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AVERAGING OF NONCLASSICAL DIFFUSION EQUATIONS LACKING INSTANTANEOUS DAMPING ON $\mathbb{R}^N$ WITH MEMORY AND SINGULARLY OSCILLATING FORCES

DANG THANH SON♮ AND NGUYEN DUONG TOAN

ABSTRACT. In this paper, we consider for $\rho \in [0,1)$ and $\varepsilon, \varsigma > 0$, the following nonclassical diffusion equations on $\mathbb{R}^N$, $N \geq 3$ with hereditary memory and singularly oscillating external force

$$u_t - \Delta u_t - \int_0^\infty \kappa_\varepsilon(s)\Delta u(t-s)ds + f(x,u) = g_0(t) + \varsigma^{-\rho} g_1(t/\varsigma),$$

together with the *averaged* equation

$$u_t - \Delta u_t - \int_0^\infty \kappa_\varepsilon(s)\Delta u(t-s)ds + f(x,u) = g_0(t)$$

formally corresponding to the limiting case $\varsigma = 0$. The main characteristics of the model is that the equation does not contain a term of the form $-\Delta u$, which contributes to an instantaneous damping. We first prove the existence of uniform attractors $\mathscr{A}_\varsigma^\varepsilon$ in the space $H^1(\mathbb{R}^N) \times L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^1(\mathbb{R}^N))$. Then, we show that the model converges to the nonclassical diffusion equation with lacking instantaneous damping when $\varepsilon \to 0$ as $t \to \infty$. The uniform (w.r.t. ς) boundedness as well as the convergence of the uniform attractor $\mathscr{A}_\varsigma^\varepsilon$ of the first equation to the uniform attractor $\mathscr{A}_0^\varepsilon$ of the second equation as $\varsigma \to 0^+$ are also studied.

1. Introduction

The main goal of this paper is to discuss the following nonclassical diffusion equation with memory

$$(1.1)\qquad \begin{cases} u_t - \Delta u_t - \int_0^\infty \kappa_\varepsilon(s)\Delta u(x,t-s)ds + f(x,u) = g^\varsigma(t), & x \in \mathbb{R}^N, t > \tau, \\ u(x,t) = u_\tau(x), & x \in \mathbb{R}^N, t \leq \tau, \\ u(x,\tau - s) = q_\tau(x,s), & x \in \mathbb{R}^N, s > 0, \end{cases}$$

where $u_\tau(x)$ and $q_\tau(x,s)$ are initial data, the function $\kappa_\varepsilon(s) : [0,\infty) \to \mathbb{R}$ is called a memory kernel (see [8, 12]), which is a continuous non-negative function, smooth decreasing on $(0,\infty)$, vanishing at infinity and satisfying

$$\kappa_\varepsilon(s) = \frac{1}{\varepsilon}\kappa\left(\frac{s}{\varepsilon}\right), \quad \varepsilon \in (0,1],$$

and

$$\int_0^\infty \kappa(s)ds = k_0 < \infty.$$

Corresponding author: ♮ dangthanhson@tcu.edu.vn.
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As $\varepsilon \to 0$, the function $\kappa_\varepsilon(s)$ converges in the sense of the distributions to the Dirac mass at zero. In addition, the nonlinearity f and the external force $g^\varsigma(x,t) = g_0(x,t) + \varsigma^{-\rho} g_1(x,t/\varsigma)$ satisfy the following conditions hold:

(H1) We define

$$\mu(s) = -\kappa'(s), \quad \mu_\varepsilon(s) = -\kappa'_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right),$$

and assume that $\mu(s) \geq 0$ is absolutely continuous, decreasing, ie, $\mu'(s) \leq 0$ almost everywhere, and the Dafermos condition

$$\mu'(s) + \delta\mu(s) \leq 0,$$

is satisfied for some $\delta > 0$, $\forall s \in \mathbb{R}^+$. Noting the definition of μ_ε, we calculate μ_ε satisfying

$$\varepsilon\mu'_\varepsilon(s) + \delta\mu_\varepsilon(s) \leq 0. \tag{1.2}$$

Since μ is decreasing, and the Gronwall inequality implies the exponential decay

$$\mu(s) \leq \delta\mu(s_0)e^{-\delta(s-s_0)}, \quad \forall s \geq s_0 > 0, \tag{1.3}$$

and $\mu(s)$ can be confirmed to be integrable,

$$\int_0^\infty \mu(s)ds = k_0, \quad \text{then } \int_0^\infty \mu_\varepsilon(s)ds = \frac{k_0}{\varepsilon}. \tag{1.4}$$

To avoid the presence of unnecessary constants, from now on we assume $k_0 = 1$ which can be always obtained by rescaling the memory kernel.

(H2) The continuous nonlinearity $f(x,u)$, with $f(\cdot,0) \in L^2(\mathbb{R}^N)$, satisfies

$$f'(x,u) \geq -\ell, \tag{1.5}$$

$$|f'(x,u)| \leq C\left(\phi_1(x) + |u|^{\frac{4}{N-2}}\right), \tag{1.6}$$

for some $\ell > 0$, and $\phi_1(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$ is nonnegative functions, along with the dissipation conditions

$$\langle F(x,u), 1\rangle \geq -C_f, \tag{1.7}$$

$$\langle f(x,u), u\rangle \geq \nu_0\langle F(x,u), 1\rangle - C_f, \tag{1.8}$$

where $C_f \geq 0, \nu_0 > 0$ and $F(x,u) = \int_0^u f(x,s)ds$ is a primitive of f.

(H3) The functions $g_0, g_1 \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$, the space of translation bounded functions in $L^2_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^N))$, that is,

$$\|g_i\|^2_{L^2_b} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_i(s)\|^2_{L^2(\mathbb{R}^N)} ds = M_i^2 < \infty \ (i = 0, 1). \tag{1.9}$$

A straightforward consequence of (1.9) is

$$\int_t^{t+1} \|g_1(y/\varsigma)\|^2 dy = \varsigma \int_{t/\varsigma}^{(t+1)/\varsigma} \|g_1(y)\|^2 dy \leq \varsigma(1 + 1/\varsigma)M_1^2 \leq 2M_1^2,$$

thus

$$\|g_1(\cdot/\varsigma)\|^2_{L^2_b} \leq 2M_1^2, \ \forall \varsigma \in (0,1].$$

Hence,

$$\|g^\varsigma\|^2_{L^2_b} \leq 2\|g_0\|^2_{L^2_b} + 2\varsigma^{-2\rho}\|g_1(\cdot/\varsigma)\|^2_{L^2_b} \leq 2M_0^2 + 4M_1^2\varsigma^{-2\rho}.$$

For $g^\varsigma \in L^2_b\left(\mathbb{R}; L^2(\mathbb{R}^N)\right)$, we denote by $\mathscr{H}_w(g^\varsigma)$ the closure of the set $\{g^\varsigma(\cdot+h)|h\in\mathbb{R}\}$ in $L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$ with the weak topology. Noting that, as in [6, Chapter 5, Proposition 4.2], we have: for all $\sigma \in \mathscr{H}_w(g^\varsigma)$ and any fixed positive number ς, then $\|\sigma\|^2_{L^2_b} \leq \|g^\varsigma\|^2_{L^2_b}$.

The nonclassical diffusion equation is a mathematical model that arises in a variety of physical phenomena, such as non-Newtonian flows, soil mechanics, and heat conduction theory (see, e.g., [1, 16, 17, 20]). It was first proposed by Aifantis in [1], and later modified by Jäckle [13] to include a memory term in the study of heat conduction and relaxation of high-viscosity liquids. The inclusion of a memory term in the diffusion equation leads to a faster rate of energy dissipation and a more accurate description of the phenomena, as the conduction of energy is affected not only by present external forces but also by historic external forces. On the other hand, equations with memory are more difficult to solve than the corresponding ones without memory. In recent years, there has been a significant amount of research on the existence and long-time behavior of solutions to nonclassical diffusion equations with memory, for both autonomous case (see [2, 4, 5, 9, 10, 21, 22, 23, 25]) and non-autonomous case (see [3, 14, 23, 24]). However, most of the existing results on nonclassical diffusion equations with memory have been obtained for bounded domains, except for the two recent results [18, 19]. In [19], the authors studied a class of nonclassical diffusion equations on $\mathbb{R}^N$ with hereditary memory (independent on ε), in presence of singularly oscillating external forces depending on a positive parameter ε and a new class of nonlinearities, which have no restriction on the upper growth of the nonlinearity.

More recently, Conti et al. [7] considered the nonclassical diffusion equation with hereditary memory lacking instantaneous damping

$$u_t - \Delta u_t - \int_0^\infty \kappa(s)\Delta u(x,t-s)ds + f(u) = g(x)$$

on a bounded three-dimensional domain. After that, Toan [18] extended some results of [7] to the non-autonomous case in unbouded domains.

As an effort to improve and extend the results of [7] and [18], in this paper, we will consider the nonclassical diffusion equation with memory lacking instantaneous damping and singularly oscillating external force. As we know, there are three main difficulties in studying problem (1.1) on $\mathbb{R}^N$. Firstly, equation (1.1) is the absence of the term $-\Delta u$, which makes the nonclassical diffusion is lacking instantaneous damping. Secondly, the problem is considered on the whole $\mathbb{R}^N$, which means that Sobolev embeddings are no longer compact and the Poicaré inequality is not satisfied. Thirdly, we rescale $\kappa(s)$ by a (small) parameter ε, i.e., the memory kernel $\kappa(s)$ is dependent on ε, which makes some computations more complicated. Moreover, the presence of the term $-\Delta u_t$ in the equation means that the solution has no higher regularity, similar to hyperbolic equations. These difficulties make it challenging to prove the existence of uniform attractors for the problem and to study the singular limit when the memory kernel collapses into the Dirac mass at zero, and finally, the uniform boundedness (w.r.t. ς) and the convergence of uniform attractors $\mathscr{A}^\varepsilon_\varsigma$ as ς tends to 0.

The paper is organized as follows. In Section 2, we introduce some necessary notations, functional spaces, and a Gronwall-type lemma. In Section 3, we prove the existence of uniform attractors $\mathscr{A}^\varepsilon$ for the family of processes generated by the model. In Section 4, we prove that the model converges (in an appropriate sense) to the nonclassical diffusion equation with lacking instantaneous damping when the scaling parameter ε of the memory kernel tends to zero. Finally, in Sections 5, we prove the uniform boundedness (w.r.t ς) and the convergence of the uniform attractors. Our results have extended some results in Conti et al. (2020) [7] to the non-autonomous case on the whole space, and the results of Toan (2020) [18] to the case of the memory kernel term that depends on ε and singularly oscillating external force.

