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POSITIVE PERIODIC SOLUTION FOR LIÉNARD EQUATION WITH DEVIATING ARGUMENT

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ABSTRACT. In this paper, we consider a kind of Liénard equations with deviating argument. Using the fixed point theorem in cone and analytical technique, we obtain the existence of positive periodic solution to the problem under the appropriate conditions. Some examples are presented to illustrate our results.

1. Introduction

In this paper, we study the existence of positive periodic solution for the following equation

$$x''(t)+f(x(t))x'(t)+g(t,x(t-\tau(t)))=0, \tag{1.1}$$

where τ is a ω-periodic continuous function, $g(t,u)$ is ω-periodic in t and $\omega>0$.

The existence of periodic solution is an important aspect in qualitative analysis for differential equation. Much work about periodic solutions for differential equations has been done by using various theorems and methods of nonlinear functional analysis, see [1–3, 6, 7, 10, 15, 20, 22, 24–26] and the references therein. When $\tau\equiv 0$, (1.1) is reduced as the usual Liénard equation. There are many work about periodic solution(s) or positive periodic solution(s) for Liénard equation, see [4, 5, 8, 9, 13, 14, 17–19, 23, 28] and references therein. Recently, some researchers have focused on periodic solutions to Liénard equation of deviating argument(s). Zhao and Nagy [27] discussed the special cases of (1.1)

$$x''(t)+h(x(t))x'(t)+k(x(t-r))=0, \tag{1.2}$$

where $h,k\in C^4$, $h(0)>0, k(0)=0, k'(0)=1$, $r>0$ is a constant. The authors proved the existence of the Hopf bifurcation by center manifold analysis.

Zhou and Long [29] discussed the Liénard equation with two deviating arguments of the form

$$x''(t)+h(x(t))x'(t)+g_1(t,x(t-\tau_1(t)))+g(t,x(t-\tau_2(t)))=v(t), \tag{1.3}$$

where $h,\tau_1,\tau_2,v:\mathbb{R}\to\mathbb{R}$ and $g_1,g_2:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, τ_1,τ_2 and v are ω-periodic, g_1 and g_2 are ω-periodic in the first argument. By using coincidence degree theory, they established some results on the existence and uniqueness of ω-periodic solution for (1.3). One can refer to [12, 16] for similar method.

Compared with periodic solution(s) of Liénard equation, the existence of positive periodic solutions for Liénard equation with deviating argument is considerably less often. Due to the influence of delay, some common methods for dealing with periodic solution problem cannot been directly applied to

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study the existence of positive periodic solution for Liénard equation with deviating argument. To the best of our knowledge, work about positive periodic solution for the following linear differential equation

$$x''(t)+cx'(t)+p(t)x(t-\tau(t))=q(t) \tag{1.4}$$

are few so far, here c is a constant and p,q,τ are ω-periodic continuous functions.

The main purpose of this paper is to show the existence of positive periodic solution for (1.1) by means of fixed point theorem in cone. To this end, we transform the original equation into first order functional differential system. With proper transformation, we need to consider only one first-order nonlinear functional differential equation. The existence of single positive periodic solution for (1.1) has been established under suitable behavior of f and g on some closed sets. So some information on the location of positive periodic solution is also obtained, leading to multiplicity results.

The paper is organized as follows. In Section 2, we prove a key lemma by using Schauder's fixed point theorem. In Section 3, the existence of positive periodic solution is studied with the help of the fixed point theorem in cone of Banach space. In Section 4, by applying the result obtained, we give some conditions guaranteeing that (1.4) has at least a positive ω-periodic solution.

2. Preliminaries

Let $X=\{u\in C(\mathbb{R},\mathbb{R}):u(t+\omega)=u(t),t\in\mathbb{R}\}$ with the norm $\|u\|=\max_{t\in[0,w]}|u(t)|$.

Consider the equations

$$u'(t)=k(t)u(t)-q(t), \tag{2.1}$$

$$u'(t)=-k(t)u(t)+q(t), \tag{2.2}$$

$$u'(t)=-a(u(t))+h(t), \tag{2.3}$$

where $k,q,h\in X$.

Lemma 2.1. *[25] Assume that $\int_0^\omega k dt\neq 0$, then (2.1) has a unique periodic solution*

$$x_1(t)=\int_t^{t+\omega}G_1^k(t,s)q(s)ds$$

and (2.2) has a unique periodic solution

$$x_2(t)=\int_t^{t+\omega}G_2^k(t,s)q(s)ds,$$

where

$$G_1^k(t,s)=\frac{\exp\int_s^{t+\omega}k(r)dr}{\exp\int_0^\omega k(r)dr-1},\quad G_2^k(t,s)=\frac{\exp\int_t^{s}k(r)dr}{\exp\int_0^\omega k(s)dr-1}.$$

We need the following well-known Schauder's fixed point theorem in our arguments.

Lemma 2.2. *Let X be a Banach space with $D\subset X$ closed and convex. Assume that $T:D\to D$ is a completely continuous operator, then T has a fixed point in D.*

Lemma 2.3. *Assume that there are $M > m$ such that $a \in C^1[m,M]$ and for all $t \in \mathbb{R}$,*

$$a(m) \le h \le a(M) \ \text{ or } \ a(M) \le h \le a(m).$$

Then (2.3) has at least one ω-periodic solution $\bar{u}$ with $m \le \bar{u} \le M$. Further suppose that a is strictly monotone in $\mathbb{R}$, the periodic solution of (2.3) is unique.

Proof. Since $a \in C^1[m,M]$, there exists $L > 0$ such that

$$Ls + a(s), \ \ Ls - a(s)$$

are strictly increasing in $[m,M]$.

Define operators T_1 and T_2 in X by

$$(T_1u)(t) = \int_t^{t+\omega} G_1^L(t,s)[Lu(s) + a(u(s)) - h(s)]ds,$$

$$(T_2u)(t) = \int_t^{t+\omega} G_2^L(t,s)[h(s) + Lu(s) - a(u(s))]ds.$$

It is easy to check that the fixed points of T_1, T_2 on X are the periodic solutions of (2.3). Let $\Lambda = \{u \in X, \ \ m \le u \le M\}$.

If $a(m) \le h \le a(M)$, for $\forall u \in \Lambda$, we have

$$Lm \le h(t) + Lu(t) - a(u(t)) \le h(t) + LM - a(M) \le LM, \ \ \forall t \in \mathbb{R},$$

$$\begin{aligned} m &= \int_t^{t+\omega} G_2^L(t,s)Lmds \\ &\le \int_t^{t+\omega} G_2^L(t,s)[h(s) + Lu(s) - a(u(s))]ds \\ &\le \int_t^{t+\omega} G_2^L(t,s)LMds = M, \end{aligned}$$

which implies that $T_2(\Lambda) \subset \Lambda$. Similarly, $T_1(\Lambda) \subset \Lambda$ if $a(M) \le h \le a(m)$.

In addition,

$$(T_1u)'(t) = L(T_1u)(t) - Lu(t) - a(u(t)) + h(t),$$
$$(T_2u)'(t) = -L(T_2u)(t) + Lu(t) + h(t) - a(u(t)).$$

from which it follows that there exists $C > 0$ such that

$$|(T_1u)'| \le C, \ \ |(T_2u)'| \le C$$

for $\forall u \in \Lambda$. Hence, $\forall u \in \Lambda$ and $\forall t_1, t_2 \in \mathbb{R}$,

$$|(T_1u)(t_1) - (T_1u)(t_2)| \le C|t_1 - t_2|,$$
$$|(T_2u)(t_1) - (T_2u)(t_2)| \le C|t_1 - t_2|,$$

which imply that $T_1, T_2 : \Lambda \to \Lambda$ are completely continuous. Using Lemma 2.3, T_1 or T_2 has a fixed point on Λ, which is a periodic solution of (2.3).

