On the localization of solutions for third-order nonlinear impulsive BVPs on an infinite interval

Ali Zerki[1,2*†]

[1]National Higher School of Hydraulics, Blida, ALGERIA.
[2]Department of Mathematics, Laboratory of Fixed-Point Theorem and Applications, Higher Normal School of Kouba, BP 92, Kouba, 16006, Algiers, ALGERIA.

Corresponding author(s). E-mail(s): alizerki28@gmail.com & a.zerki@ensh.dz;
[†]These authors contributed equally to this work.

Abstract

The following third-order nonlinear impulsive BVP is analyzed in this study to demonstrate the existence and location of solutions. We apply the upper and lower solutions method with a Nagumo-type condition, utilizing the Schaefer fixed-point theorem, when the nonlinearity is a Carathéodory function,

$$\begin{cases} v'''(t) + g(t, v(t), v'(t), v''(t)) = 0, \ \ t \in [0, +\infty)\backslash\{t_1, t_2, \dots\}, \\ v(0) = A, v'(0) = B, \quad v''(+\infty) = C, \\ \triangle v(t_k) = I_{1k}(t_k, v(t_k), v'(t_k)), \\ \triangle v'(t_k) = I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k)), \\ \triangle v''(t_k) = I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k)), \\ v''(t_k^+) = I_{3k}^+(t_k^+, v(t_k^+), v'(t_k^+), v''(t_k^+)). \end{cases}$$

Keywords: Impulsive conditions, Nagumo condition, differential equations, upper and lower solutions.

MSC Classification: 34A36 , 34B15 , 34B37 , 47H10.

1 May 2024 15:31:48 PDT
231121-Zerki Version 2 - Submitted to Rocky Mountain J. Math.

1 Introduction

Third-order differential equations, irrespective of their forms, have several mathematical uses in the fields of mechanics, physics, ... etc. This motivated researchers to broaden the scope of their research utilizing various methods, to find solutions or numerically calculate the solutions for third-order BVPs, by studying the existence of solutions or locating them, as shown in [1, 3, 6, 8, 9, 11, 12, 18].

Recent years have witnessed an increase in interest in impulsive BVPs, leading to the development of new studies and the attainment of significant conclusions regarding these problems (see some references [5, 7, 10, 15, 21, 23]).

Numerous articles provide results for these kinds of problems using the upper and lower solutions method, with or without Nagumo-type conditions. This method allows us to locate the solution and its derivatives. For example, we reference [4, 13, 14, 17, 19].

In [16], Minhós and Carapinha studied the existence and location of solutions using this method with the Nagumo condition for the following third-order ϕ-Laplacian impulsive boundary value problem on a bounded interval with finite impulse moments.

$$\begin{cases} (\phi(x''))'(t) + q(t)f(t, x(t), x'(t), x''(t)) = 0, \ \ t \in [a, b]\backslash\{t_1, \dots, t_n\}, \\ x(a) = A, \ x'(a) = B, \quad x''(b) = C, \\ \triangle x(t_k) = I_{1k}(t_k, x(t_k), x'(t_k)), \\ \triangle x'(t_k) = I_{2k}(t_k, x(t_k), x'(t_k), x''(t_k)), \\ \phi(x''(t_k^+)) = I_{3k}(t_k, x(t_k), x'(t_k), x''(t_k)), \end{cases}$$

where ϕ is an increasing homeomorphism with $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, f is a continuous function, and q is a positive continuous function such that $\int_a^b q(s)ds < +\infty$. The Nagumo condition enables us to prove that the last derivative of the unknown can be estimated from its remaining derivatives. Moreover, in [17], they proved the existence and localization of solutions adopting this method, without the Nagumo condition, combined with the Schauder fixed-point theorem, when the nonlinearity is an L^1-Carathéodory function and the impulsive conditions are given by Carathéodory sequences, for the problem

$$\begin{cases} x'''(t) + f(t, x(t), x'(t), x''(t)) = 0, \ \ t \in [0, +\infty)\backslash\{t_1, t_2, \dots\}, \\ x(0) = A, \ x'(0) = B, \quad x''(+\infty) = C, \\ \triangle x(t_k) = I_{1k}(t_k, x(t_k), x'(t_k), x''(t_k)), \\ \triangle x'(t_k) = I_{2k}(t_k, x(t_k), x'(t_k), x''(t_k)), \\ \triangle x''(t_k) = I_{3k}(t_k, x(t_k), x'(t_k), x''(t_k)). \end{cases}$$

Our work is based on the results of the previous two studies, to show the existence and localization of solutions for the next third-order impulsive boundary value problem (1), (2), (3) on the half-line, with infinite impulse moments, using the Schaefer fixed-point theorem and the upper and lower solutions method with a Nagumo-type