2. Preliminaries

At first, following Dafermos [11], we consider a new variable which reflects the history of (1.1), that is

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)dr,\ s \geq 0,$$

then we can check that

$$\partial_t \eta^t(x,s) = u(x,t) - \partial_s \eta^t(x,s),\ s \geq 0.$$

Since $\mu_\varepsilon(s) = -\kappa_\varepsilon'(s)$, problem (1.1) can be transformed into the following system

$$(2.1)\qquad \begin{cases} u_t - \Delta u_t - \int_0^\infty \mu_\varepsilon(s)\Delta\eta^t(x,s)ds + f(x,u) = g^\varsigma(t), & x\in\mathbb{R}^N, t>\tau, \\ \partial_t\eta^t(x,s) = -\partial_s\eta^t(x,s) + u(x,t), & x\in\mathbb{R}^N, t>\tau, s\geq 0, \\ u(x,t) = u_\tau(x), & x\in\mathbb{R}^N, t\leq\tau, \\ \eta^\tau(x,s) = \eta_\tau(x,s) := \int_\tau^s q(x,r)dr, & x\in\mathbb{R}^N, s>0. \end{cases}$$

Let $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ be the norm and scalar product in $L^2(\mathbb{R}^N)$, respectively. For $i=1,2$, we define the *history spaces*

$$\mathscr{M}_\varepsilon^i = \begin{cases} L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^i(\mathbb{R}^N)), & \varepsilon>0, \\ \{0\}, & \varepsilon=0, \end{cases}$$

equipped with inner product and norm, respectively,

$$\langle\varphi_1,\varphi_2\rangle_{\mathscr{M}_\varepsilon^i} = \int_0^\infty \mu_\varepsilon(s)\langle\varphi_1(s),\varphi_2(s)\rangle_{H^i(\mathbb{R}^N)}ds,$$
$$\|\varphi\|^2_{\mathscr{M}_\varepsilon^i} = \int_0^\infty \mu_\varepsilon(s)\|\varphi\|^2_{H^i(\mathbb{R}^N)}ds.$$

We now introduce the following Hilbert spaces

$$\mathscr{H}_\varepsilon^i = H^i(\mathbb{R}^N)\times\mathscr{M}_\varepsilon^i,\quad i=1,2,$$

with the norms

$$\|(u,\eta)\|^2_{\mathscr{H}_\varepsilon^1} = \|u\|^2 + \|\nabla u\|^2 + \|\eta\|^2_{\mathscr{M}_\varepsilon^1},$$
$$\|(u,\eta)\|^2_{\mathscr{H}_\varepsilon^2} = \|u\|^2_{H^2(\mathbb{R}^N)} + \|\eta\|^2_{\mathscr{M}_\varepsilon^2}.$$

Since our the main purpose is to consider $\varepsilon \to 0$, we must give the equation when $\varepsilon = 0$,

$$\begin{cases} u_t - \Delta u_t + f(x,u) = g^{\varsigma}(t), & x \in \mathbb{R}^N, t > \tau, \\ u(x,t) = u_\tau(x), & x \in \mathbb{R}^N, t \le \tau. \end{cases} \tag{2.2}$$

And in order to be consistent with the memory equation, let η^t satisfy the Cauchy problem in $\mathscr{H}^1_\varepsilon$,

$$\begin{cases} \partial_t \eta^t(x,s) = -\partial_s \eta^t(x,s) + u(x,t), & x \in \mathbb{R}^N, t > \tau, s \ge 0, \\ \eta^\tau(x,s) = \eta_\tau(x,s) := \int_\tau^s q(x,r)dr, & x \in \mathbb{R}^N, s > 0. \end{cases} \tag{2.3}$$

Let $z_\tau = (u_\tau, \eta_\tau)$ and let $U^\varepsilon(t,\tau)z_\tau = z(t) = (u(t),\eta^t)$, be the solution of (2.1) and

$$U^0_\varsigma(t,\tau)u_\tau = u(t), \quad U^0_\varsigma(t,\tau)\eta_\tau = \eta^t, \quad \varepsilon = 0,$$

represent the solutions of (2.2) and (2.3), respectively.

The following Gronwall-type lemma is the main tool in the proof.

Lemma 2.1. [15] *Let Λ_ε be a family of absolutely continuous nonnegative functions on $[\tau,\infty)$ satisfying for some $\gamma > 0$, $C > 0$ and for any $\varepsilon \in (0,\varepsilon_0)$, for some small $\varepsilon_0 > 0$, the differential inequality*

$$\frac{d}{dt}\Lambda_\varepsilon(t) + \gamma\varepsilon\Lambda_\varepsilon(t) \le c\varepsilon^p[\Lambda_\varepsilon(t)]^q + \frac{C}{\varepsilon^r},$$

where the nonnegative parameters p,q,r fulfill

$$p - 1 > (q-1)(1+r) \ge 0.$$

Moreover, let E be a continuous non-negative function on $[\tau,\infty)$ such that

$$\frac{1}{m}E(t) \le \Lambda_\varepsilon(t) \le mE(t)$$

for every $\varepsilon > 0$ small and some $m \ge 1$. Then, there exist $\nu > 0$ and an increasing positive function $\mathscr{Q}(\cdot)$ such that

$$E(t) \le \mathscr{Q}(E(\tau))e^{-\nu(t-\tau)} + C.$$

By the Faedo-Galerkin method, arguing as in the proof of [18, Theorem 2.1], we obtained the following results.

Theorem 2.2. *Assume that hypotheses* **(H1)-(H3)** *hold. Then for any fixed nonnegative number ς, any $z_\tau = (u_\tau,\eta_\tau) \in \mathscr{H}^1_\varepsilon$ and $T > \tau$, $\tau \in \mathbb{R}$ given, problem (2.1) has a unique weak solution $z = (u,\eta^t)$ on the interval $[\tau,T]$ satisfying $z \in C([\tau,T];\mathscr{H}^1_\varepsilon)$. Moreover, the weak solutions depend continuously on the initial data.*

Accordingly, the problem (2.1) generates a dynamical system, we define a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathscr{H}_w(g^\varsigma)}$ as follows

$$U_\sigma(t,\tau) : \mathscr{H}^1_\varepsilon \to \mathscr{H}^1_\varepsilon,$$

where $U_\sigma(t,\tau)z_\tau$ is the unique weak solution of (1.1) (with σ in place of g^ς) at the time t with the initial datum z_τ at τ.

3. Existence of an uniform attractor

3.1. *Existence of an absorbing set.* In order to deal with the (possible) singularity of $\mu_\varepsilon(s)$ at zero, given any $\nu \in (0, 1/4)$, we choose $s_* = s_*(\nu) > 0$ such that

$$\int_0^{s_*} \mu_\varepsilon(s)ds \leq \nu, \tag{3.1}$$

and we define $\mu_{\varepsilon\nu} : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\mu_{\varepsilon\nu}(s) = \mu_\varepsilon(s_*)\chi_{(0,s_*]}(s) + \mu_\varepsilon(s)\chi_{(s_*,\infty]}(s),$$

where χ denotes the characteristic function.

The proof of existence of an uniform absorbing set exploits in a crucial way the following technical lemma.

Lemma 3.1. *Assume that $z(t) = (u(t), \eta^t)$ is a sufficiently regular solution to (2.1). Then, for any $\nu \in (0, \frac{1}{4})$, the functional*

$$\Lambda_j(t) = -\int_0^\infty \mu_{\varepsilon\nu}(s)\langle u(t), j\eta^t(s)\rangle ds - \int_0^\infty \mu_{\varepsilon\nu}(s)\langle \nabla u(t), \nabla\eta^t(s)\rangle ds, \quad j = 0, 1,$$

fulfills the differential inequality

$$\begin{aligned}\frac{d}{dt}\Lambda_j(t) + \frac{1}{2\varepsilon}\|u(t)\|^2_{H^1(\mathbb{R}^N)} \leq \frac{\varepsilon}{4}\|u_t(t)\|^2_{H^1(\mathbb{R}^N)} + \frac{1}{\varepsilon^2}\int_0^\infty \mu_\varepsilon(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds \\ - \frac{\varepsilon\mu_\varepsilon(s_*)}{2}\int_0^\infty \mu'_\varepsilon(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds.\end{aligned} \tag{3.2}$$

Besides, we have the control

$$|\Lambda_j(t)| \leq \frac{1}{\varepsilon}E_j, \tag{3.3}$$

where $E_j = \|u\|^2 + \|\nabla u\|^2 + \int_0^\infty \mu_\varepsilon(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds, \quad j = 0, 1.$

Proof. Firstly, from the definition of $\Lambda_j(t)$, immediately, we deduce the inequality (3.3). Indeed,

$$\begin{aligned}|\Lambda_j(t)| &= \left|\int_0^\infty \mu_{\varepsilon\nu}(s)\langle u(t), j\eta^t(s)\rangle ds + \int_0^\infty \mu_{\varepsilon\nu}(s)\langle \nabla u(t), \nabla\eta^t(s)\rangle ds\right| \\ &\leq \frac{1}{\sqrt{\varepsilon}}\|u(t)\|\left(\int_0^\infty \mu_\varepsilon(s)j\|\eta^t(s)\|^2 ds\right)^{1/2} + \frac{1}{\sqrt{\varepsilon}}\|\nabla u(t)\|\left(\int_0^\infty \mu_\varepsilon(s)\|\nabla\eta^t(s)\|^2 ds\right)^{1/2} \\ &\leq \frac{1}{\varepsilon}\left(\|u\|^2 + \|\nabla u\|^2\right) + \int_0^\infty \mu_\varepsilon(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right)ds \leq \frac{1}{\varepsilon}E_j.\end{aligned}$$

Secondly, we will prove the inequality (3.2). The time-derivative Λ_j, we get

$$\begin{aligned}\frac{d}{dt}\Lambda_j = &-\int_0^\infty \mu_{\varepsilon\nu}(s)\langle u_t(t), j\eta^t(s)\rangle ds - \int_0^\infty \mu_{\varepsilon\nu}(s)\langle \nabla u_t(t), \nabla\eta^t(s)\rangle ds \\ &- \int_0^\infty \mu_{\varepsilon\nu}(s)\langle u(t), j\eta^t_t(s)\rangle ds - \int_0^\infty \mu_{\varepsilon\nu}(s)\langle \nabla u(t), \nabla\eta^t_t(s)\rangle ds.\end{aligned} \tag{3.4}$$

Using the Young inequality, we have

$$\begin{aligned}
&-\int_0^\infty \mu_{\varepsilon\nu}(s)\langle u_t(t), j\eta^t(s)\rangle ds - \int_0^\infty \mu_{\varepsilon\nu}(s)\langle \nabla u_t(t), \nabla\eta^t(s)\rangle ds \\
&\quad\le \frac{1}{\sqrt{\varepsilon}}\|u_t(t)\|\left(\int_0^\infty \mu_\varepsilon(s) j\|\eta^t(s)\|^2 ds\right)^{1/2} + \frac{1}{\sqrt{\varepsilon}}\|\nabla u_t(t)\|\left(\int_0^\infty \mu_\varepsilon(s)\|\nabla\eta^t(s)\|^2 ds\right)^{1/2} \\
&\quad\le \frac{\varepsilon}{4}\left(\|u_t(t)\|^2 + \|\nabla u_t(t)\|^2\right) + \frac{1}{\varepsilon^2}\int_0^\infty \mu_\varepsilon(s)\left(j\|\eta^t(s)\|^2 + \|\nabla\eta^t(s)\|^2\right) ds.
\end{aligned}$$

Recalling that $-\eta_t^t = \eta_s^t - u(t)$, from the definition of $\mu_{\varepsilon\nu}(s)$ and (3.1), we have

$$\begin{aligned}
&-\langle u(t), j\eta_t^t\rangle_{\mu_{\varepsilon\nu}} - \langle \nabla u(t), \nabla\eta_t^t\rangle_{\mu_{\varepsilon\nu}} \\
&\qquad = \langle u(t), j\eta_s^t\rangle_{\mu_{\varepsilon\nu}} + \langle \nabla u(t), \nabla\eta_s^t\rangle_{\mu_{\varepsilon\nu}} - \int_0^\infty \mu_{\varepsilon\nu}(s)(\|u\|^2 + \|\nabla u\|^2) ds \\
&\qquad \le -\int_{s_*}^\infty \mu_\varepsilon'(s)\big(\langle u, j\eta^t(s)\rangle + \langle \nabla u, \nabla\eta^t(s)\rangle\big) ds - \int_{s_*}^\infty \mu_\varepsilon(s) ds \|u(t)\|_{H^1(\mathbb{R}^N)}^2 \\
&\qquad \le \left(-\int_{s_*}^\infty \mu_\varepsilon'(s) ds\right)^{1/2}\|u(t)\|\left(-\int_{s_*}^\infty \mu_\varepsilon'(s) j\|\eta^t(s)\|^2 ds\right)^{1/2} \\
&\qquad + \left(-\int_{s_*}^\infty \mu_\varepsilon'(s) ds\right)^{1/2}\|\nabla u(t)\|\left(-\int_{s_*}^\infty \mu_\varepsilon'(s)\|\nabla\eta^t(s)\|^2 ds\right)^{1/2} - \left(\frac{1}{\varepsilon} - \nu\right)\|u(t)\|_{H^1(\mathbb{R}^N)}^2 \\
&\qquad \le -\frac{\varepsilon\mu_\varepsilon(s_*)}{2}\int_0^\infty \mu_\varepsilon'(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t(s)\|^2\right) ds - \frac{1}{2\varepsilon}\|u(t)\|_{H^1(\mathbb{R}^N)}^2.
\end{aligned}$$