Further suppose that a is strictly monotone in $\mathbb{R}$. Assume that u_1 and u_2 are two ω-periodic solutions of (2.3), we have

$$(u_1 - u_2)' + a(u_1(t)) - a(u_2(t)) = 0,$$

$$\int_0^\omega [a(u_1(s)) - a(u_2(s))](u_1(s) - u_2(s))ds = 0,$$

which implies that

$$[a(u_1(s)) - a(u_2(s))][u_1(s) - u_2(s)] \equiv 0, \ \ \forall s,$$

$$u_1 \equiv u_2,$$

since $[a(u_1) - a(u_2)](u_1 - u_2)$ is of sign. Hence, the periodic solution of (2.3) is unique. □

Remark 2.1. *Under the conditions of Lemma 2.3, one can define an operator T_a by*

$$T_a(h) = \bar{u}.$$

Moreover, $T_a : X \to X$ is continuous.

3. Main results

At this section, we always assume that the following condition is satisfied

(H) There are $0 \le a < b \le +\infty$ such that $f \in C[a,b], f(a) \ne 0, f(b) \ne 0$ and

$$F(x) = \int_a^x f(t)dt$$

is strictly monotone, where $f(b) = \lim_{t\to+\infty} f(t)$ if $b = +\infty$.

Let

$$\widetilde{f}(t) = \begin{cases} f(b), & t \ge b, \\ f(t), & a \le t < b, \\ f(a), & t < a, \end{cases} \quad \widetilde{F}(t) = \int_a^t \widetilde{f}(s)ds.$$

then $\widetilde{F}$ is strictly monotone on $\mathbb{R}$, $\lim_{|t|\to\infty} \widetilde{F}(t) = \infty$ and thus the inverse $\widetilde{F}^{-1}$ of $\widetilde{F}$ exists,

$$\widetilde{F}^{-1}(t) = F^{-1}(t), \ \ \forall \min\{0, F(b)\} \le t \le \max\{0, F(b)\},$$

where F^{-1} is the inverse of F.

Consider the equation

$$u''(t) + \widetilde{f}(u(t))u'(t) + g(t, u(t - \tau(t))) = 0, \tag{3.1}$$

If the periodic solution u of (3.1) satisfies that $a \le u \le b$, u is also periodic solution of (1.1). Now, suppose that u is a periodic solution of (3.1) and $y = \exp(u'(t) + \widetilde{F}(u(t)))$, then $y > 0$ and

$$\begin{cases} u'(t) + \widetilde{F}(u(t)) = \ln y(t), \\ y'(t) = -y(t)g(t, u(t - \tau(t))). \end{cases}$$

Noting that $\widetilde{F}$ is strictly monotone, by Lemma 2.3, we have

$$\min\{\widetilde{F}^{-1}(\ln y_*), \widetilde{F}^{-1}(\ln y^*)\} \le T_{\widetilde{F}}(\ln y) \le \max\{\widetilde{F}^{-1}(\ln y_*), \widetilde{F}^{-1}(\ln y^*)\},$$

where

$$y^* = \max_{t\in\mathbb{R}} y(t), \quad y_* = \min_{t\in\mathbb{R}} y(t).$$

Moreover,

$$y' = \mu y - y[\mu + g(t, T_{\widetilde{F}}(\ln y)(t-\tau(t)))],$$

or

$$y' = -\mu y + y[\mu - g(t, T_{\widetilde{F}}(\ln y)(t-\tau(t)))],$$

where $\mu \in \mathbb{R}$.

Define the operators and the cone in X by

$$(Ay)(t) = \int_t^{t+\omega} G_1^{\mu}(t,s)y(s)[\mu + g(s, T_{\widetilde{F}}(\ln y)(s-\tau(s)))]ds,$$

$$(By)(t) = \int_t^{t+\omega} G_2^{\mu}(t,s)y(s)[\mu - g(s, T_{\widetilde{F}}(\ln y)(s-\tau(s)))]ds,$$

$$K = \{u \in X, u \geq k\|u\|\}, \ \ k = e^{-\mu\omega}.$$

Theorem 3.1. *There exist $\mu > 0$ and $R > r > 0$ such that*

(H_1) $\min\{0, F(b)\} \leq \ln kr, \ln R \leq \max\{0, F(b)\}$,

(H_2) $g : \mathbb{R} \times [\min\{F^{-1}(\ln kr), F^{-1}(\ln R)\}, \max\{F^{-1}(\ln R), F^{-1}(\ln kr)\}] \to \mathbb{R}$ *is continuous and*

$$g(t,u) > -\mu \ \ or \ \ g(t,u) < \mu$$

for $(t,u) \in \mathbb{R} \times [\min\{F^{-1}(\ln kr), F^{-1}(\ln R)\}, \max\{F^{-1}(\ln R), F^{-1}(\ln kr)\}]$.

(H_3) $\max_{\Lambda_\alpha \leq s \leq \bar{\Lambda}_\alpha} \int_0^{\omega} g(t,s)dt < 0 < \min_{\Lambda_\beta \leq s \leq \bar{\Lambda}_\beta} \int_0^{\omega} g(t,s)dt$, *where* $\{\alpha, \beta\} = \{R, r\}$,

$$\bar{\Lambda}_p = \max\{F^{-1}(\ln p), F^{-1}(\ln kp)\}, \ \ \Lambda_p = \min\{F^{-1}(\ln p), F^{-1}(\ln kp)\}.$$

Then (1.1) *has a positive solution $x \in X$ with*

$$\min\{F^{-1}(\ln R), F^{-1}(\ln kr)\} \leq x \leq \max\{F^{-1}(\ln R), F^{-1}(\ln kr)\}.$$

Remark 3.1. *$F^{-1}(\ln R), F^{-1}(\ln r), F^{-1}(\ln kr)$ and $F^{-1}(\ln kR)$ are well-defined since (H) and (H_1) are satisfied.*

Proof. Without loss of generality, we assume that $\alpha = R, \beta = r$ and

$$g(t,u) < \mu$$

for $(t,u) \in \mathbb{R} \times [\min\{F^{-1}(\ln kr), F^{-1}(\ln R)\}, \max\{F^{-1}(\ln kr), F^{-1}(\ln R)\}]$.

Let $\Omega_l = \{v \in X : \|u\| < l\}$. At first, we show that $B : K \cap (\bar{\Omega}_R/\Omega_r) \to K$. For any $u \in K \cap (\bar{\Omega}_R/\Omega_r), kr \leq u \leq R$ for $t \in \mathbb{R}$ and

$$\min\{F^{-1}(\ln kr), F^{-1}(\ln R)\} \leq T_{\widetilde{F}}(\ln u) \leq \max\{F^{-1}(\ln kr), F^{-1}(\ln R)\},$$

$$g(t, T_{\widetilde{F}}(\ln u)(t-\tau(t))) < \mu, \ \ \forall t, \tag{3.2}$$

$$\begin{aligned}(Bu)(t+\omega) &= \int_{t+\omega}^{t+2\omega} \frac{e^{t+\omega-s}}{e^{\mu\omega}-1} u(s) \left[\mu - g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))\right] ds \\ &= \int_t^{t} \frac{e^{t+\omega-s}}{e^{\mu\omega}-1} u(s+\omega) \left[\mu - g(s, T_{\widetilde{F}}(\ln u)(s+\omega-\tau(s+\omega)))\right] ds = (Bu)(t),\end{aligned}$$

$$
\begin{aligned}
(Bu)(t) &= \int_t^{t+\omega} G_2^{\mu}(t,s)u(s)\left[\mu - g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))\right] ds \\
&\geq \frac{1}{e^{\mu\omega}-1}\int_t^{t+\omega} u(s)\left[\mu - g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))\right] ds, \\
(Bu)(t) &\leq \frac{e^{\mu\omega}}{e^{\mu\omega}-1}\int_t^{t+\omega} u(s)\left[\mu - g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))\right] ds,
\end{aligned}
$$

$$(Bu)(t) \geq k\|Bu\|.$$

Thus $B : K\cap\bar{\Omega}_R/\Omega_r \to K$. It is easy to check that $B : K\cap\bar{\Omega}_R/\Omega_r \to K$ is completely continuous.

