigg| \\
&\leq \mu_2\mu_2^*\phi_q(\mu_1\mu_1^*)\int_1^e m_{\sigma_2}(s)\phi_q\bigg(\int_1^e m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\bigg)\frac{ds}{s} \\
&= \mu_2\mu_2^*\phi_q(\mu_1\mu_1^*)\int_1^e m_{\sigma_2}(s)\frac{ds}{s}\phi_q\bigg(\int_1^e m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\bigg),
\end{aligned}$$

and

$$\begin{aligned}
|{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(z)| &= \bigg| -\phi_q\bigg(\int_1^e H(s,\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\bigg)\bigg| \\
&\leq \phi_q(\mu_1\mu_1^*)\phi_q\bigg(\int_1^e m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\bigg).
\end{aligned}$$

Thus, we ensure

$$||A\kappa|| \leq \max\{1, \mu_2\mu_2^*\int_1^e m_{\sigma_2}(s)\frac{ds}{s}\}\phi_q(\mu_1\mu_1^*)\phi_q\bigg(\int_1^e m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\bigg).$$

Therefore,

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$$
\begin{aligned}
\Lambda(A\kappa) &= \min_{z\in[\eta,e]} A\kappa(z) \\
&\geq V_2(\log\eta)^{\sigma_2-1}\int_1^e m_{\sigma_2}(s)\phi_q\Big(\int_1^e V_1(\log s)^{\sigma_1-1}m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} \\
&\geq V_2(\log\eta)^{\sigma_2-1}\phi_q(V_1(\log\eta)^{\sigma_1-1})\int_\eta^e m_{\sigma_2}(s)\frac{ds}{s}\phi_q\Big(\int_1^e m_{\sigma_1}(\tau)\varrho(\tau,\kappa(\tau),{}^H\mathfrak{D}_{1^+}^{\sigma_2}\kappa(\tau))\frac{d\tau}{\tau}\Big) \\
&\geq \frac{N}{\max\{1,\mu_2\mu_2^*\int_1^e m_{\sigma_2}(s)\frac{ds}{s}\}\phi_q(\mu_1\mu_1^*)}||A\kappa|| \\
&> \frac{NK}{\max\{1,\mu_2\mu_2^*\int_1^e m_{\sigma_2}(s)\frac{ds}{s}\}\phi_q(\mu_1\mu_1^*)} \\
&\geq L.
\end{aligned}
$$

Then, $(B3)$ of Theorem 2.1 is also verified. Then, all conditions of Theorem 2.1 are confirmed, the problem (1.1) has at least three positive solutions such that

$$||\kappa_1|| < h, \quad L < \Lambda(\kappa_2) \text{ and } h < ||\kappa_3|| \text{ with } \Lambda(\kappa_3) < L.$$

□

Example 3.1 *Let us consider the following HFBVP*

$$
\begin{cases}
{}^H\mathfrak{D}_{1^+}^{14/5}(\phi_2({}^H\mathfrak{D}_{1^+}^{14/5}\kappa(z))) = \varrho(z,\kappa(z),{}^H\mathfrak{D}_{1^+}^{14/5}\kappa(z)), \quad z\in(1,e), \\
\kappa(1)=\kappa'(1)={}^H\mathfrak{D}_{1^+}^{14/5}\kappa(1)=(\phi_2({}^H\mathfrak{D}_{1^+}^{14/5}\kappa(1))'=0, \\
\phi_2({}^H\mathfrak{D}_{1^+}^{14/5}\kappa(e)) = \phi_2({}^H\mathfrak{D}_{1^+}^{14/5}\kappa(e^{1/2}))+3\phi_2({}^H\mathfrak{D}_{1^+}^{14/5}\kappa(e^{1/4}))+\int_1^e\phi_2({}^H\mathfrak{D}_{1^+}^{14/5}\kappa(z))\frac{dz}{z}, \\
\kappa(e)=\frac{1}{3}\kappa(e^{1/2})+\frac{2}{3}\kappa(e^{1/4})+\int_1^e\kappa(z)\frac{dz}{z},
\end{cases}
\tag{3.1}
$$

where $p=2$, $\sigma_1,\sigma_2=\frac{14}{5}$, $k=q=2$, $\eta_1=e^{1/2}$, $\eta_2=e^{1/4}$, $\xi_1=e^{1/2}$, $\xi_2=e^{1/4}$, $a_1=1$, $a_2=3$, $b_1=\frac{1}{3}$, $b_2=\frac{2}{3}$, $\eta=e^{7/10}$, $g(z)=h(z)=1$ *for* $z\in[1,e]$ *and*

$$
\varrho(z,\kappa,\iota)=\begin{cases}
e^{-10z}+\frac{2\kappa}{300}+\frac{|\iota|}{10^5}, & \kappa\in[0,0.3], \\
e^{-10z}+(10\kappa-3)(10^5-0.002)+0.002+\frac{|\iota|}{10^5}, & \kappa\in[0.3,0.4], \\
e^{-10z}+\frac{1000\kappa-400}{899999.6}+10^5+\frac{|\iota|}{10^5}, & \kappa\in[0.4,\infty),
\end{cases}
$$

for $z\in[1,e],\iota\in[0,\infty)$. *After calculating directly, we obtain* $B\approx 5.9268$, $C\approx 8.8956$, $N\approx 0.0002$, $F\approx 0.000004$. *If* $h=0.3$, $L=0.4$, $K=200000$ *and* $r=900000$,

$$
\begin{aligned}
&\varrho(z,\kappa,\iota) > \phi_p(\frac{L}{F}) \approx 100000 \text{ for } (z,\kappa,\iota)\in[e^{7/10},e]\times[0.4,200000]\times[-200000,0], \\
&\varrho(z,\kappa,\iota) \leq \min\{\phi_p(\frac{r}{B}),\phi_p(\frac{r}{C})\} \approx 101173 \text{ for } (z,\kappa,\iota)\in[1,e]\times[0,900000]\times[-900000,0], \\
&\varrho(z,\kappa,\iota) < \min\{\phi_p(\frac{h}{B}),\phi_p(\frac{h}{C})\} \approx 0.0034 \text{ for } (z,\kappa,\iota)\in[1,e]\times[0,0.3]\times[-0.3,0].
\end{aligned}
$$

i.e., ϱ *satisfies the conditions of Theorem 3.1. By Theorem 2.1, the problem* (3.1) *has at least three positive solutions* κ_j *for* $j\in\{1,2,3\}$ *with*

$$||\kappa_1|| < 0.3, \ \Lambda(\kappa_2) > 0.4, \quad 0.3 < ||\kappa_3|| \text{ with } \Lambda(\kappa_3) < 0.4.$$

Conclusion

This study examined the existence results of fractional boundary value problem by the Hadamard fractional derivative. With the help of the Green's function's properties and the fixed point theory, we obtained the appropriate conditions to ensure the existence results for the HFBVP including p-Laplacian operator. To our knowledge, few researchers worked the HFBVP with p-Laplacian operator. Moreover, the function ϱ which consists of the Hadamard derivative operator ${}^{H}\mathfrak{D}_{1^+}^{\sigma_2}$ has been studied by only a few researchers.

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