Collecting two estimates above, inserting them on the right-hand side of (3.4), we obtain the desired differential inequality (3.2). The proof is completed. □

Lemma 3.2. *Under the assumptions* **(H1)-(H3)***, for* $\varepsilon \in (0,1]$*, any fixed nonnegative number* ς *and any initial datum* $z_\tau \in \mathscr{H}_\varepsilon^1$*, the family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathscr{H}_w(g^\varsigma)}$ *associated to problem* (2.1) *has an* ($\mathscr{H}_\varepsilon^1$ *,*$\mathscr{H}_\varepsilon^1$ *)-uniform absorbing set.*

Proof. At first, we replace g^ς with σ in (2.1), and then multiplying the first and second equation of (2.1) by $u(t)$ in $L^2(\mathbb{R}^N)$ and by $j\eta^t(s)$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, L^2(\mathbb{R}^N))$, respectively, and adding the results, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|u\|^2 + \|\nabla u\|^2 + j\int_0^\infty \mu_\varepsilon(s)\|\eta^t\|^2 ds\right) + \langle f(x,u), u\rangle + \int_0^\infty \mu_\varepsilon(s)\langle \nabla\eta^t(s), \nabla u\rangle ds \\
&- \frac{j}{2}\int_0^\infty \partial_s\mu_\varepsilon(s)\|\eta^t\|^2 ds = j\int_0^\infty \mu_\varepsilon(s)\langle \eta^t(s), u\rangle ds + (\sigma, u).
\end{aligned} \tag{3.5}$$

Similarly, multiplying the second equation of (2.1) by $-\Delta\eta^t(s)$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, L^2(\mathbb{R}^N))$, we have

$$\begin{aligned}
\int_0^\infty \mu_\varepsilon(s)\int_{\mathbb{R}^N} \nabla\eta^t\nabla u dx ds &= \int_0^\infty \mu_\varepsilon(s)\int_{\mathbb{R}^N} \nabla\eta^t\nabla\eta_t^t dx ds + \int_0^\infty \mu_\varepsilon(s)\int_{\mathbb{R}^N} \nabla\eta^t\nabla\eta_s^t dx ds \\
&= \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu_\varepsilon(s)\int_{\mathbb{R}^N} |\nabla\eta^t|^2 dx ds - \frac{1}{2}\int_0^\infty \partial_s\mu_\varepsilon(s)\int_{\mathbb{R}^N} |\nabla\eta^t|^2 dx ds.
\end{aligned} \tag{3.6}$$

Using assumptions (1.8), (1.4) and the Cauchy inequality, we have

$$\begin{aligned} \langle f(x,u),u\rangle &\geq \nu_0\langle F(x,u),1\rangle - C_f, \\ 2(\sigma,u) &\leq \frac{\delta}{4}\|u\|^2 + C\|\sigma\|^2, \quad \text{where } 0<\delta<1, \\ 2j\int_0^\infty \mu_\varepsilon(s)\langle \eta^t(s),u\rangle ds &\leq \frac{j}{\delta\varepsilon}\|u\|^2 + j\delta\int_0^\infty \mu_\varepsilon(s)\|\eta^t\|^2 ds. \end{aligned} \tag{3.7}$$

Summation of (3.5), (3.6) and then combining with (3.7), we get

$$\begin{aligned} &\frac{d}{dt}E_j - 2\int_0^\infty \partial_s\mu_\varepsilon(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t\|^2\right) ds + 2\nu_0\left(\langle F(x,u),1\rangle - \frac{C_f}{\nu_0}\right) \\ &\leq \left(\frac{\delta}{4} + \frac{j}{\delta\varepsilon}\right)\|u\|^2 + \delta j\int_0^\infty \mu_\varepsilon(s)\|\eta^t\|^2 ds + C\|\sigma\|^2, \end{aligned}$$

where $E_j = \|u\|^2 + \|\nabla u\|^2 + \int_0^\infty \mu_\varepsilon(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t\|^2\right) ds,\ j=0,1.$

Besides, multiplying the first equation of (2.1) by u_t in $L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned} \|u_t\|^2 + \|\nabla u_t\|^2 + \frac{d}{dt}\big\langle F(x,u),1\big\rangle &= \langle\sigma(t),u_t\rangle - \int_0^\infty \mu_\varepsilon(s)\langle\nabla\eta^t(s),\nabla u_t\rangle ds \\ &\leq \frac{1}{2}(\|u_t\|^2 + \|\nabla u_t\|^2) + \frac{1}{2\varepsilon}\int_0^\infty \mu_\varepsilon(s)\|\nabla\eta^t(s)\|^2 ds + \frac{1}{2}\|\sigma(t)\|^2. \end{aligned}$$

Thus,

$$\frac{d}{dt}\big\langle F(x,u),1\big\rangle + \frac{1}{2}(\|u_t\|^2 + \|\nabla u_t\|^2) \leq \frac{1}{2\varepsilon}\int_0^\infty \mu_\varepsilon(s)\|\nabla\eta^t(s)\|^2 ds + \frac{1}{2}\|\sigma(t)\|^2.$$

Now, we define the functional

$$\Phi_j(t) = E_j + \delta\big\langle F(x,u),1\big\rangle + \delta\varepsilon\Lambda_j(t),\ j=0,1,$$

where $\Lambda_j(t)$ is defined in Lemma 3.1. Besides, using condition (1.6), (1.7) and (3.3) in Lemma 3.1, we have

$$E_j - \delta C_f \leq \Phi_j \leq 2\Big(E_j + \delta\big\langle F(x,u),1\big\rangle\Big) \leq CE_1^{\frac{N}{N-2}} + C. \tag{3.8}$$

Using the condition (1.3), we can see that $-\mu_\varepsilon'(s) \geq \frac{\delta}{\varepsilon}\mu_\varepsilon(s)$, then Φ_j satisfies the differential inequality

$$\begin{aligned} \frac{d}{dt}\Phi_j + \frac{\delta}{4}\|u\|^2_{H^1(\mathbb{R}^N)} + \frac{\delta-\delta\varepsilon^2}{4}\|u_t\|^2_{H^1(\mathbb{R}^N)} + 2\nu_0\big\langle F(x,u),1\big\rangle + \frac{\delta}{8\varepsilon}\int_0^\infty \mu_\varepsilon(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t\|^2\right) ds \\ - \frac{1-4\varepsilon\mu_\varepsilon(s_*)}{8}\int_0^\infty \mu_\varepsilon'(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t\|^2\right) ds \\ \leq \frac{j}{\delta\varepsilon}\|u\|^2 + C\|\sigma(t)\|^2 + 2C_f. \end{aligned}$$

Thus, there exist constant $\gamma > 0$ such that

$$\frac{d}{dt}\Phi_j + \gamma\Phi_j \leq \frac{j}{\delta\varepsilon}\Phi_0 + C\|\sigma(t)\|^2 + 2C_f. \tag{3.9}$$

From (3.9), let $j=0$, and then applying Gronwall inequality, we get

$$\Phi_0(t) \le \Phi_0(\tau)e^{-\gamma(t-0)} + C\int_\tau^t e^{-\gamma(t-r)}\|\sigma(r)\|^2 dr + 2C_f, \tag{3.10}$$

with

$$\begin{aligned}\int_\tau^t e^{-\gamma(t-r)}\|\sigma(r)\|^2 dr &\le \left(\int_{t-1}^t e^{-\gamma(t-r)}\|\sigma(r)\|^2 ds + \int_{t-2}^{t-1} e^{-\gamma(t-r)}\|\sigma(r)\|^2 dr + \dots\right)\\ &\le \left(1+e^{-\gamma}+e^{-2\gamma}+\dots\right)\|\sigma\|^2_{L^2_b} \le \frac{1}{1-e^{-\gamma}}\|g^\varsigma\|^2_{L^2_b},\end{aligned} \tag{3.11}$$

where we have used the fact that $\|\sigma\|^2_{L^2_b} \le \|g^\varsigma\|^2_{L^2_b}$ for all $\sigma \in \mathscr{H}_w(g^\varsigma)$.
Combining (3.8), (3.10) and (3.11), we get

$$\Phi_0(t) \le C\left(E_1^{\frac{N}{N-2}}(\tau)+1\right)e^{-\gamma(t-\tau)} + \frac{C}{1-e^{-\gamma}}\|g^\varsigma\|^2_{L^2_b} + 2C_f \le \rho_0. \tag{3.12}$$

We now consider (3.9) for $j=1$, using (3.12) and the Gronwall inequality, we obtain

$$\begin{aligned}\Phi_1(t) &\le \Phi_1(\tau)e^{-\gamma(t-\tau)} + C\rho_0 + C\int_0^t e^{-\gamma(t-r)}\|\sigma(r)\|^2 dr\\ &\le \Phi_1(\tau)e^{-\gamma(t-\tau)} + C\rho_0 + \frac{C}{1-e^{-\gamma}}\|g^\varsigma\|^2_{L^2_b} + C.\end{aligned}$$

Thus,

$$E_1(t) \le C\left(E_1^{\frac{2N}{N-2}}(\tau)+1\right)e^{-\gamma(t-\tau)} + C\rho_0 + \frac{C}{1-e^{-\gamma}}\|g^\varsigma\|^2_{L^2_b} + C.$$

Hence there exists $\rho_1 > 0$ such that

$$E_1(t) \le \rho_1 \text{ or } \|z(t)\|^2_{\mathscr{H}^1_\varepsilon} \le \rho_1, \tag{3.13}$$

for all $z_\tau \in B$, $\sigma \in \mathscr{H}_w(g^\varsigma)$ and for all $t \ge T_B$, where B is an arbitrary bounded subset of $\mathscr{H}^1_\varepsilon$. This completes the proof. □

Combining Lemma 3.2 with Theorem 2.2, we can obtain the result as follows.

Lemma 3.3. *Under the assumption of Lemma 3.2, then for any bounded (in $\mathscr{H}^1_\varepsilon$) subset B, there exists a constant $N_B = N(\|B\|_{\mathscr{H}^1_\varepsilon}, \|g^\varsigma\|_{L^2_b})$, such that for any $\tau \in \mathbb{R}, z_\tau \in B$,*

$$\|U_\sigma(t,\tau)z_\tau\|^2_{\mathscr{H}^1_\varepsilon} \le N_B, \quad \text{as } t \ge \tau.$$

3.2. ***Asymptotic compactness.*** The main difficulty of the problem is that the embeddings are no longer compact and the whole dissipation is contributed by the convolution term only. In order to show that $U^\varsigma_\varepsilon(t,\tau)z_\tau$ is uniformly asymptotically compact in $\mathscr{H}^1_\varepsilon$, we perform a standard decomposition of the solution into two summands, one of which is shown to be arbitrarily small in the long time (see Lemma 3.4) and the other of which is compact (see Lemma 3.6). This yields the desired result.