1 May 2024 15:31:48 PDT
231121-Zerki Version 2 - Submitted to Rocky Mountain J. Math.

condition.

$$v'''(t) + g(t, v(t), v'(t), v''(t)) = 0,\ a.e.,\ \ t \in [0, +\infty)\backslash\{t_1, t_2, \ldots\}, \tag{1}$$

where $g : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, the boundary conditions are

$$v(0) = A,\ v'(0) = B, \quad v''(+\infty) = C, \tag{2}$$

$A, B, C \in \mathbb{R}$ and $v''(+\infty) := \lim\limits_{t\to+\infty} v''(t)$. The impulsive conditions are defined as follows:

$$\begin{cases} \triangle v(t_k) = I_{1k}(t_k, v(t_k), v'(t_k)), \\ \triangle v'(t_k) = I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k)), \\ \triangle v''(t_k) = I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k)), \\ v''(t_k^+) = I_{3k}^+(t_k^+, v(t_k^+), v'(t_k^+), v''(t_k^+)), \end{cases} \tag{3}$$

where $\triangle v^{(i)}(t_k) = v^{(i)}(t_k^+) - v^{(i)}(t_k^-)$, $i = 0, 1, 2$ such that $I_{2k},\ I_{3k} \in C([0, +\infty) \times \mathbb{R}^3, \mathbb{R})$, and $I_{1k} \in C([0, +\infty) \times \mathbb{R}^2, \mathbb{R})$, where $0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$, and $t_k \to +\infty$ as $k \to +\infty$. $J = [0, +\infty)$ and $J' = [0, +\infty)\backslash\{t_1, t_2, \ldots\}$. Take $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$ for $k \in \mathbb{N}^*$.

The definitions of the upper and lower solutions and the bilateral Nagumo conditions are presented in Section 2. We justify our fundamental result in Section 3. Finally, in Section 4, we show a clear example illustrating a possible application of the existence theorem.

2 Definitions and preliminary results

Let the set

$$\begin{aligned} PC^m[0, +\infty) =&\{v \in C^m([0, +\infty), \mathbb{R}),\ \text{for } t \neq t_k,\ v^{(i)}(t_k) = v^{(i)}(t_k^-),\ v^{(i)}(t_k^+) \text{ exist} \\ &\text{for } k = 1, 2, 3, \cdots,\ \text{for } i = 0, 1, \cdots, m\}. \end{aligned}$$

In this work, we consider the following space.

$$X = \left\{ v \in PC^2[0, +\infty) : \lim_{t\to+\infty} v''(t) \text{ exists in } \ \mathbb{R} \ \right\},$$

with the norm $\|x\|_X := \max\{\|x\|_0, \|x'\|_1, \|x''\|_2\}$, where

$$\|\omega^{(i)}\|_i = \sup_{0\leq t<+\infty} \left| \frac{\omega^{(i)}(t)}{1 + t^{2-i}} \right|,\ i = 0, 1, 2.$$

From [5, 25], we have that $(X, \|\cdot\|_X)$ is a Banach space, and for all $v \in X$, $\lim\limits_{t\to+\infty} v''(t) = \lim\limits_{t\to+\infty} \frac{v'(t)}{1+t}$ and $\frac{1}{2} \lim\limits_{t\to+\infty} v''(t) = \lim_{t\to+\infty} \frac{v(t)}{1+t^2}$.

The following definition presents the assumptions regarding nonlinearity.

1 May 2024 15:31:48 PDT
231121-Zerki Version 2 - Submitted to Rocky Mountain J. Math.

Definition 1. A function $g : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is called a Carathéodory function if it satisfies:

(i) for each $(z_1, z_2, z_3) \in \mathbb{R}^3$, $t \mapsto g(t, z_1, z_2, z_3)$ is measurable on $[0, +\infty)$;
(ii) for almost every $t \in [0, +\infty), (z_1, z_2, z_3) \mapsto g(t, z_1, z_2, z_3)$ is continuous in $\mathbb{R}^3$.

The following lemma provides the solution to the linear impulsive boundary value problem associated with (1), (2), (3).

Lemma 1. *Let $\eta \in L^1[0, +\infty)$, and suppose that the series $\sum_{k=1}^{+\infty} I_{1k}(t_k, v(t_k), v'(t_k))$, $\sum_{k=1}^{+\infty} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k))$ and $\sum_{k=1}^{+\infty} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k))$ are convergent. Then the linear impulsive boundary value problem*

$$v'''(t) + \eta(t) = 0, \; a.e., \;\; t \in J', \tag{4}$$

with boundary conditions (2), (3), has a unique solution in X expressed as

$$\begin{aligned} v(t) =& A + Bt + \frac{t^2}{2}C + \sum_{t_k<t} I_{1k}(t_k, v(t_k), v'(t_k)) \\ &+ \sum_{t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k))(t - t_k) \\ &+ \int_0^t (t-s) \left(\int_s^{+\infty} \eta(\tau)d\tau - \sum_{t_k>s} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k)) \right) ds. \end{aligned} \tag{5}$$

In order, we provide an *a priori* estimation for v''.