Next, we show that

$$u \neq \lambda Bu, \quad u \in K\cap\partial\Omega_r \text{ and } 0 < \lambda \leq 1. \tag{3.3}$$

If it is not true, there exist $u \in K\cap\partial\Omega_r$ and $0 < \lambda \leq 1$ such that $u = \lambda Bu$. Noting that

$$(Bu)'(t) + \mu(Bu)(t) = u(t)[\mu - g(t, T_{\widetilde{F}}(\ln u)(t-\tau(t)))],$$

we have

$$
\begin{aligned}
u'(t) + \mu u(t) &= \lambda u(t)[\mu - g(t, T_{\widetilde{F}}(\ln u)(t-\tau(t)))], \\
(\ln u(t))' + \mu &= \lambda[\mu - g(t, T_{\widetilde{F}}(\ln u)(t-\tau(t)))].
\end{aligned}
$$

Integrating the equation above from 0 to w, we obtain

$$\lambda\int_0^{\omega} g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))ds = (\lambda-1)\mu w \leq 0, \tag{3.4}$$

where we use the fact $0 < \lambda \leq 1$ and $u(0) = u(w)$. Since $kr \leq u \leq r$ for $u \in K\cap\partial\Omega_r$, $\ln kr \leq \ln u \leq \ln r$ and

$$\Lambda_r = \min\{F^{-1}(\ln r), F^{-1}(\ln kr)\} \leq T_{\widetilde{F}}(\ln u) \leq \max\{F^{-1}(\ln r), F^{-1}(\ln kr)\} = \bar{\Lambda}_r. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\min_{\Lambda_r\leq s\leq\bar{\Lambda}_r}\int_0^{\omega} g(t,s)dt \leq 0, \tag{3.6}$$

which is a contradiction. So $u \neq \lambda Bu, \forall u \in K\cap\partial\Omega_r$ and $0 < \lambda \leq 1$.

Next, we show that

$$\inf\|Bu\| > 0, \quad u \neq \lambda Bu \quad \text{for} \quad u \in K\cap\partial\Omega_R, \lambda \geq 1.$$

Suppose that $\inf\|Bu\| = 0$ for $u \in K\cap\partial\Omega_R$, there exists sequence $\{u_n\} \subset \partial\Omega_R$ such that $\|Bu_n\| \to 0$ as $n \to \infty$. Hence, $kR \leq u_n(t) \leq R$ and

$$0 \leq \int_t^{t+\omega} G_2^{\mu}(t,s)u_n(s)[\mu - g(s, T_{\widetilde{F}}(\ln u_n)(s-\tau(s)))]ds \leq \|Bu_n\| \to 0 \text{ as } n \to \infty.$$

One easily obtain that

$$g(s, T_{\widetilde{F}}(\ln u_n)(s-\tau(s))) \to \mu \quad \text{as} \quad n \to \infty \tag{3.7}$$

for $\forall s$, which contradicts (3.2). Hence, $\inf\|Bu\| > 0$ for $u \in K\cap\partial\Omega_R$.

Suppose that $u = \lambda Bu$ for $u \in K \cap \partial\Omega_R$ and $\lambda \geq 1$. Similar to (3.4), we have

$$\lambda \int_0^{\omega} g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))ds = (\lambda - 1)\mu w \geq 0,$$

$$\int_0^{w} g(s, T_{\widetilde{F}}(\ln u)(s-\tau(s)))ds \geq 0. \tag{3.8}$$

Noting that $\ln kR \leq \ln u \leq \ln R$ for $u \in K \cap \partial\Omega_R$,

$$T_{\widetilde{F}}(\ln u) = T_F(\ln u) \in [\min\{F^{-1}(\ln R), F^{-1}(\ln kR)\}, \max\{F^{-1}(\ln R), F^{-1}(\ln kR)\}],$$

we have

$$\max_{\Lambda_R \leq s \leq \bar{\Lambda}_R} \int_0^{\omega} g(t,s)dt \geq 0,$$

which is a contradiction.

Therefore, there exists $u \in K \cap \bar{\Omega}_R / \Omega_r$ such that $Bu = u$. Moreover, $kr \leq u \leq R$ for all $t \in \mathbb{R}$. Let $x = T_{\widetilde{F}}(\ln u)$, then

$$\begin{cases} (Bu)' + \mu Bu = u[\mu - g(t, (T_{\widetilde{F}}(\ln u))(t-\tau(t)))], \\ x' + \widetilde{F}(x) = \ln u, \end{cases}$$

that is,

$$\begin{cases} u'(t) = -u(t)g(t, x(t-\tau(t))), \\ x'(t) + \widetilde{F}(x(t)) = \ln u(t), \end{cases}$$

$$x''(t) + \widetilde{f}(x(t))x'(t) = -g(t, x(t-\tau(t))).$$

Again $x = T_{\widetilde{F}}(\ln u) \in [a,b]$, we have

$$x''(t) + f(x(t))x'(t) + g(t, x(t-\tau(t))) = 0,$$

which implies that x is a positive periodic solution of (1.1).

If $g(t,u) > -\mu$, we consider the operator A in the similar way. □

Using the same idea, for the equation

$$x''(t) + f(x(t))x'(t) + G(t, x(t-\tau_1(t)), \dots, x(t-\tau_m(t))) = 0, \tag{3.9}$$

where $G(t+\omega, v_1, \dots, v_m) = G(t, v_1, \dots, v_m)$, τ_i are continuous ω-periodic functions, we have

Theorem 3.2. *There exist $\mu > 0, R > r > 0$ such that (H_1) holds and*

(H_4) *$G : \mathbb{R} \times [\min\{F^{-1}(\ln kr), F^{-1}(\ln R)\}, \max\{F^{-1}(\ln R), F^{-1}(\ln kr)\}]^m \to \mathbb{R}$ is continuous and*

$$G(t, v_1, .., v_m) > -\mu \ \ or \ \ G(t, v_1, \dots, v_m) < \mu$$

for $(t, v_1, \dots, v_m) \in \mathbb{R} \times [\min\{F^{-1}(\ln kr), F^{-1}(\ln R)\}, \max\{F^{-1}(\ln kr), F^{-1}(\ln R)\}]^m$.

(H_5)

$$\max_{\Lambda_\alpha \leq v_i \leq \bar{\Lambda}_\alpha, i=1,2,\dots m} \int_0^{\omega} G(t, v_1, , \dots, v_m)dt < 0 < \min_{\Lambda_\beta \leq v_i \leq \bar{\Lambda}_\beta, i=1,2,\dots m} \int_0^{\omega} G(t, v_1, \dots, v_m)dt,$$

where $\{\alpha, \beta\} = \{R, r\}$.

Then (3.9) *has a positive periodic solution x with*

$$\min\{F^{-1}(\ln R), F^{-1}(\ln kr)\} \le x \le \max\{F^{-1}(\ln R), F^{-1}(\ln kr)\}.$$

Example 3.1. Consider the equation

$$x''(t) + \frac{x'(t)}{x(t)} + \lambda \frac{2\sin t + \sin x(t-\tau)}{2 + x(t-\tau)} = 0, \tag{3.10}$$

where $\tau \in \mathbb{R}$.