3.2.1. *Decomposition of the equation.* For any $r > 0$, as in [18], we introduce two smooth positive functions $\phi_r^i : \mathbb{R}^N \to \mathbb{R}^+$, $i = 1,2$, such that

$$\phi_r^1(x) + \phi_r^2(x) = 1 \quad \forall x \in \mathbb{R}^N,$$

and

$$\phi_r^1(x) = 0 \text{ if } |x| \le r,$$
$$\phi_r^2(x) = 0 \text{ if } |x| \ge r+1.$$

Putting $\sigma_i(x,t) = \sigma(x,t)\phi_r^i(x),\ i = 1,2$. The dependence on r of σ_i is omitted for simplicity of notation. Therefore, we can check that

$$\lim_{r\to\infty} \|\sigma_1\| = 0,$$
$$\sigma_2(x,t) = 0, \text{ as } |x| \ge r+1.$$

For the nonlinearity f, we decompose $f = f_0 + f_1$, where $f_0, f_1 \in C(\mathbb{R})$ satisfy

$$f_0(x,u)u \ge 0, \quad F_0(x,u) = \int_0^u f_0(x,y)dy \ge 0 \quad \forall u \in \mathbb{R}, \tag{3.14}$$

$$|f_0(x,u)| \le C(\phi_1(x)|u| + |u|^{\frac{N+2}{N-2}}), \quad \forall u \in \mathbb{R}, \tag{3.15}$$

and

$$|f_1(x,u)| \le C(\phi_1(x) + |u|^q), \quad \forall u \in \mathbb{R}, \text{ and } 0 < q < \frac{N+2}{N-2}. \tag{3.16}$$

To make the asymptotic regular estimates, we decompose the solution $U_\sigma(t,\tau)z_\tau = z(t) = (u(t),\eta^t)$ (where $z_\tau = (u_\tau,\eta^\tau)$) of problem (2.1) into the sum

$$U_\sigma(t,\tau)z_\tau = D(t,\tau)z_\tau + K_\sigma(t,\tau)z_\tau,$$

where $D(t,\tau)z_\tau = z_1(t)$ and $K_\sigma(t,\tau)z_\tau = z_2(t)$, that is, $z = (u,\eta^t) = z_1 + z_2$, the decomposition is as follows

$$u = v + w, \quad \eta^t = \zeta^t + \xi^t,$$
$$z_1 = (v,\zeta^t), \quad z_2 = (w,\xi^t),$$

where $z_1(t)$ solves the following equation

$$\begin{cases} v_t - \Delta v_t - \int_0^\infty \mu_\varepsilon(s)\Delta\zeta^t(s)ds + f_0(x,v) = \sigma_1(t), & x \in \mathbb{R}^N, t > \tau, \\ \partial_t\zeta^t = -\partial_s\zeta^t + v, & x \in \mathbb{R}^N, t > \tau, s \ge 0, \\ v(x,t) = u_\tau(x), & x \in \mathbb{R}^N, t \le \tau, \\ \zeta^\tau(x,s) = \eta_\tau(x,s) := \int_0^s g_\tau(x,r)dr, & x \in \mathbb{R}^N, s > 0, \end{cases} \tag{3.17}$$

and $z_2(t)$ is the unique solution of the following problem

$$
(3.18)\qquad \begin{cases} w_t - \Delta w_t - \int_0^\infty \mu_\varepsilon(s)\Delta\xi^t(s)ds + f(x,u) - f_0(x,v) = \sigma_2(t), & x\in\mathbb{R}^N, t>\tau,\\ \partial_t\xi^t = -\partial_s\xi^t + w, & x\in\mathbb{R}^N, t>\tau, s\geq 0,\\ w(x,t)=0, & x\in\mathbb{R}^N, t\leq\tau,\\ \xi^\tau(x,s)=\xi_\tau(x,s)=0, & x\in\mathbb{R}^N, s>0. \end{cases}
$$

By using similar arguments as in the proof of Theorem 2.2, one can prove the existence and uniqueness of solutions to problems (3.17) and (3.18).

We begin with the decay estimate for solutions of (3.17). By using similar arguments as in the proof of Lemma 3.2, replacing $\sigma(t)$ with $\sigma_1(t)$ and $f(x,u)$ with $f_0(x,u)$, we obtain the lemma as follows.

Lemma 3.4. *Assume that hypotheses (3.14), (3.15) and* **(H2)-(H3)** *hold, for any $\tau\in\mathbb{R}$, the solutions of equation (3.17) satisfy the following estimate: there is a constant $\gamma_1>0$ and there exist $T>\tau$ large enough, such that*

$$\|D(t,\tau)z_\tau\|^2_{\mathscr{H}_1} \leq \mathscr{Q}(\|z_\tau\|_{\mathscr{H}_1})e^{-\gamma_1(t-\tau)} + \rho_2,$$

where $\mathscr{Q}$ is an increasing function on $[0,\infty)$ and ρ_2 depends on $\|\sigma_1\|_{L^2_b}$.

About the solution $z_2(t)$ of (3.18), arguing as in the proof of [18, Lemmas 3.7, 3.8], we obtain the following results.

Lemma 3.5. *Let B be a bounded subset in $\mathscr{H}_1$. Then for any $\omega>0$, there exist $T_\omega>0$ and $K_\omega>0$ such that*

$$\int_{|x|\geq K_\omega}(|w|^2+|\nabla w|^2)dx + \int_0^\infty \mu_\varepsilon(s)\int_{|x|\geq K_\omega}\left(|\xi^t(s)|^2+|\nabla\xi^t(s)|^2\right)dxds < \omega, \forall t\geq T_\omega, \forall z_\tau\in B.$$

Lemma 3.6. *Let* **(H1)-(H3)** *and (3.16) hold and $\alpha=\min\{\frac{1}{4},\frac{N+2-q(N-2)}{2}\}$. For each time $T>\tau$ and $R>r_B$, there exists a positive constant N^* which depends on T, $\|\sigma_1\|_{L^2_b}$ and $\|z_\tau\|_{\mathscr{H}^1_\varepsilon}$, such that*

$$\|K_\sigma(T,\tau)z_\tau\|^2_{\mathscr{H}^{1+\alpha}_\varepsilon} \leq N^*.$$

Therefore, we get the following lemma.

Lemma 3.7. *Let $\{K_\sigma(t,\tau)z_\tau\}_{t\geq\tau}$ be the solution process of (3.18). Then under the assumption of* **(H1)-(H3)** *and (3.16), for $T>\tau$ large enough, such that*

$$K_\sigma(t,\tau)B_0 \text{ is relatively compact in } \mathscr{H}^1_\varepsilon.$$

By Lemma 3.2, the family of processes $U_\sigma(t,\tau)$ has a bounded absorbing B_0 in $\mathscr{H}^1_\varepsilon$. Moreover, $U_\sigma(t,\tau)$ is uniform asymptotically compact in $\mathscr{H}^1_\varepsilon$ due to Lemmas 3.4 and 3.7. Therefore, we obtain the following theorem.

Theorem 3.8. *Assume that hypotheses* **(H1)-(H3)** *hold. Then for any fixed positive number ε, the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathscr{H}_w(g^\varsigma)}$ associated to (1.1) possesses an uniform attractor $\mathscr{A}^\varepsilon$ in the space $\mathscr{H}^1_\varepsilon$. Moreover,*

$$\mathscr{A}^\varepsilon = \bigcup_{\sigma\in\mathscr{H}_w(g^\varsigma)}\mathscr{K}_\sigma(s), \quad \forall s\in\mathbb{R},$$

where $\mathscr{K}_\sigma(s)$ is the kernel section at time s of the process $U_\sigma(t,\tau)$.

4. The singular limit

In this section, for any fixed positive number ς, we consider the singular limit of the system with $g^\varsigma = 0$ when $t \to +\infty$. Let

$$U_\varsigma^\varepsilon(t,\tau)z_\tau = \hat{z}(t) = (\hat{u}(t), \hat{\eta}^t),$$
$$U_\varsigma^0(t,\tau)z_\tau = z(t) = (u(t), \eta^t),$$

denote the solutions of system of (2.1) and (2.2)-(2.3), respectively. Set

$$\bar{u}(t) = \hat{u}(t) - u(t), \quad \text{and} \quad \bar{\eta}^t = \hat{\eta}^t - \eta^t,$$

then $\bar{z} = (\bar{u}, \bar{\eta}^t)$ fulfills the system

$$\begin{cases} \bar{u}_t - \Delta\bar{u}_t - \int_0^\infty \mu_\varepsilon(s)\Delta\bar{\eta}^t(s)ds + f(x,\hat{u}) - f(x,u) = 0, & x \in \mathbb{R}^N, t > \tau, \\ \partial_t\bar{\eta}^t = -\partial_s\bar{\eta}^t + \bar{u}, & x \in \mathbb{R}^N, t > \tau, s \geq 0, \\ \bar{u}(x,t) = 0, & x \in \mathbb{R}^N, t \leq \tau, \\ \bar{\eta}^\tau(x,s) = 0, & x \in \mathbb{R}^N, s > 0. \end{cases} \tag{4.1}$$

From Lemma 3.2 and using assumption (1.2), we immediately obtain the following lemma.

Lemma 4.1. *For $g^\varsigma = 0$, we have*

$$\left(\|U_\varsigma^\varepsilon(t,\tau)z_\tau\|^2_{\mathscr{H}_1^\varepsilon}, \|U_\varsigma^0(t,\tau)z_\tau\|^2_{\mathscr{H}_1}\right) \leq \mathscr{Q}(z_\tau)e^{-\gamma(t-\tau)}, \quad \forall t \geq \tau.$$

We refer to Conti et al [8] for the proof of the following lemma.

Lemma 4.2. [8] *For all $\varepsilon \in (0,1]$, we have*

$$\max\left\{\|\hat{\eta}^t\|^2_{\mathscr{M}_1^\varepsilon}, \|\eta^t\|^2_{\mathscr{M}_1^\varepsilon}\right\} \leq \mathscr{Q}(z_\tau)e^{\frac{-\delta(t-\tau)}{2\varepsilon}} + C\varepsilon, \quad \forall t \geq \tau.$$

Now we have the main theorem of this section.