We claim that for any $l \in \mathbb{N}$, there exists $\lambda_l > 0$ such that (3.10) has at least l positive 2π-periodic solutions for $|\lambda| \le \lambda_l$.

Clearly, $x \equiv C > 0$ is positive 2π-periodic solution for $\lambda = 0$. Now, let $\lambda \ne 0$ and choose $a = 1, b = +\infty$, then $F(t) = \int_1^u \frac{1}{t} dt = \ln u$ is strictly increasing for $u \ge 1$ and $F^{-1}(u) = e^u, u \ge 0$. Let

$$\lambda_l = \frac{1}{8\pi} \ln \frac{4l+4}{4l+3}, \quad \mu = 2|\lambda|, \quad r_n = (2n+1)\pi - \frac{\pi}{6}, \quad R_n = 2(n+1)\pi - \frac{\pi}{6}, \quad n = 0, \cdots, l-1.$$

Then,

$$|g(t,u)| < 2|\lambda|, \quad \forall t \in \mathbb{R}, u \in \mathbb{R},$$
$$k = e^{-4\pi|\lambda|} \ge e^{-\frac{1}{2}\ln\frac{4l+4}{4l+3}} > e^{-\frac{1}{8l+6}} > \frac{8l+5}{8l+6} > \frac{(2l+1)\pi + \frac{\pi}{2}}{R_l}, \quad kr_0 > 1,$$
$$2n\pi + \frac{\pi}{2} < kr_n < r_n, \quad (2n+1)\pi + \frac{\pi}{2} < kR_n < R_n, \; n = 0, 1, \ldots, l,$$
$$kr_n > R_{n-1}, \quad n = 1, 2, \ldots, l-1.$$

If $0 < \lambda \le \lambda_l$,

$$\begin{aligned}
\max_{\Lambda_{R_n} \le s \le \bar{\Lambda}_{R_n}} \int_0^\omega g(t,s)dt &\le \frac{2\lambda \int_0^\pi \sin t dt}{2 + kR_n} + \frac{2\lambda \int_\pi^{2\pi} \sin t dt}{2 + R_n} + \max_{kR_n \le s \le R_n} \frac{2\pi\lambda \sin s}{2+s} \\
&\le \frac{4\lambda(1-k)R_n}{(2+R_n)(2+kR_n)} - \frac{\pi\lambda}{2+R_n} < 0,
\end{aligned}$$

$$\begin{aligned}
\min_{\Lambda_{r_n} \le s \le \bar{\Lambda}_{r_n}} \int_0^\omega g(t,s)dt &\ge \frac{2\lambda \int_0^\pi \sin t dt}{2 + r_n} + \frac{2\lambda \int_\pi^{2\pi} \sin t dt}{2 + kr_n} + \min_{kr_n \le s \le r_n} \frac{2\pi\lambda \sin s}{2+s} \\
&\ge \frac{4\lambda(k-1)r_n}{(2+r_n)(2+kr_n)} + \frac{\pi\lambda}{2+r_n} > 0.
\end{aligned}$$

If $-\lambda_l \le \lambda < 0$,

$$\max_{\Lambda_{r_n} \le s \le \bar{\Lambda}_{r_n}} \int_0^\omega g(t,s)dt < 0 < \min_{\Lambda_{R_n} \le s \le \bar{\Lambda}_{R_n}} \int_0^\omega g(t,s)dt.$$

From Theorem 3.1, (3.10) has one positive 2π-periodic solution $x_i \in [kr_i, R_i], 0 \le i \le l-1$.

Example 3.2. Consider the equation

$$x''(x) + 2x'(t) + \frac{1}{100}\left[x(t)x(t-\tau_1)(2 - x(t-\tau_2)) - \frac{1}{4}(1 + \sin 2\pi t)\right] = 0 \tag{3.11}$$

where $\tau_1, \tau_2 \in \mathbb{R}$.

In fact,

$$w=1,\ \ f(t)=2,\ \ g(t,v_1,v_2,v_3)=\frac{1}{100}[v_1v_2(2-v_3)-0.25(1+\sin 2\pi t)].$$

Let

$$a=0,\ \ b=+\infty,\ \ \mu=\frac{13}{100},\ \ k=e^{-\mu w}\approx 0.87,$$

$$r_1=k^{-1}+0.01,\ \ r_2=e^2,\ \ r_3=e^3,\ \ r_4=e^5,$$

then

$$F^{-1}(u)=\frac{u}{2},\ \ u>0,$$

$$|g(t,v_1,v_2,v_3)|\le\frac{1}{100}\left(\frac{5}{2}\times\frac{5}{2}\times 2+\frac{1}{2}\right)=\mu,\ \ \forall t\in\mathbb{R},\ \frac{\ln kr_1}{2}\le v_i\le\frac{\ln r_4}{2}(1\le i\le 3).$$

In addition,

$$\max_{\Lambda_{r_1}\le s_i\le\bar{\Lambda}_{r_1}}\int_0^\omega g(t,s_1,s_2,s_3)dt\le\frac{1}{100}\left[\frac{\ln^2 r_1}{4}\cdot 2-\frac{1}{4}\right]\approx\frac{1}{400}(\mu^2\times 2-1)<0,$$

$$\min_{\Lambda_{r_2}\le s_i\le\bar{\Lambda}_{r_2}}\int_0^\omega g(t,s_1,s_2,s_3)dt\ge\frac{1}{100}\left[\frac{(4-\ln r_2)\ln^2 kr_2}{8}-\frac{1}{4}\right]=\frac{(2-\mu)^2-1}{400}>0,$$

$$\min_{\Lambda_{r_3}\le s_i\le\bar{\Lambda}_{r_3}}\int_0^\omega g(t,s_1,s_2,s_3)dt\ge\frac{1}{100}\left[\frac{(4-\ln r_3)\ln^2 kr_3}{8}-\frac{1}{4}\right]=\frac{(3-\mu)^2-2}{800}>0,$$

$$\max_{\Lambda_{r_4}\le s_i\le\bar{\Lambda}_{r_4}}\int_0^\omega g(t,s_1,s_2,s_3)dt\le\frac{1}{100}\left[\frac{(4-\ln kr_4)\ln^2 kr_4}{8}-\frac{1}{4}\right]<0.$$

By Theorem 3.1, (3.11) has positive 1-periodic solutions x_1 and x_2 with

$$\frac{\ln kr_1}{2}\le x_1\le\frac{\ln r_2}{2}=1<\frac{\ln kr_3}{2}=\frac{3-\mu}{2}\le x_2\le\frac{\ln r_4}{2}=\frac{5}{2}.$$

4. Application

In this section, we apply Theorem 3.1 to (1.4) and give some conditions guaranteeing that (1.4) has at least a positive ω-periodic solution. For a given continuous function h, let

$$\overline{h}=\int_0^w h(s)ds,\ \ h^+=\max\{h,0\},\ \ h^-=\max\{-h,0\}=-h+h^+.$$

Theorem 4.1. (1.4) *has at least a positive ω-periodic solution if one of the following conditions is satisfied*

(1) $c>0,\ \overline{p}>0,\ \overline{q}>0$ *and* $p\ge 0,\ \frac{c\overline{q}}{\omega\overline{p}}>q$ *for all* $t\in\mathbb{R}$;
(2) $c>0,\ \overline{p}<0,\ \overline{q}<0$ *and* $p\le 0,\ \frac{c\overline{q}}{\omega\overline{p}}+q>0$ *for all* $t\in\mathbb{R}$;
(3) $c<0,\ \overline{p}<0,\ \overline{q}<0$ *and* $p\le 0,\ q>\frac{c\overline{q}}{\omega\overline{p}}$ *for all* $t\in\mathbb{R}$;
(4) $c<0,\ \overline{p}>0,\ \overline{q}>0$ *and* $p\ge 0,\ \frac{c\overline{q}}{\omega\overline{p}}<-q$ *for all* $t\in\mathbb{R}$.

Proof. Clearly $f(t)=c,\ g(t,u)=p(t)u-q(t)$. Let $a=0, b=+\infty$, then

$$F(u)=\int_0^u cdt = cu(u>0), \quad F^{-1}(u)=\frac{u}{c}(u>0).$$

Assume that case 1 or case 2 holds. Let R be sufficiently large and

$$0<\varepsilon<\frac{2}{3}\min\left\{\frac{c\bar q}{\bar p},\ \min_{t\in\mathbb{R}}\left\{\frac{c\bar q}{\bar p}-\omega q\right\},\ \min_{t\in\mathbb{R}}\left\{\frac{c\bar q}{\bar p}+\omega q\right\}\right\},$$

$$r=e^{\frac{c\bar q}{\bar p}-\varepsilon},\ \ \mu=\frac{1}{w}\left(\frac{c\bar q}{\bar p}-\frac{3}{2}\varepsilon\right),$$

then

$$kr=e^{(\frac{c\bar q}{\bar p}-\varepsilon-\frac{c\bar q}{\bar p}+\frac{3}{2}\varepsilon)}=e^{\frac{\varepsilon}{2}}>1,$$

$$F^{-1}(\ln kr)>0, F^{-1}(\ln R)>0,$$

$$p(t)x-q(t)\ge -q(t)>-\mu \ \text{ for } \ t\in\mathbb{R},\ \frac{\ln kr}{c}\le x\le\frac{\ln R}{c}, \quad \text{if case 1},$$

$$p(t)x-q(t)\le -q(t)<\mu \ \text{ for } \ t\in\mathbb{R},\ \frac{\ln kr}{c}\le x\le\frac{\ln R}{c}, \quad \text{if case 2}.$$

If case 1 is satisfied,

$$\max_{\Lambda_r\le s\le\bar\Lambda_r}\int_0^\omega g(t,s)dt\le\bar p\frac{\ln r}{c}-\bar q=\frac{\bar p}{c}\left(\frac{c\bar q}{\bar p}-\varepsilon\right)-\bar q=-\frac{\bar p}{c}\varepsilon<0,$$

$$\min_{\Lambda_R\le s\le\bar\Lambda_R}\int_0^\omega g(t,s)dt\ge\bar p\frac{\ln kR}{c}-\bar q=\frac{\bar p}{c}\ln kR-\bar q>0.$$

If case 2 is satisfied,

$$\max_{\Lambda_R\le s\le\bar\Lambda_R}\int_0^\omega g(t,s)dt\le\bar p\frac{\ln kR}{c}-\bar q=-|\frac{\bar p}{c}|\cdot\ln kR-\bar q<0,$$

$$\min_{\Lambda_r\le s\le\bar\Lambda_r}\int_0^\omega g(t,s)dt\ge\bar p\frac{\ln r}{c}-\bar q=\frac{\bar p}{c}\left(\frac{c\bar q}{\bar p}-\varepsilon\right)-\bar q=-\frac{\bar p}{c}\varepsilon=|\frac{\bar p}{c}|\varepsilon>0.$$

Assume that case 3 or case 4 holds. Let

$$0<\varepsilon<\frac{1}{2}\min\left\{-\frac{c\bar q}{\bar p},\min_{t\in\mathbb{R}}\left\{q\omega-\frac{c\bar q}{\bar p}\right\},\min_{t\in\mathbb{R}}\left\{q\omega+\frac{c\vec{q}}{\bar p}\right\}\right\},$$

$$r=e^{\frac{c\bar q}{\bar p}-\varepsilon},\quad R=e^{-\varepsilon},\ \ \mu=\frac{1}{\omega}\left(-\frac{c\bar q}{\bar p}-2\varepsilon\right),$$

then

$$r<R<1,\ \ \mu>0,\ \ \frac{\ln kr}{c}>\frac{\ln R}{c}>0,$$

$$p(t)x-q(t)\le -q<\mu \ \text{ for } \ t\in\mathbb{R},\ \frac{\ln R}{c}\le x\le\frac{\ln kr}{c}, \quad \text{if case 3},$$

$$p(t)x-q(t)\ge -q>-\mu \ \text{ for } \ t\in\mathbb{R},\ \frac{\ln R}{c}\le x\le\frac{\ln kr}{c}, \quad \text{if case 4}.$$

If case 3 holds,

$$\max_{\Lambda_r\le s\le \bar{\Lambda}_r}\int_0^{\omega} g(t,s)dt \le \overline{p}\frac{\ln r}{c}-\overline{q}=\frac{\overline{p}}{c}\left(\frac{c\overline{q}}{\overline{p}}-\varepsilon\right)-\overline{q}=-\frac{\overline{p}}{c}\varepsilon<0,$$

$$\min_{\Lambda_R\le s\le \bar{\Lambda}_R}\int_0^{\omega} g(t,s)dt \ge \overline{p}\frac{\ln kR}{c}-\overline{q}=\frac{\overline{p}}{c}(-\mu w-\varepsilon)-\overline{q}=\frac{\overline{p}}{c}\left(\frac{c\overline{q}}{\overline{p}}+\varepsilon\right)-\overline{q}=\frac{\overline{p}}{c}\varepsilon>0.$$

If case 4 holds,

$$\max_{\Lambda_R\le s\le \bar{\Lambda}_R}\int_0^{\omega} g(t,s)dt \le \overline{p}\frac{\ln kR}{c}-\overline{q}=\frac{\overline{p}}{c}(-\mu w-\varepsilon)-\overline{q}=\frac{\overline{p}}{c}\varepsilon<0,$$

$$\min_{\Lambda_r\le s\le \bar{\Lambda}_r}\int_0^{\omega} g(t,s)dt \ge \overline{p}\frac{\ln r}{c}-\overline{q}=\frac{\bar{p}}{c}\left(\frac{c\overline{q}}{\overline{p}}-\varepsilon\right)-\overline{q}=-\frac{\overline{p}}{c}\varepsilon>0.$$

Hence, by Theorem 3.1, (1.4) has at least a positive w-periodic solution. □

Next, we consider the case that p may change sign.

Theorem 4.2. (1.4) *has at least a positive ω-periodic solution if one of the following conditions is satisfied*

(1) $c>0,\ \overline{p}>0,\ \overline{q}>0,\ \omega|p|\overline{p^+}<c\overline{p}$ *for all* $t\in\mathbb{R}$ *and* $\frac{c\overline{q}}{\omega\overline{p^+}}>\min\{A_1,A_2\}$, *where*

$$A_1=\max_{t\in\mathbb{R}}\{\varphi_1(t)\},\ \ A_2=\max_{t\in\mathbb{R}}\{-\varphi_1(t)\},\ \ \varphi_1(t)=\frac{c(p\overline{q}-q\overline{p})}{c\overline{p}-|p|\omega\overline{p^+}}.$$

(2) $c>0,\ \overline{p}<0,\ \overline{q}<0,\ \omega|p|\overline{p^-}<-c\overline{p}$ *for all* $t\in\mathbb{R}$ *and* $\frac{-c\overline{q}}{\omega\overline{p^-}}>\min\{B_1,B_2\}$, *where*

$$B_1=\max_{t\in\mathbb{R}}\{\varphi_2(t)\},\ \ B_2=\max_{t\in\mathbb{R}}\{-\varphi_2(t)\},\ \ \varphi_2(t)=\frac{c(p\overline{q}-q\overline{p})}{c\overline{p}+|p|\omega\overline{p^-}}.$$