Theorem 4.3. *For any $z_\tau \in B_{\mathscr{H}_1^\varepsilon}$ and any $t \geq \tau$,*

(i) *for every fixed $\varepsilon > 0$, there holds*

$$\lim_{t\to+\infty} \|U^\varepsilon(t,\tau)z_\tau - U^0(t,\tau)z_\tau\|^2_{\mathscr{H}_1^\varepsilon} \leq C\varepsilon.$$

(ii) *$\forall\varepsilon > 0$, there holds*

$$\lim_{\varepsilon\to 0^+}\lim_{t\to\infty} \|U^\varepsilon(t,\tau)z_\tau - U^0(t,\tau)z_\tau\|^2_{\mathscr{H}_1^\varepsilon} = 0.$$

Proof. Multiplying the first equation of (4.1) by $\bar{u}(t) + \varepsilon^2\bar{u}_t$ in $L^2(\mathbb{R}^N)$, the second equation of (4.1) by $j\bar{\eta}^t(s)$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, L^2(\mathbb{R}^N))$, and adding the results, we get

$$\begin{aligned} &\frac{d}{dt}\left(\|\bar{u}\|^2 + \|\nabla\bar{u}\|^2 + \int_0^\infty \mu_\varepsilon(s)(j\|\bar{\eta}^t\|^2 + \|\nabla\bar{\eta}^t\|^2)ds\right) - 2\int_0^\infty \partial_s\mu_\varepsilon(s)(j\|\bar{\eta}^t\|^2 + \|\nabla\bar{\eta}^t\|^2)ds \\ &+ 2\varepsilon^2(\|\bar{u}_t\|^2 + \|\nabla\bar{u}_t\|^2) + 2\varepsilon^2\int_0^\infty \mu_\varepsilon(s)\int_{\mathbb{R}^N} \nabla\bar{\eta}^t\nabla\bar{u}_t dxds + \langle f(x,\hat{u}) - f(x,u), 2\bar{u} + 2\varepsilon^2\bar{u}_t\rangle \\ &= 2j\int_0^\infty \mu_\varepsilon(s)\langle\bar{\eta}^t(s), \bar{u}\rangle ds. \end{aligned} \tag{4.2}$$

Using the Hölder inequality, (3.13), (1.6) and $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ continuously, we get

$$\begin{aligned}\langle f(x,\hat{u}) - f(x,u), 2\varepsilon^2 \bar{u}_t\rangle &\le C\varepsilon^2 \int_{\mathbb{R}^N} \left(|\hat{u}|^{\frac{4}{N-2}} + |u|^{\frac{4}{N-2}} + |\phi(x)|\right)|\bar{u}||\bar{u}_t|dx \\ &\le C\varepsilon^2 \left(\|\hat{u}\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}} + \|u\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}} + \|\phi(x)\|_{L^{\frac{N}{2}}}\right)\|\bar{u}\|_{L^{\frac{2N}{N-2}}}\|\bar{u}_t\|_{L^{\frac{2N}{N-2}}} \\ &\le C\varepsilon^2 \left(\|\hat{u}\|^{\frac{4}{N-2}}_{H^1(\mathbb{R}^N)} + \|u\|^{\frac{4}{N-2}}_{H^1(\mathbb{R}^N)} + \|\phi(x)\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}\right)\|\bar{u}\|_{H^1(\mathbb{R}^N)}\|\bar{u}_t\|_{H^1(\mathbb{R}^N)} \\ &\le C\varepsilon^2\|\bar{u}\|^2_{H^1(\mathbb{R}^N)} + \frac{\varepsilon^2}{2}\|\bar{u}_t\|^2_{H^1(\mathbb{R}^N)},\end{aligned}$$

and similarly

$$\begin{aligned}\langle f(x,\hat{u}) - f(x,u), 2\bar{u}\rangle &\le C\int_{\mathbb{R}^N}\left(|\hat{u}|^{\frac{N+2}{N-2}} + |u|^{\frac{N+2}{N-2}} + |\phi_1(x)|(|u| + |\hat{u}|)\right)|\bar{u}|dx \\ &\le C\left(\|\hat{u}\|^{\frac{N+2}{N-2}}_{L^{\frac{2N}{N-2}}} + \|u\|^{\frac{N+2}{N-2}}_{L^{\frac{2N}{N-2}}} + \|\phi(x)\|_{L^{\frac{N}{2}}}(\|u\|_{L^{\frac{2N}{N-2}}} + \|\hat{u}\|_{L^{\frac{2N}{N-2}}})\right)\|\bar{u}\|_{L^{\frac{2N}{N-2}}} \\ &\le C\left(\|\hat{u}\|^{\frac{N+2}{N-2}}_{H^1(\mathbb{R}^N)} + \|u\|^{\frac{N+2}{N-2}}_{H^1(\mathbb{R}^N)} + \|\phi(x)\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}\left(\|u\|_{H^1(\mathbb{R}^N)} + \|\hat{u}\|_{H^1(\mathbb{R}^N)}\right)\right)\|\bar{u}\|_{H^1(\mathbb{R}^N)} \\ &\le \varepsilon\|\bar{u}\|^2_{H^1(\mathbb{R}^N)} + \frac{C}{\varepsilon},\end{aligned}$$

and

$$\begin{aligned}2\varepsilon^2\int_0^\infty \mu_\varepsilon(s)\int_{\mathbb{R}^N}\nabla\bar{\eta}^t\nabla\bar{u}_t dxds &\le 2\varepsilon^2\int_0^\infty \mu_\varepsilon(s)\|\nabla\bar{\eta}^t\|\|\nabla\bar{u}_t\|ds \\ &\le 2\varepsilon^2\left(\frac{1}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)\|\nabla\bar{\eta}^t\|^2ds + \frac{\varepsilon}{4}\int_0^\infty\mu_\varepsilon(s)ds\|\nabla\bar{u}_t\|^2\right) \\ &\le 2\varepsilon\int_0^\infty\mu_\varepsilon(s)\|\nabla\bar{\eta}^t\|^2ds + \frac{\varepsilon^2}{2}\|\nabla\bar{u}_t\|^2, \text{ where } \int_0^\infty\mu_\varepsilon(s)ds = \frac{1}{\varepsilon},\end{aligned}$$

and

$$2j\int_0^\infty\mu_\varepsilon(s)\|\bar{\eta}^t(s)\|\|\bar{u}\|ds \le \frac{j\delta}{\varepsilon}\int_0^\infty\mu_\varepsilon(s)\|\bar{\eta}^t(s)\|^2ds + \frac{j}{\delta}\|\bar{u}\|^2.$$

Combining the above inequalities, we obtain

$$\begin{aligned}&\frac{d}{dt}E_{1j}(t) - 2\int_0^\infty\partial_s\mu_\varepsilon(s)(j\|\bar{\eta}^t\|^2 + \|\nabla\bar{\eta}^t\|^2)ds + 2\varepsilon^2(\|\bar{u}_t\|^2 + \|\nabla\bar{u}_t\|^2) \\ &\quad\le 2\varepsilon\int_0^\infty\mu_\varepsilon(s)\|\nabla\bar{\eta}^t\|^2ds + (C\varepsilon^2+\varepsilon)\|\bar{u}\|^2_{H^1(\mathbb{R}^N)} + \frac{C}{\varepsilon} + \frac{j\delta}{2\varepsilon}\int_0^\infty\mu_\varepsilon(s)\|\bar{\eta}^t(s)\|^2ds + \frac{j}{\delta}\|\bar{u}\|^2,\end{aligned}$$

where $E_{1j}(t) = \|\bar{u}\|^2_{H^1(\mathbb{R}^N)} + \int_0^\infty\mu_\varepsilon(s)(j\|\bar{\eta}^t\|^2 + \|\nabla\bar{\eta}^t\|^2)ds$ and $E_{1j}(\tau) = 0$. Putting $\Phi_{1j}(t) = E_{1j}(t) + \nu_1\varepsilon\Lambda_{1j}(t)$, where Λ_{1j} is defined as in Lemma 3.1 but with $(\bar{u},\bar{\eta})$ instead of (u,η). Besides,

$$\varepsilon E_{1j}(t) \le \Phi_{1j}(t) \le \frac{2}{\varepsilon}E_{1j}(t).$$

Therefore, using (1.2), we get

$$\begin{aligned}&\frac{d}{dt}\Phi_{1j}(t)+\frac{\nu_1-2\varepsilon}{2}\|\bar{u}\|^2_{H^1(\mathbb{R}^N)}+\frac{\delta-2\nu_1-\varepsilon^2}{2\varepsilon}\int_0^\infty\mu_\varepsilon(s)(j\|\bar{\eta}^t\|^2+\|\nabla\bar{\eta}^t\|^2)ds\\&-\left(1-4\varepsilon^2\mu_\varepsilon(s_*)\right)\int_0^\infty\partial_s\mu_\varepsilon(s)(j\|\bar{\eta}^t\|^2+\|\nabla\bar{\eta}^t\|^2)ds+\frac{\varepsilon^2(2-\nu_1)}{4}(\|\bar{u}_t\|^2+\|\nabla\bar{u}_t\|^2)\\\leq{}& C\varepsilon^2\|\bar{u}\|_1^2+\frac{C}{\varepsilon}+\frac{j}{\delta}\|\bar{u}\|.\end{aligned}$$

Choosing $\nu_1>0$ is small enough such that $\nu_1<\min\{\frac{\delta}{2},2\}$, therefore

$$\frac{d}{dt}\Phi_{1j}(t)+\gamma_1\varepsilon\Phi_{1j}(t)\leq\varepsilon^2\Phi_{1j}(t)+\frac{C}{\varepsilon}+\frac{j}{\delta}\Phi_{10}(t). \tag{4.3}$$

Putting $j=0$ in (4.3) and using Lemma 2.1, we obtain

$$E_{10}(t)\leq C,$$

where $E_{1j}(\tau)=0$. For $g^\varsigma=0$ and by Lemmas 4.1 and 4.2, we deduce for every fixed ε,

$$\lim_{t\to+\infty}E_{10}(t)\leq C\varepsilon,\ \text{where } E_{10}(\tau)=0.$$

Similarly, we get

$$\lim_{t\to+\infty}E_{11}(t)\leq C\varepsilon\quad\text{and}\ \lim_{\varepsilon\to0^+}\lim_{t\to+\infty}E_{11}(t)=0.$$

We complete the proof. □

5. Uniform boundedness and convergence of the uniform attractors

In this section, we will prove the following facts concerning the family $\mathscr{A}^\varepsilon_\varsigma$ of uniform attractors of the processes generated by (1.1):

(i) The family $\mathscr{A}^\varepsilon_\varsigma$ is uniformly (w.r.t. ς) bounded in $\mathscr{H}^1_\varepsilon$:

$$\sup_{\varsigma\in[0,1]}\|\mathscr{A}^\varepsilon_\varsigma\|_{\mathscr{H}^1_\varepsilon}<\infty.$$

(ii) The attractor $\mathscr{A}^\varepsilon_\varsigma$ converges to $\mathscr{A}^\varepsilon_0$ as $\varsigma\to0^+$ in the standard Hausdorff semi-distance in $\mathscr{H}^1_\varepsilon$:

$$\lim_{\varsigma\to0^+}\{\text{dist}_{\mathscr{H}^1_\varepsilon}(\mathscr{A}^\varepsilon_\varsigma,\mathscr{A}_0)\}=0.$$

To prove the above results, we add the assumption for the following nonlinear function:

$$\langle f(x,u),u\rangle\geq d_0\|u\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}}-C, \tag{5.1}$$

for some $d_0>0$. Then, from (1.6) and (5.1), we deduce that there exists $d_1>0$ such that

$$d_1\|u\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}}-C\leq\langle F(x,u),1\rangle\leq C\|u\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}}+C. \tag{5.2}$$

5.1. *Uniform boundedness of the uniform attractors.* To this end, setting $G(t,\tau)=\int_\tau^t g_1(s)ds, t\geq\tau$, we assume that

$$\sup_{t\geq\tau,\tau\in\mathbb{R}} \|G(t,\tau)\|^2 \leq m^2. \tag{5.3}$$

Proposition 5.1. *Assume that* $g_1\in L_b^2(\mathbb{R};L^2(\mathbb{R}^N))$ *and satisfies* (5.3). *Then, the solution* $v(t)$ *to the problem*

$$\begin{cases} v_t-\Delta v_t-\int_0^\infty \mu_\varepsilon(s)\Delta\eta_1^t(s)ds=g_1(t/\varsigma),\\ \partial_t\eta_1^t=-\partial_s\eta_1^t+v,\\ (v(\tau),\eta_1^\tau)=(0,0), \end{cases} \tag{5.4}$$

with $\varepsilon\in(0,1]$, *satisfies the inequality*

$$\|v(t)\|^2_{H^1(\mathbb{R}^N)}+\|\eta_1^t(s)\|^2_{1,\mu_\varepsilon}\leq Cm^2\varsigma^2, \quad \forall t\geq\tau, \tag{5.5}$$

where C *is a constant independent of* g_1.