(3) $c<0,\ \overline{p}>0,\ \overline{q}>0,\ \omega|p|\overline{p^+}<-c\overline{p}$ *for all* $t\in\mathbb{R}$ *and* $\frac{-c\overline{q}}{\omega\overline{p^+}}>\min\{C_1,C_2\}$, *where*

$$C_1=\max_{t\in\mathbb{R}}\{\varphi_3(t)\},\ \ C_2=\max_{t\in\mathbb{R}}\{-\varphi_3(t)\},\ \ \varphi_3(t)=\frac{c(p\overline{q}-q\overline{p})}{c\overline{p}+|p|\omega\overline{p^+}}.$$

(4) $c<0,\ \overline{p}<0,\ \overline{q}<0,\ \omega|p|\overline{p^-}<c\overline{p}$ *for all* $t\in\mathbb{R}$ *and* $\frac{c\overline{q}}{\omega\overline{p^-}}>\min\{D_1,D_2\}$, *where*

$$D_1=\max_{t\in\mathbb{R}}\{\varphi_4(t)\},\ \ D_2=\max_{t\in\mathbb{R}}\{-\varphi_4(t)\},\ \ \varphi_2(t)=\frac{c(p\overline{q}-q\overline{p})}{c\overline{p}-|p|\omega\overline{p^-}}.$$

Proof. (1) Let

$$0<\varepsilon_1<\frac{1}{2}\min\left\{1,\frac{c\overline{q}-\overline{p^+}\omega\min\{A_1,A_2\}}{\overline{p}+\overline{p^+}\omega},\left(\max_{t\in\mathbb{R}}\frac{|p|\overline{p}}{c\overline{p}-|p|\overline{p^+}}\right)^{-1}\right\},$$

$$R=e^{\frac{c\overline{q}}{\overline{p}}+\frac{\overline{p^+}}{\overline{p}}\mu\omega+\varepsilon_1^2},\ \ r=e^{\frac{c\overline{q}}{\overline{p}}-\frac{\overline{p^-}}{\overline{p}}\mu\omega-\varepsilon_1^2},\ \ \mu=\begin{cases}A_1+\varepsilon_1, A_1\le A_2,\\ A_2+\varepsilon_1, A_1>A_2.\end{cases}$$

Then

$$\begin{aligned}\ln kr &= \frac{c\bar q}{\bar p} - \frac{\overline{p^-}}{\bar p}\mu\omega - \mu\omega - \varepsilon_1^2\\ &= \frac{1}{\bar p}\left(c\bar q - \overline{p^+}\omega\min\{A_1,A_2\} - \overline{p^+}\omega\varepsilon_1 - \bar p\varepsilon_1^2\right)\\ &> \frac{1}{\bar p}\left(c\bar q - \overline{p^+}\omega\min\{A_1,A_2\} - (\overline{p^+}\omega + \bar p)\varepsilon_1\right) > 0.\end{aligned}$$

For $\frac{\ln kr}{c} \le x \le \frac{\ln R}{c}$,

$$\begin{aligned}p(t)x - q(t) &\le p^+\frac{\ln R}{c} - p^-\frac{\ln kr}{c} - q\\ &= \frac{p^+}{c}\left(\frac{c\bar q}{\bar p} + \frac{\overline{p^+}}{\bar p}\mu\omega + \varepsilon_1^2\right) - \frac{p^-}{c}\left(-\mu\omega + \frac{c\bar q}{\bar p} - \frac{\overline{p^-}}{\bar p}\mu\omega - \varepsilon_1^2\right) - q\\ &= p\frac{\bar q}{\bar p} - q + \frac{p^+\overline{p^+} + p^-\bar p + p^-\overline{p^-}}{c\bar p}\mu\omega + \frac{p^+ + p^-}{c}\varepsilon\\ &= p\frac{\bar q}{\bar p} - q + \frac{|p|\overline{p^+}}{c\bar p}\mu\omega + \frac{|p|}{c}\varepsilon_1^2 < A_1 + \varepsilon_1 \quad \text{if } A_1 \le A_2,\\ p(t)x - q(t) &\ge p^+\frac{\ln kr}{c} - p^-\frac{\ln R}{c} - q\\ &= \frac{p^+}{c}\left(\frac{c\bar q}{\bar p} - \mu\omega - \frac{\overline{p^-}}{\bar p}\mu\omega - \varepsilon_1^2\right) - \frac{p^-}{c}\left(\frac{c\bar q}{\bar p} + \frac{\overline{p^+}}{\bar p}\mu\omega + \varepsilon_1^2\right) - q\\ &= p\frac{\bar q}{\bar p} - q - \frac{|p|\overline{p^+}}{c\bar p}\mu\omega - \frac{|p|}{c}\varepsilon_1^2 > -(A_2 + \varepsilon_1) \quad \text{if } A_1 > A_2.\end{aligned}$$

In addition,

$$\begin{aligned}\max_{\Lambda_r \le s \le \bar\Lambda_r}\int_0^\omega g(t,s)dt &\le \overline{p^+}\frac{\ln r}{c} - \overline{p^-}\frac{\ln kr}{c} - \bar q = \frac{\bar p}{c}\ln r + \frac{\overline{p^-}}{c}\mu\omega - \bar q\\ &= \frac{\bar p}{c}\left(\frac{c\bar q}{\bar p} - \frac{\overline{p^-}}{\bar p}\mu\omega - \varepsilon_1^2\right) + \frac{\overline{p^-}}{c}\mu\omega - \bar q = -\frac{\bar p}{c}\varepsilon_1^2 < 0,\end{aligned}$$

$$\begin{aligned}\min_{\Lambda_R \le s \le \bar\Lambda_R}\int_0^\omega g(t,s)dt &\ge \frac{\overline{p^+}}{c}\ln kR - \frac{\overline{p^-}}{c}\ln R - \bar q = \frac{\bar p}{c}\ln R - \frac{\overline{p^+}}{c}\mu\omega - \bar q\\ &= \frac{\bar p}{c}\left(\frac{c\bar q}{\bar p} + \frac{\overline{p^+}}{\bar p}\mu\omega + \varepsilon_1^2\right) - \frac{\overline{p^+}}{c}\mu\omega - \bar q = \frac{\bar p}{c}\varepsilon_1^2 > 0.\end{aligned}$$

(2) Let

$$0 < \varepsilon_2 < \frac{1}{2}\min\left\{1, \frac{-(c\bar q + \overline{p^-}\omega\min\{B_1,B_2\})}{\overline{p^-}\omega - \bar p}, \left(\max_{t\in\mathbb{R}}\frac{c|p|\overline{p}}{c\bar p + |p|\overline{p^-}\omega}\right)^{-1}\right\},$$

$$R = e^{\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega + \varepsilon_2^2}, \quad r = e^{\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega - \varepsilon_2^2}, \quad \mu = \begin{cases} B_1+\varepsilon_2, B_1 \le B_2, \\ B_2+\varepsilon_2, B_1 > B_2. \end{cases}$$

Then

$$\begin{aligned}
\ln kr &= \frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega - \mu\omega - \varepsilon_2^2 = \frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^-}}{\bar{p}}\mu\omega - \varepsilon_2^2 \\
&= \frac{1}{\bar{p}}[c\bar{q} + \overline{p^-}\omega\min\{B_1,B_2\} + \overline{p^-}\omega\varepsilon_2 - \bar{p}\varepsilon_2^2] \\
&\ge \frac{1}{\bar{p}}[c\bar{q} + \overline{p^-}\omega\min\{B_1,B_2\} + (\overline{p^-}\omega - \bar{p})\varepsilon_2] > 0.
\end{aligned}$$

For $\frac{\ln kr}{c} \le x \le \frac{\ln R}{c}$,

$$\begin{aligned}
p(t)x - q(t) &\le \frac{p^+\ln R}{c} - \frac{p^-\ln kr}{c} - q \\
&\le \frac{p^+}{c}\left(\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega + \varepsilon_2^2\right) - \frac{p^-}{c}\left(\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega - \mu\omega - \varepsilon_2^2\right) - q \\
&= p\frac{\bar{q}}{\bar{p}} - q - \frac{|p|\overline{p^-}}{c\bar{p}}\mu\omega + \frac{|p|}{c}\varepsilon_2^2 < B_1 + \varepsilon_2, \text{ if } B_1 \le B_2, \\
p(t)x - q(t) &\ge \frac{p^+}{c}\ln kr - \frac{p^-}{c}\ln R - q \\
&= p\frac{\bar{q}}{\bar{p}} - q + \frac{|p|\overline{p^-}}{c\bar{p}}\mu\omega - \frac{|p|}{c}\varepsilon_2^2 > -(B_2 + \varepsilon_2) \text{ if } B_1 > B_2, \\
\max_{\Lambda_R \le s \le \bar{\Lambda}_R}\int_0^\omega g(t,s)dt &\le \frac{\overline{p^+}}{c}\ln R - \frac{\overline{p^-}}{c}\ln kR - \bar{q} = \frac{\bar{p}}{c}\ln R + \frac{\overline{p^-}}{c}\mu\omega - \bar{q} \\
&= \frac{\bar{p}}{c}\left(\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega + \varepsilon_2^2\right) + \frac{\overline{p^-}}{c}\mu\omega - \bar{q} = \frac{\bar{p}}{c}\varepsilon_2^2 < 0, \\
\min_{\Lambda_r \le s \le \bar{\Lambda}_r}\int_0^\omega g(t,s)dt &\ge \frac{\overline{p^+}}{c}\ln kr - \frac{\overline{p^-}}{c}\ln r - \bar{q} \\
&= \frac{\bar{p}}{c}\ln r - \frac{\overline{p^+}}{c}\mu\omega - \bar{q} = -\frac{\bar{p}}{c}\varepsilon_2^2 > 0.
\end{aligned}$$

(3) Let

$$0 < \varepsilon_3 < \frac{1}{2}\min\left\{1, \frac{-c\bar{q} - \overline{p^+}\omega\min\{C_1,C_2\}}{\overline{p^+}\omega + \bar{p}}, \left(\max_{t\in\mathbb{R}}\frac{-|p|\bar{p}}{c\bar{p} + |p|\overline{p^+}\omega}\right)^{-1}\right\},$$

$$R = e^{\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega + \varepsilon_3^2}, \quad r = e^{\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega - \varepsilon_3^2}, \quad \mu = \begin{cases} C_1+\varepsilon_3, C_1 \le C_2, \\ C_2+\varepsilon_3, C_1 > C_2. \end{cases}$$

Then

$$\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega + \varepsilon_3^2 = \frac{1}{\bar{p}}(c\bar{q} + \overline{p^+}\omega\min\{C_1,C_2\} + \overline{p^+}\omega\varepsilon_3 + \bar{p}\varepsilon_3^2) < 0,$$

$$0 < r < R < 1.$$

For $\frac{\ln R}{c} \leq x \leq \frac{\ln kr}{c}$,

$$\begin{aligned}
p(t)x - q(t) &\leq p^+\frac{\ln kr}{c} - p^-\frac{\ln R}{c} - q \\
&= \frac{p^+}{c}\left(-\mu\omega + \frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega - \varepsilon_3^2\right) - \frac{p^-}{c}\left(\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega + \varepsilon_3^2\right) \\
&= p\frac{\bar{q}}{\bar{p}} - q - \frac{|p|\overline{p^+}}{c\bar{p}}\mu\omega - \frac{|p|}{c}\varepsilon_3^2 < \mu \quad \text{if } C_1 \leq C_2, \\
p(t)x - q(t) &\geq p^+\frac{\ln R}{c} - p^-\frac{\ln kr}{c} - q \\
&= p\frac{\bar{q}}{\bar{p}} - q + \frac{|p|\overline{p^+}}{c\bar{p}}\mu\omega + \frac{|p|}{c}\varepsilon_3^2 > -\mu \quad \text{if } C_1 > C_2, \\
\max_{\Lambda_R \leq s \leq \bar{\Lambda}_R}\int_0^\omega g(t,s)dt &\leq \overline{p^+}\frac{\ln kR}{c} - \overline{p^-}\frac{\ln R}{c} - \bar{q} \\
&= \frac{\bar{p}}{c}\ln R - \frac{\overline{p^+}}{c}\mu\omega - \bar{q} = \frac{\bar{p}}{c}\varepsilon_3^2 < 0, \\
\min_{\Lambda_r \leq s \leq \bar{\Lambda}_r}\int_0^\omega g(t,s)dt &\geq \overline{p^+}\frac{\ln kr}{c} - \overline{p^-}\frac{\ln R}{c} - \bar{q} \\
&= \frac{\bar{p}}{c}\ln r + \frac{\overline{p^-}}{c}\mu\omega - \bar{q} = -\frac{\bar{p}}{c}\varepsilon_3^2 > 0.
\end{aligned}$$

(4) Let

$$0 < \varepsilon_4 < \frac{1}{2}\min\left\{1, \frac{c\bar{q} - \overline{p^-}\omega\min\{D_1,D_2\}}{\overline{p^-}\omega - \bar{p}}, \left(\max_{t\in\mathbb{R}}\frac{-|p|\bar{p}}{c\bar{p} - |p|\overline{p^-}\omega}\right)^{-1}\right\},$$

$$R = e^{\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega + \varepsilon_4^2}, \quad r = e^{\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega - \varepsilon_4^2}, \quad \mu = \begin{cases} D_1 + \varepsilon_4, D_1 \leq D_2, \\ D_2 + \varepsilon_4, D_1 > D_2. \end{cases}$$

Then

$$\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu\omega + \varepsilon_4^2 = \frac{1}{\bar{p}}[c\bar{q} - \overline{p^-}\omega\min\{D_1,D_2\} - \overline{p^-}\omega\varepsilon + \bar{p}\varepsilon^2] < 0,$$

$$r < R < 1.$$

For $\frac{\ln R}{c} \le x \le \frac{\ln kr}{c}$,

$$\begin{aligned}
p(t)x - q(t) &\le p^+ \frac{\ln kr}{c} - p^- \frac{\ln R}{c} - q \\
&= \frac{p^+}{c}\left(\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega - \varepsilon_4^2 - \mu\omega\right) - \frac{p^-}{c}\left(\frac{c\bar{q}}{\bar{p}} - \frac{\overline{p^-}}{\bar{p}}\mu w + \varepsilon_4^2\right) - q \\
&= p\frac{\bar{q}}{\bar{p}} - q + \frac{|p|\overline{p^-}}{c\bar{p}}\mu\omega - \frac{|p|}{c}\varepsilon_4^2 < D_1 + \varepsilon_4 \quad \text{if } D_1 \le D_2,
\end{aligned}$$

$$\begin{aligned}
p(t)x - q(t) &\ge p^+ \frac{\ln R}{c} - p^- \frac{\ln kr}{c} - q \\
&= p\frac{\bar{q}}{\bar{p}} - q - \frac{|p|\overline{p^-}}{c\bar{p}}\mu\omega + \frac{|p|}{c}\varepsilon_4^2 > -(D_2 + \varepsilon_4) \quad \text{if } D_1 > D_2,
\end{aligned}$$

$$\begin{aligned}
\max_{\Lambda_r \le s \le \bar{\Lambda}_r} \int_0^\omega g(t,s)dt &= \overline{p^+}\frac{\ln kr}{c} - \overline{p^-}\frac{\ln r}{c} - \bar{q} = \frac{\bar{p}}{c}\ln r - \frac{\overline{p^+}}{c}\mu\omega - \bar{q} \\
&= \frac{\bar{p}}{c}\left(\frac{c\bar{q}}{\bar{p}} + \frac{\overline{p^+}}{\bar{p}}\mu\omega - \varepsilon_4^2\right) - \frac{\overline{p^+}}{c}\mu\omega - \bar{q} = -\frac{\bar{p}}{c}\varepsilon_4^2 < 0,
\end{aligned}$$

$$\min_{\Lambda_R \le s \le \bar{\Lambda}_R} \int_0^\omega g(t,s)dt \ge \overline{p^+}\frac{\ln R}{c} - \overline{p^-}\frac{\ln kR}{c} - \bar{q} = \frac{\bar{p}}{c}\ln R + \frac{\overline{p^-}}{c}\mu\omega - \bar{q} = \frac{\bar{p}}{c}\varepsilon_4^2 > 0.$$