Proof. Without loss of generality, we may assume $\tau=0$, then

$$V_t(t)=v(t)=\int_0^t v_t(s)ds, \text{ because } v(0)=0,$$

$$\partial_t\bar\eta_1^t(x,s)=\eta_1^t(x,s)=\int_0^t \partial_t\eta_1^r(x,s)dr \text{ because } \eta_1^0=0.$$

Integrating (5.4) in time, we see that the function $V(t)$ solves the problem

$$\begin{cases} V_t-\Delta V_t-\int_0^\infty \mu_\varepsilon(s)\Delta\bar\eta_1^t(x,s)ds=G_\varsigma(t),\\ \partial_t\bar\eta_1^t+\partial_s\bar\eta_1^t=V, \end{cases} \tag{5.6}$$

where

$$V|_{t=0}=0, \qquad \bar\eta_1^t|_{t=0}=0,$$

$$G_\varsigma(t)=\int_0^t g_1(s/\varsigma)ds=\varsigma\int_0^{t/\varsigma} g_1(s)ds=\varsigma G(t/\varsigma,0).$$

From (5.3), we deduce that

$$\sup_{t\geq 0}\|G_\varsigma(t)\|\leq m\varsigma. \tag{5.7}$$

Thus,

$$\begin{aligned}\int_t^{t+1}\|G_\varsigma(s)\|^2ds&=\varsigma^2\int_{t/\varsigma}^{(t+1)/\varsigma}\|G(s,0)\|^2ds\\ &\leq\varsigma^2(1+\frac{1}{\varsigma})\sup_{t\geq0}\left(\int_t^{t+1}\|G(s,0)\|^2ds\right)\leq 2m^2\varsigma^2,\end{aligned}$$

i.e.

$$\sup_{t\geq0}\int_t^{t+1}\|G_\varsigma(s)\|^2ds\leq 2m^2\varsigma^2.$$

For $\gamma > 0$, as estimated in (3.11), we get

$$\int_0^t e^{-\gamma(t-s)}\|G_\varsigma(s)\|^2 ds \le Cm^2\varsigma^2.$$

Multiplying the first equation of (5.6) by $V+\varepsilon^2 V_t$ in $L^2(\mathbb{R}^N)$, the second equation of (5.6) by $j\bar{\eta}^t$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, L^2(\mathbb{R}^N))$, and adding the results, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|V\|^2+\|\nabla V\|^2+\int_0^\infty \mu_\varepsilon(s)(j\|\bar{\eta}_1^t(s)\|^2+\|\nabla\bar{\eta}_1^t(s)\|^2)ds\right)\\
&-2\int_0^\infty \mu_\varepsilon'(s)(j\|\bar{\eta}_1^t(s)\|^2+\|\nabla\bar{\eta}_1^t(s)\|^2)ds+2\varepsilon^2(\|V_t\|^2+\|\nabla V_t\|^2)\\
&=2\langle G_\varsigma(t), V+\varepsilon^2 V_t\rangle-2\varepsilon^2\int_0^\infty \mu_\varepsilon(s)\langle\nabla\bar{\eta}_1^t(s),\nabla V_t\rangle ds+2j\int_0^\infty \mu_\varepsilon(s)\langle\bar{\eta}_1^t(s),V\rangle ds.
\end{aligned}$$

Using the Hölder and Young inequality, we have

$$\begin{aligned}
&2\langle G_\varsigma(t), V+\varepsilon^2 V_t\rangle-2\varepsilon^2\int_0^\infty \mu_\varepsilon(s)\langle\nabla\bar{\eta}_1^t(s),\nabla V_t\rangle ds+2j\int_0^\infty \mu_\varepsilon(s)\langle\bar{\eta}_1^t(s),V\rangle ds\\
&\le C\|G_\varsigma(t)\|^2+\frac{\nu_2}{4}\|V\|^2+\varepsilon^2\|V_t\|^2+\varepsilon\int_0^\infty \mu_\varepsilon(s)(j\|\bar{\eta}_1^t(s)\|^2+\|\nabla\bar{\eta}_1^t(s)\|^2)ds+\varepsilon^2\|\nabla V_t\|^2\\
&+\frac{j\delta}{2\varepsilon}\int_0^\infty \mu_\varepsilon(s)\|\bar{\eta}_1^t(s)\|^2 ds+\frac{2j}{\delta}\|V\|^2.
\end{aligned}$$

Now, putting $E_{2j}=\|V\|^2+\|\nabla V\|^2+\int_0^\infty \mu_\varepsilon(s)(j\|\bar{\eta}_1^t(s)\|^2+\|\nabla\bar{\eta}_1^t(s)\|^2)ds$, $\Phi_{2j}=E_{2j}+\nu_2\varepsilon\Lambda_{2j}$, where Λ_{2j} is defined in Lemma 3.1 (with $(V,\bar{\eta}_1^t)$ in place of (u,η^t)), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Phi_{2j}+\frac{\nu_2}{4}\|V\|^2_{H^1(\mathbb{R}^N)}+\frac{\varepsilon^2(4-\nu_2)}{4}\|V_t\|^2_{H^1(\mathbb{R}^N)}+\left(\frac{\delta-2\nu_2-\varepsilon^2}{2\varepsilon}\right)\int_0^\infty \mu_\varepsilon(s)(j\|\bar{\eta}_1^t(s)\|^2+\|\nabla\bar{\eta}_1^t(s)\|^2)ds\\
&-\left(1-\frac{\nu_2\varepsilon^2\mu_\varepsilon(s_*)}{2}\right)\int_0^\infty \mu_\varepsilon'(s)(j\|\bar{\eta}_1^t(s)\|^2+\|\nabla\bar{\eta}_1^t(s)\|^2)ds\le\frac{2j}{\delta}\|V\|^2+C\|G_\varsigma(t)\|^2.
\end{aligned}$$

Choosing $\nu_2,\gamma_2>0$ is small enough, we have

$$\frac{d}{dt}\Phi_{2j}+2\gamma_2 E_{2j}\le\frac{2j}{\delta}\|V\|^2+C\|G_\varsigma(t)\|^2.$$

Up to further reducing γ_2, we also have

$$\varepsilon E_{2j}\le\Phi_{2j}\le\frac{2}{\varepsilon}E_{2j}.$$

Thus,

$$\frac{d}{dt}\Phi_{2j}+\gamma_2\varepsilon\Phi_{2j}\le\frac{2j}{\delta}\Phi_{20}+C\|G_\varsigma(t)\|^2. \tag{5.8}$$

Putting $j=0$ in (5.8) and subsequently substituting the result into (5.8) with $j=1$, we deduce that

$$\Phi_{21}(t)\le\Phi_{21}(0)e^{-\gamma t}+Cm^2\varsigma^2+C\int_\tau^t e^{-\gamma(t-r)}\|G_\varsigma(t)\|^2 dr\le Cm^2\varsigma^2.$$

Thus,

$$E_{21} = \|V\|^2 + \|\nabla V\|^2 + \int_0^\infty \mu_\varepsilon(s) \left(\|\bar{\eta}_1^t\|^2 + \|\nabla \bar{\eta}_1^t(s)\|^2\right) ds \leq Cm^2\varsigma^2. \tag{5.9}$$

Now, multiplying (5.6) by V_t, applying the Hölder and Cauchy inequalities, we obtain

$$\begin{aligned}\|V_t\|^2 + \|\nabla V_t\|^2 &\leq (G_\varsigma(t), V_t) + \left|\int_0^\infty \mu_\varepsilon(s) \left(\nabla \bar{\eta}_1^t(s), \nabla V_t\right) ds\right| \\ &\leq C\|G_\varsigma(t)\|^2 + \frac{1}{2}\left(\|V_t\|^2 + \|\nabla V_t\|^2\right) + C\int_0^\infty \mu_\varepsilon(s)\|\nabla \bar{\eta}_1^t(s)\|^2 ds.\end{aligned}$$

Using (5.9) and (5.7), we deduce that

$$\|V_t\|^2 + \|\nabla V_t\|^2 \leq Cm^2\varsigma^2, \text{ i.e., } \|v\|^2 + \|\nabla v\|^2 \leq Cm^2\varsigma^2. \tag{5.10}$$

Multiplying the second equation in (5.6) by η_1^t in $L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^1(\mathbb{R}^N))$, we get

$$\frac{d}{dt}\|\eta_1^t\|^2_{1,\mu_\varepsilon} + 2\langle \partial_s \eta_1^t, \eta_1^t\rangle^2_{1,\mu_\varepsilon} = 2\int_0^\infty \mu_\varepsilon(s)\langle \eta_1^t(s), V_t\rangle_{H^1(\mathbb{R}^N)} ds.$$

Using (1.2) we have

$$\frac{d}{dt}\|\eta_1^t\|^2_{1,\mu_\varepsilon} + \frac{\delta}{\varepsilon}\|\eta_1^t\|^2_{1,\mu_\varepsilon} \leq C\|V_t\|_1^2.$$

Using (5.10) and applying the Gronwall inequality, we have

$$\|\eta_1^t\|^2_{1,\mu_\varepsilon} \leq Cm^2\varsigma^2. \tag{5.11}$$

Combining (5.10) and (5.11), we get (5.5) as desired.

This completes the proof. □

Theorem 5.2. *Assume* **(H1)-(H3)** *and* (5.3) *hold. Then the uniform attractors $\mathscr{A}_\varsigma^\varepsilon$ are uniformly (w.r.t. ς) bounded in $\mathscr{H}_\varepsilon^1$, that is,*

$$\sup_{\varsigma \in [0,1]} \|\mathscr{A}_\varsigma^\varepsilon\|_{\mathscr{H}_\varepsilon^1} < \infty.$$

Proof. Let $z(t) = (u(t), \eta^t(s))$ be the solution to (1.1) with initial datum $z_\tau \in \mathscr{H}_\varepsilon^1$. Firstly, for $\varsigma > 0$, we consider the problem

$$\begin{cases} v_t - \Delta v_t - \int_0^\infty \mu_\varepsilon(s)\Delta \eta_1^t(s) ds = \varsigma^{-\rho} g_1(t/\varsigma), \\ \partial_t \eta_1^t = -\partial_s \eta_1^t + v, \\ (v(\tau), \eta_1^\tau) = (0,0). \end{cases}$$

From Proposition 5.1, we have

$$\|v\|^2_{H^1(\mathbb{R}^N)} + \|\eta_1^t\|^2_{1,\mu_\varepsilon} \leq Cm^2\varsigma^{2(1-\rho)}, \quad \forall t \geq \tau. \tag{5.12}$$

Then, the function $(w(t), \eta_2^t) = z(t) - (v(t), \eta_1^t)$ clearly satisfies the equation

$$\begin{cases} w_t - \Delta w_t - \int_0^\infty \mu_\varepsilon(s)\Delta \eta_2^t(s) ds + f(x,w) = -(f(x,w+v) - f(x,w)) + g_0(t), \\ \partial_t \eta_2^t = -\partial_s \eta_2^t + w, \\ (w(\tau), \eta_2^\tau) = (u_\tau, \eta_\tau). \end{cases} \tag{5.13}$$