In either case, by Theorem 3.1, (1.4) has at least a positive w-periodic solution. □

Example 4.1. Consider the equation

$$x''(t) + cx'(t) + 2x(t - \frac{\pi}{2}) = d + \sin t, \tag{4.1}$$

where $c, d \in \mathbb{R}$.

Clearly, $d > 0$ if (4.1) has positive 2π-periodic solution. From Theorem 4.1, if

$$\frac{|c|d}{4\pi} > d + 1 \quad \text{and} \quad d > 0,$$

(4.1) has positive 2π-periodic solution. Let

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

be 2π-periodic solution of (4.1). Substituting into (4.1), we have

$$a_0 = \frac{d}{2}, \quad a_n = b_n = 0 (n \ge 2), \quad a_1 = \frac{2-c}{1+(c-2)^2}, \quad b_1 = \frac{-1}{1+(c-2)^2},$$

which implies that x changes sign if

$$\frac{|d|}{2} < \frac{1}{\sqrt{1+(c-2)^2}}$$

and (4.1) has a unique positive 2π-periodic solution if

$$\frac{d}{2} > \frac{1}{\sqrt{1+(c-2)^2}}.$$

References

[1] D. Anderson, R. Douglas, R.I. Avery, Richard I, *Existence of a periodic solution for continuous and discrete periodic second-order equations with variable potentials*, J. Appl. Math. Comput. **37** (2011), no. 1-2, 297–312.

[2] M. Belaid, A. Ardjouni, A. Djoudi, *Periodicity and positivity in nonlinear neutral integro-dynamic equations with variable delay*, Mathematica, **62** (2020), no. 2, 117–132.

[3] A. Boucherif, N. Daoudi-Merzagui, *Periodic solutions of singular nonautonomous second order differential equations*, NoDEA Nonlinear Differential Equations Appl. **15** (2008), no. 1-2, 147–158.

[4] T. Carletti, G. Villari, F. Zanolin, *Existence of harmonic solutions for some generalisation of the non-autonomous Liénard equations*, Monatsh. Math. **199** (2022), no. 2, 243–257.

[5] M. V. Demina, *Invariant algebraic curves for Liénard dynamical systems revisited*, Appl. Math. Lett. **84** (2018), 42-48

[6] R. Eswari, J. Alzabut, S. Jehad, M. E. Samei, C. Tunç, J.M. Jonnalagadda, *New results on the existence of periodic solutions for Rayleigh equations with state-dependent delay*, Nonauton. Dyn. Syst. **9** (2022), no. 1, 103–115.

[7] M. Feng, N. Deng, *Multiple positive doubly periodic solutions to nonlinear telegraph systems*, Appl. Math. Lett. **133** (2022),108233, 6 pp.

[8] V. A. Gaiko, *Global bifurcation analysis of generalized Linéard polynomial dynamical system*, J. Math. Sci. **270** (2023), no. 5, 674–682.

[9] F. Gao, S. Lu, M. Yao, *Periodic solutions for Liénard type equation with time-variable coefficient*, Adv. Difference Equ. **2015** (2015), 9 pp.

[10] Z. Guo, J. Yu, *Multiplicity results for periodic solutions to delay differential equations via critical point theory*, J. Differential Equations, **218** (2005), no. 1, 15–35.

[11] D. Guo, V.Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL(1988)

[12] X. Hou, and Z. Wu, *Existence and uniqueness of periodic solutions for a kind of Liénard equation with multiple deviating arguments*, J. Appl. Math. Comput. **38** (2012) no.1-2, 181-193.

[13] S. Lu , Y. Guo, L. Chen, *Periodic solutions for Liénard equation with an indefinite singularity*, Nonlinear Anal: RWA. **45** (2019) 542-556

[14] S. Lu, S. Zhou, X. Yu, *Homoclinic solutions for a class of relativistic Liénard equations*, J. Fixed Point Theory Appl. **23** (2021), no. 3, Paper No. 39, 15 pp.

[15] S. Lu, *Existence of homoclinic solutions for a class of neutral functional differential equations*, Acta Math. Sin. (Engl. Ser.). **28** (2012), no. 6, 1261–1274.

[16] B. Liu, *Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with two deviating arguments*, Comput. Math. Appl. **55** (2008), no. 9, 2108–2117.

[17] J. Mawhin, and R. E. Gains, *Coincidence degree and nonlinear differential equations*, (Lecture Notes in Mathematics, vol. 568) Springer,New York, 1977.

[18] J. Mawhin, R. Nussbaum, P. Fitzpatrick, and M. Martelli, *Lecture on topological methods for ordinary differential equations*, Heidelberg: Springer, Berlin, 1993.

[19] J. Mawhin, Periodic solutions, *workshop on nonlinear boundary value problems for ordinary differential equations an applications*, ICTP SMR/42/1, Trieste, 1977.

[20] S. Padhi, S. Pati, *Positive periodic solutions for a nonlinear functional differential equation*, Mem. Differ. Equ. Math. Phys. **51** (2010), 109–118.

[21] J. Šremr, *Existence and exact multiplicity of positive periodic solutions to forced non-autonomous Duffing type differential equations, Electron*, J. Qual Theo. **62** (2021), 33 pp.

[22] S. Srivastava, *Periodic solutions of delayed difference equations*, Bull. Math. Anal. Appl. **4** (2012), no. 4, 18–28.

[23] G. Villari, F. Zanolin, *On the uniqueness of the limit cycle for the Liénard equation via a comparison method for the energy level curves*, Dyn. Syst. Appl. **25** (2016), 321–334.

[24] W. Wang, J. Shen, *Positive periodic solutions for neutral functional differential equations*, Appl. Math. Lett. **102** (2020), 106154, 6 pp.

[25] W. Wang, Y. Luo, *Periodic solutions for a class of second order differential equation*, Electron J. Qual Theo. **41** (2013), 10pp

[26] J. Wang, B. Wang, Y. Miao, X. Yu, *Existence and multiplicity of positive periodic solutions to a class of Liénard equations with repulsive singularities*, J. Fixed Point Theory Appl. **24** (2022), no. 3, Paper No. 64.

[27] S. Zhao and T. K. Nagy, *Center manifold analysis of the delayed Liénard equation*, Delay Differential Equations: Recent Advances and New Directions, Springer Science+Business Media LLC 2009, 203–219.

[28] B. Zhang, *Periodic solutions of the retarded Liénard equation*, Ann. Mat. Pura Appl. **172** (1997),no. 4,25–42.

[29] Q. Zhou, F. Long, *Existence and uniqueness of periodic solutions for a kind of Liénard equation with two deviating arguments*, J. Comput. Appl. Math. **206** (2007), no. 2, 1127–1136.

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