Multiplying the first equation of (5.13) by $w+a^2w_t$ in $L^2(\mathbb{R}^N)$, the second equation of (5.13) by $j\eta_2^t$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, L^2(\mathbb{R}^N))$, and adding the results, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|w\|^2+\|\nabla w\|^2+\int_0^\infty \mu_\varepsilon(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds+2a^2\langle F(x,w),1\rangle\right)\\&+2a^2\|w_t\|^2_{H^1(\mathbb{R}^N)}-2\int_0^\infty \mu_\varepsilon'(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds+2\langle f(x,w),w\rangle+2a^2\int_0^\infty\mu_\varepsilon(s)\langle\nabla\eta_2^t(s),\nabla w_t\rangle ds\\=&-2\langle f(x,w+v)-f(x,w),w+a^2w_t\rangle+2\langle g_0(t),w+a^2w_t\rangle+2j\int_0^\infty\mu_\varepsilon(s)\langle\eta_2^t(s),w\rangle ds.\end{aligned}$$

From (1.6), (5.1), (5.2) and the embedding $H^1(\mathbb{R})\hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we get

$$\langle f(x,w),w\rangle\geq \nu_0\langle F(x,w),1\rangle-C_f,$$

$$\langle f(x,w),w\rangle\geq d_0\|w\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}}-C,$$

and

$$\begin{aligned}&|\langle f(x,w+v)-f(x,w),w\rangle|\leq C\int_{\mathbb{R}^N}\left(|\phi(x)|+|v|^{\frac{4}{N-2}}+|w|^{\frac{4}{N-2}}\right)|v||w|dx\\&\leq C\|\phi\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)}\|w\|_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{N+2}{N-2}}_{H^1(\mathbb{R}^N)}\|w\|_{H^1(\mathbb{R}^N)}+C\|v\|_{H^1(\mathbb{R}^N)}\|w\|^{\frac{N+2}{N-2}}_{H^1(\mathbb{R}^N)}\\&\leq C\|\phi\|^2_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2(N+2)}{N-2}}_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2N}{N-2}}_{H^1(\mathbb{R}^N)}+\frac{\nu_3}{4}\|w\|^2_{H^1(\mathbb{R}^N)}+\frac{d_0}{2}\|w\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)},\end{aligned}$$

similarly,

$$\begin{aligned}&|\langle f(x,w+v)-f(x,w),a^2w_t\rangle|\\&\leq Ca^2\int_{\mathbb{R}^N}\left(|\phi(x)|+|v|^{\frac{4}{N-2}}+|w|^{\frac{4}{N-2}}\right)|v||w_t|dx\\&\leq Ca^2\|\phi\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)}\|w_t\|_{H^1(\mathbb{R}^N)}+Ca^2\|v\|^{\frac{N+2}{N-2}}_{H^1(\mathbb{R}^N)}\|w_t\|_{H^1(\mathbb{R}^N)}+Ca^2\|v\|_{H^1(\mathbb{R}^N)}\|w\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}\|w_t\|_{H^1(\mathbb{R}^N)}\\&\leq Ca^2\|\phi\|^2_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+Ca^2\|v\|^{\frac{2(N+2)}{N-2}}_{H^1(\mathbb{R}^N)}+Ca^2\|v\|^{\frac{2N}{N-2}}_{H^1(\mathbb{R}^N)}+\left(\frac{a^2}{4}+\|v\|^2_{H^1(\mathbb{R}^N)}\right)\|w_t\|^2_{H^1(\mathbb{R}^N)}+Ca^4\|w\|^{\frac{8}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)},\end{aligned}$$

and

$$2j\int_0^\infty\mu_\varepsilon(s)\langle\eta_2^t(s),w\rangle ds\leq\frac{j\nu_3}{\varepsilon}\int_0^\infty\mu_\varepsilon(s)\|\eta_2^t(s)\|^2ds+\frac{j}{\nu_3}\|w\|^2,$$

and the last

$$\langle g_0(t),w+a^2w_t\rangle\leq C\|g_0(t)\|^2+\frac{\nu_3}{4}\|w\|^2+\frac{a^2}{4}\|w_t\|^2.$$

Combining all the above inequalities, we obtain

(5.14)
$$\frac{d}{dt}E_{3j}+a^2\|w_t\|^2_{H^1(\mathbb{R}^N)}-2\int_0^\infty \mu_\varepsilon'(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds+\nu_0\langle F(x,w),1\rangle+\frac{d_0}{2}\|w\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}$$
$$\leq C\|\phi\|^2_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2(N+2)}{N-2}}_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2N}{N+2}}_{H^1(\mathbb{R}^N)}+\frac{j\nu_3}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)\|\eta_2^t(s)\|^2ds+\frac{j}{\nu_3}\|w\|^2$$
$$+\frac{\nu_3}{4}\|w\|^2_{H^1(\mathbb{R}^N)}+Ca^4\|w\|^{\frac{8}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}+C(\|g_0(t)\|^2+1),$$

where $E_{3j}=\|w\|^2+\|\nabla w\|^2+\int_0^\infty \mu_\varepsilon(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds+2a^2\langle F(x,u),1\rangle,\ j=0,1.$
Putting
$$\Phi_{3j}(t)=E_{3j}+\nu_3\varepsilon\Lambda_{3j},$$
where Λ_{3j} is defined in Lemma 3.1 with (w,η_2^t) in place of (u,η^t).
From (5.14) and (1.2), we obtain

(5.15)
$$\frac{d}{dt}\Phi_{3j}+\frac{\delta-2\nu_3}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds+\frac{\nu_3}{4}\|w\|^2_{H^1(\mathbb{R}^N)}+\nu_0\langle F(x,w),1\rangle$$
$$+\left(a^2-\frac{\varepsilon^2\nu_3}{4}\right)\|w_t\|^2_{H^1(\mathbb{R}^N)}-\left(1-\frac{\nu_3\varepsilon^2\mu_\varepsilon(s_*)}{2}\right)\int_0^\infty \mu_\varepsilon'(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds+\frac{d_0}{2}\|w\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}$$
$$\leq C\|\phi\|^2_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2(N+2)}{N-2}}_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2N}{N-2}}_{H^1(\mathbb{R}^N)}+\frac{j}{\nu_3}\|w\|^2+Ca^4\|w\|^{\frac{8}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}+C(\|g_0(t)\|^2+1).$$

We consider two cases:
Case 1: $N\geq 4$. We have $\frac{8}{N-2}\leq\frac{2N}{N-2}$ for all $N\geq 4$, then

$$Ca^4\|w\|^{\frac{8}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}\leq\frac{d_0}{2}\|w\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}+C. \tag{5.16}$$

Combining with (5.15) and (5.16), choosing $\nu_3,\gamma_3>0$ is small enough, we obtain
$$\frac{d}{dt}\Phi_{3j}+\gamma_3E_{3j}\leq C\|\phi\|^2_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2(N+2)}{N-2}}_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2N}{N+2}}_{H^1(\mathbb{R}^N)}+\frac{j}{\nu_3}\|w\|^2+C(\|g_0(t)\|^2+1),$$
where $-\int_0^\infty \mu_\varepsilon'(s)(j\|\eta_2^t(s)\|^2+\|\nabla\eta_2^t(s)\|^2)ds>0$ can be neglected.
On the other hand, $\varepsilon E_{3j}\leq\Phi_{3j}\leq\frac{1}{\varepsilon}E_{3j}$ we get

(5.17)
$$\frac{d}{dt}\Phi_{3j}+\gamma_3\varepsilon\Phi_{3j}\leq\frac{j}{\nu_3}\Phi_{30}+C\|\phi\|^2_{L^{\frac{N}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2(N+2)}{N-2}}_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{2N}{N+2}}_{H^1(\mathbb{R}^N)}+C(\|g_0(t)\|^2+1).$$

Putting $j=0$ in (5.17) and subsequently substituting the result into (5.17) with $j=1$, we deduce that
$$\Phi_{31}(t)\leq C\Phi_{31}(\tau)e^{-\gamma_3(t-\tau)}+Cm^2\varsigma^{2(1-\rho)}+Cm^{\frac{2(N+2)}{N-2}}\varsigma^{\frac{2(N+2)(1-\rho)}{N-2}}+Cm^{\frac{2N}{N+2}}\varsigma^{\frac{2N(1-\rho)}{N-2}}$$
$$+C\int_\tau^t e^{-\gamma_3(t-r)}\|g_0(r)\|^2dr+C.$$

Arguing as in the proof of (3.11), we have

$$C\int_\tau^t e^{-\gamma_3(t-r)}\|g_0(r)\|^2dr \leq \frac{1}{1-e^{-\gamma_3}}\|g_0\|_{L_b^2} \leq CM_0^2, \tag{5.18}$$

thus,

$$\Phi_{31}(t) \leq C\Phi_{31}(\tau)e^{-\gamma_3(t-\tau)} + C\left(1+m^2+m^{\frac{2(N+2)}{N-2}}+m^{\frac{2N}{N+2}}+M_0^2\right). \tag{5.19}$$

Case 2: $N=3$**.** Using (5.2), we have

$$\begin{aligned} Ca^4\|w\|^{\frac{8}{N-2}}_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} &= Ca^4\|w\|^8_{L^6(\mathbb{R}^N)} \leq Ca^4(\langle F(x,w),1\rangle + C)^{\frac{4}{3}} \\ &\leq Ca^4\Phi_{3j}^{\frac{4}{3}}(t)+C. \end{aligned} \tag{5.20}$$

Combining with (5.15) and (5.20), choosing $\nu_3>0$ is small enough such that $0<\nu_3<\min\{\frac{\delta}{2},1\}$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Phi_{3j}+\gamma_4 a\Phi_{3j}(t) \\ &\leq Ca^4\Phi_{3j}^{\frac{4}{3}}(t)+\frac{j}{\nu_3}\Phi_{30}(t)+C\|\phi\|^2_{L^{\frac{3}{2}}(\mathbb{R}^N)}\|v\|^2_{H^1(\mathbb{R}^N)}+C\|v\|^5_{H^1(\mathbb{R}^N)}+C\|v\|^{\frac{6}{5}}_{H^1(\mathbb{R}^N)}+C(\|g_0(t)\|^2+1). \end{aligned} \tag{5.21}$$

From (5.21), let $j=0$, then applying Gronwall inequality in Lemma 2.1 and (5.18), we get

$$\begin{aligned} \Phi_{30}(t) &\leq \mathcal{Q}(\Phi_{30}(\tau))e^{-\gamma_4(t-\tau)}+Cm^2\varsigma^{2(1-\rho)}+Cm^5\varsigma^5(1-\rho)+Cm^{\frac{6}{5}}\varsigma^3(1-\rho) \\ &\quad +C\int_\tau^t e^{-\gamma_4(t-r)}\|g_0(r)\|^2dr+C \\ &\leq C\mathcal{Q}(\Phi_{30}(\tau))e^{-\gamma_4(t-\tau)}+C\left(1+m^2+m^5+m^{\frac{6}{5}}+M_0^2\right). \end{aligned} \tag{5.22}$$

Now consider (5.21) for $j=1$ and using (5.22) and Gronwall inequality in Lemma 2.1, we obtain

$$\Phi_{31}(t) \leq \mathcal{Q}(\Phi_{30}(\tau))e^{-\gamma_4(t-\tau)}+C\left(1+m^2+m^5+m^{\frac{6}{5}}+M_0^2\right). \tag{5.23}$$

From (5.19) and (5.23) we can see that

$$\begin{aligned} &\|w\|^2+\|\nabla w\|^2+\int_0^\infty \mu_\varepsilon(s)\left(\|\eta_2^t\|^2+\|\nabla\eta_2^t(s)\|^2\right)ds \\ &\qquad \leq \mathcal{Q}(E_{31}(\tau))e^{-\gamma_4(t-\tau)}+C\left(1+m^2+m^{\frac{2(N+2)}{N-2}}+m^{\frac{2N}{N+2}}+M_0^2\right), \quad \forall N\geq 3. \end{aligned} \tag{5.24}$$

Recalling that $z(t)=(v(t),\eta_1^t(s))+(w(t),\eta_2^t(s))$ and using (5.12) and (5.24), we have

$$\|z\|^2_{\mathcal{H}_\varepsilon^1} \leq e^{-\gamma_5(t-\tau)}\|z_\tau\|^2_{\mathcal{H}_\varepsilon^1}+C\left(1+m^2+m^{\frac{2(N+2)}{N-2}}+m^{\frac{2N}{N+2}}+M_0^2\right), \quad \forall t\geq\tau. \tag{5.25}$$

Hence the processes $U_\varsigma^\varepsilon(t,\tau)$ have an absorbing set B^*, which is independent of ς. Since $\mathscr{A}_\varsigma^\varepsilon\subset B^*$, the proof is completed.

□

5.2. *Convergence of the uniform attractors.* In this subsection, we will establish the upper semicontinuity of the uniform attractors $\mathscr{A}_\varsigma^\varepsilon$ at $\varsigma = 0$.

Theorem 5.3. *Let conditions* **(H1)-(H3)** *and* (5.3) *hold. Then, for every* $\rho \in [0,1)$, *the uniform attractor* $\mathscr{A}_\varsigma^\varepsilon$ *converges to* $\mathscr{A}_0^\varepsilon$ *as* $\varsigma \to 0^+$ *in the following sense:*

$$\lim_{\varsigma \to 0^+} \{\mathrm{dist}_{\mathscr{H}_\varepsilon^1}(\mathscr{A}_\varsigma^\varepsilon, \mathscr{A}_0^\varepsilon)\} = 0.$$

The proof of this theorem requires some steps. The first task is to compare the solutions to (1.1) corresponding to $\varsigma > 0$ and $\varsigma = 0$, respectively, starting from the same initial data. Denoting

$$z_\varsigma^\varepsilon(t) = U_\varsigma^\varepsilon(t,\tau) z_\tau,$$

where z_τ belongs to the absorbing set B^* which found in the previous section.

Besides, in order to prove the convergence of the uniform attractors, we actually need consider whole family of equations

$$\widehat{u}_t - \Delta \widehat{u}_t + f(x,u) - \int_0^\infty \mu_\varepsilon(s) \Delta \widehat{\eta}^t(s) ds = \widehat{g}^\varsigma(t), \tag{5.26}$$

with the external force $\widehat{g}^\varsigma \in \mathscr{H}_w(g^\varsigma)$. To this end, we observe that every function $\widehat{g}_1 \in \mathscr{H}_w(g_1)$ fulfills the inequality (5.3).

For any $\varsigma \in [0,1]$, we denote

$$\widehat{u^\varsigma}(t) = U_{\widehat{g}^\varsigma}(t,\tau) u_\tau,$$

where u_τ belongs to the absorbing set B^*. Therefore,

$$\widehat{z}^\varsigma(t) = (\widehat{u}^\varsigma(t), \widehat{\eta}_\varsigma^t) = U_{\widehat{g}^\varsigma}(t,\tau)\widehat{z}_\tau,$$

is the solution to (5.26) with the external force $\widehat{g}^\varsigma = \widehat{g}_0 + \varsigma^{-\rho}\widehat{g}_1(./\varsigma) \in \mathscr{H}_w(g^\varsigma)$. Since Theorem 5.2, along with the estimate of Theorem 3.8 to handle the case $\varsigma = 0$, we get the uniform bound

$$\sup_{\varsigma \in [0,1]} \|\widehat{z}^\varsigma(t)\|_{\mathscr{H}_\varepsilon^1} \le C, \quad \forall t \ge \tau.$$

Next, we define the deviation

$$\bar{z}(t) = \widehat{z}^\varsigma(t) - \widehat{z}^0(t) = (r(t), \zeta^t).$$

Lemma 5.4. *For every* $\varsigma \in (0,1]$, *we have the estimate*

$$\|\bar{z}(t)\|_{\mathscr{H}_\varepsilon^1}^2 \le C\left(\ell m^2 \varsigma^{2(1-\rho)} + m\varsigma^{1-\rho}\right) e^{C(t-\tau)} + Cm^2\varsigma^{2(1-\rho)}, \ \forall t \ge \tau, \tag{5.27}$$

for some positive constant C *independent of* $\varsigma, \tau, \widehat{g}^\varsigma$.

Proof. Let $(v(t), \eta_1^t)$ be the solution to the auxiliary problem (5.4) with null initial datum $(v_\tau, \eta_1^\tau) = (0,0)$. The difference $(w(t), \eta_2^t) = \bar{z}(t) - (v(t), \eta_1^t) = (r(t), \zeta^t) - (v(t), \eta_1^t)$ clearly satisfies the equations

$$\begin{cases} w_t - \Delta w_t - \int_0^\infty \mu_\varepsilon(s) \Delta \eta_2^t(s) ds + f(x,u^\varsigma) - f(x,u^0) = 0, \\ \partial_t \eta_2^t = -\partial_s \eta_2^t + w, \\ (w(\tau), \eta_2^\tau) = (0,0). \end{cases}$$

Multiplying the first equation of (5.2) by w in $L^2(\mathbb{R}^N)$, the second equation of (5.2) by $\eta_2^t(s)$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, L^2(\mathbb{R}^N))$, and adding the results, we get

$$\begin{aligned}\frac{d}{dt}\Big(\|w\|^2+\|\nabla w\|^2+\int_0^\infty \mu_\varepsilon(s)(\|\eta_2^t\|^2+\|\nabla\eta_2^t\|^2)ds\Big)\\ -2\int_0^\infty \mu_\varepsilon'(s)(\|\eta_2^t\|^2+\|\nabla\eta_2^t\|^2)ds+2(f(x,u^\varsigma)-f(x,u^0),w+v)\\ \le 2\left|(f(x,u^\varsigma)-f(x,u^0),v)\right|+2\int_0^\infty \mu_\varepsilon(s)\langle\eta_2^t(s),w\rangle ds.\end{aligned}$$

Using conditions (1.5) and (1.6), we obtain

$$\begin{aligned}2(f(x,u^\varsigma)-f(x,u^0),w+v)&=2\int_\Omega f'(\xi)(w+v)^2dx\\ &\ge -2\ell\|w+v\|^2\ge -C\ell(\|w\|^2+\|v\|^2),\end{aligned}$$

and

$$\begin{aligned}2\left|(f(x,u^\varsigma)-f(x,u^0),v)\right|&\le 2\int_\Omega(|f(u^\varsigma)|+|f(u^0)|)|v|dx\\ &\le 2\left(\|f(u^\varsigma)\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}+\|f(u^0)\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}\right)\|v\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}\\ &\le C\|v\|_{H^1(\mathbb{R}^N)},\end{aligned}$$

and

$$2\int_0^\infty \mu_\varepsilon(s)\langle\eta_2^t(s),w\rangle ds\le\frac{1}{\delta}\|w\|^2+\frac{\delta}{\varepsilon}\int_0^\infty\mu_\varepsilon(s)\|\eta_2^t(s)\|^2ds.$$

Besides, using (1.2), we have

$$0\le-\int_0^\infty\mu_\varepsilon'(s)(\|\eta_2^t\|^2+\|\nabla\eta_2^t\|^2)ds\le\frac{\delta}{\varepsilon}\int_0^\infty\mu_\varepsilon(s)(\|\eta_2^t\|^2+\|\nabla\eta_2^t\|^2)ds.$$

Combining all the above inequalities and using (5.25) and we get

$$\frac{d}{dt}\left(\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|^2_{1,\mu_\varepsilon}\right)\le C\left(\ell+\frac{1}{\delta}\right)\|w\|^2+C\ell\|v\|^2+C\|v\|_{H^1(\mathbb{R}^N)},$$

thus

$$\frac{d}{dt}y(t)\le Cy(t)+C\ell m^2\varsigma^{2(1-\rho)}+Cm\varsigma^{(1-\rho)},$$

where

$$y(t)=\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|^2_{1,\mu_\varepsilon}.$$

Since $(w(\tau),\eta_2^\tau)=(0,0)$, the Gronwall inequality leads to

$$\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|^2_{1,\mu_\varepsilon}\le C\left(\ell m^2\varsigma^{2(1-\rho)}+m\varsigma^{1-\rho}\right)e^{C(t-\tau)},\quad\forall t\ge\tau.$$

Finally, recalling that $(w(t),\eta_2^t)=(r(t),\zeta^t)-(v(t),\eta_1^t)$, using again (5.12), we obtain the desired estimate (5.27). □

Proof of Theorem 5.3. Although the proof of this theorem is similar in [19, Theorem 4.4], we present here another (simpler) proof for the completeness and convenience of the reader.

For $\varsigma > 0$, let $z^\varsigma \in \mathscr{A}^\varepsilon_\varsigma$. Thus, in view of (3.8), there exists a complete bounded trajectory $\widehat{z}^\varsigma(t)$ of (5.26), with the external force

$$\widehat{g}^\varsigma = \widehat{g}_0 + \varsigma^{-\rho}\widehat{g}_1(./\varsigma) \in \mathscr{H}_w(g^\varsigma), \quad \text{where } \widehat{g}_0 \in \mathscr{H}_w(g_0),\ \widehat{g}_1 \in \mathscr{H}_w(g_1),$$

such that $\widehat{z}^\varsigma(0) = z^\varsigma$.

By Lemma 5.4 with $t = 0$,

$$\|z^\varsigma - U_{\widehat{g}_0}(0,\tau)\widehat{z}^\varsigma(\tau)\|_{\mathscr{H}^1_\varepsilon} \le C\left(\ell m^2\varsigma^{2(1-\rho)} + m\varsigma^{1-\rho}\right)e^{C\tau} + Cm^2\varsigma^{2(1-\rho)}, \quad \forall \tau \le 0.$$

Besides, it is known (see e.g. [6]) that the set $\mathscr{A}^\varepsilon_0$ attracts $U_{\widehat{g}_0}(t,\tau)B^*$, uniformly not only with respect to $\tau \in \mathbb{R}$, but also with respect to $\widehat{g}_0 \in \mathscr{H}_w(g^0)$. Then, for every $\delta > 0$, there is $\tau = \tau(\delta) \le 0$ independent of ς such that

$$\operatorname{dist}_{\mathscr{H}^1_\varepsilon}\left(U_{\widehat{g}_0}(0,\tau)\widehat{z}^\varsigma(\tau), \mathscr{A}^\varepsilon_0\right) \le \delta.$$

Using the triangle inequality we get

$$\operatorname{dist}_{\mathscr{H}^1_\varepsilon}\left(z^\varsigma, \mathscr{A}^\varepsilon_0\right) \le C\left(\ell m^2\varsigma^{2(1-\rho)} + m\varsigma^{1-\rho}\right)e^{C(t-\tau)} + Cm^2\varsigma^{2(1-\rho)} + \delta.$$

Since $z^\varsigma \in A^\varepsilon_\varsigma$ is arbitrary, we reach the conclusion

$$\limsup_{\varsigma\to 0^+}\{\operatorname{dist}_{\mathscr{H}^1_\varepsilon}(\mathscr{A}^\varepsilon_\varsigma, A^\varepsilon_0)\} \le \delta.$$

Letting $\delta \to 0$ we complete the proof. □

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Foundation Sciences Faculty, Telecommunications University
101 Mai Xuan Thuong, Nha Trang, Khanh Hoa, Vietnam
Email address: dangthanhson@tcu.edu.vn, dangthanhson1810@gmail.com

Department of Mathematics, Haiphong University
171 Phan Dang Luu, Kien An, Haiphong, Vietnam
Email address: toannd@dhhp.edu.vn