Let $\gamma, \Gamma \in PC^1[0, +\infty), \gamma^{(i)}(t) \leq \Gamma^{(i)}(t), i = 0, 1$ such that $\sup_{k=0,1,2,\ldots} \left\{ \left| \frac{\Gamma'(t_{k+1}) - \gamma'(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\gamma'(t_{k+1}) - \Gamma'(t_k)}{t_{k+1} - t_k} \right| \right\} = \mu$ exists, where $t_0 = 0$.
Define the set

$$E = \left\{ (t, x_0, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^3 : \gamma^{(i)}(t) \leq x_i \leq \Gamma^{(i)}(t), i = 0, 1 \right\}.$$

Definition 2. A Carathéodory function $g : E \to \mathbb{R}$ is said to satisfy the Nagumo-type growth condition on E, if it satisfies

$$|g(t, z_0, z_1, z_2)| \leq \psi(t)h(|z_2|), \quad \forall (t, z_0, z_1, z_2) \in E, \tag{6}$$

for some positive continuous functions ψ, h, such that

$$\int_0^{+\infty} \psi(s)ds < +\infty, \; \int_0^{+\infty} \psi(s)ds < \int_\mu^{+\infty} \frac{1}{h(s)}ds. \tag{7}$$

Lemma 2. *Let $g : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be a Carathéodory function satisfying (6) on E. Then there exists $R \geq \mu > 0$ such that every v solution of (1), (2), (3) satisfying*

$$\gamma(t) \leq v(t) \leq \Gamma(t), \; \gamma'(t) \leq v'(t) \leq \Gamma'(t), \tag{8}$$

1 May 2024 15:31:48 PDT
231121-Zerki Version 2 - Submitted to Rocky Mountain J. Math.

for $t \in [0,+\infty)$*, satisfies* $\|v''\|_2 < R$.

Proof. Let v be a solution to problem (1), (2), (3) such that $\gamma(t) \le v(t) \le \Gamma(t)$ and $\gamma'(t) \le v'(t) \le \Gamma'(t)$, for $t \in [0,+\infty)$. By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ such that

$$v''(\eta_0) = \frac{v'(t_{k+1}) - v'(t_k)}{t_{k+1} - t_k}, \text{ for each } k = 0, 1, 2, \cdots . \tag{9}$$

Moreover,

$$-R \le \mu \le \frac{\gamma'(t_{k+1}) - \Gamma'(t_k)}{t_{k+1} - t_k} \le v''(\eta_0) \le \frac{\Gamma'(t_{k+1}) - \gamma'(t_k)}{t_{k+1} - t_k} \le \mu \le R.$$

If $|v''(t)| \le R$ for all $t \in (0,+\infty)$, where $R > \mu$, then the proof is complete, such that

$$\|v''\|_2 = \sup_{0<t<+\infty} \left|\frac{v''(t)}{2}\right| = \sup_{0\le t<+\infty} \left|\frac{v''(t)}{2}\right| \le \frac{R}{2} < R.$$

Assume there exists $\tau \in (0,+\infty)$ such that $|v''(\tau)| > R$. Considering that $v''(\tau) > R$, then there exists η_1 such that $v''(\eta_1) = R$, where $\int_0^{+\infty} \psi(s)ds < \int_\mu^R \frac{1}{h(s)}ds$. If $\eta_0 < \eta_1$, suppose without loss of generality that

$$v''(t) > 0 \text{ and } v''(\eta_0) \le v''(t) \le R = v''(\eta_1), \text{ for } t \in [\eta_0, \eta_1].$$

So,

$$|v'''(t)| = |g(t, v(t), v'(t), v''(t))| \le \psi(t)h(v''(t)), \text{ for } t \in [\eta_0, \eta_1].$$

From (6),

$$\int_{v''(\eta_0)}^{v''(\eta_1)} \frac{1}{h(s)}ds \le \int_{\eta_0}^{\eta_1} \frac{|v'''(s)|}{h(v''(s))}ds = \int_{\eta_0}^{\eta_1} \frac{|g(t, v(s), v'(s), v''(s))|}{h(v''(s))}ds$$
$$\le \int_{\eta_0}^{\eta_1} \psi(s)ds \le \int_0^{+\infty} \psi(s)ds,$$

since $v''(\eta_0) \le \mu < R$, and

$$\int_{v''(\eta_0)}^{v''(\eta_1)} \frac{1}{h(s)}ds \ge \int_\mu^{v''(\eta_1)} \frac{1}{h(s)}ds > \int_0^{+\infty} \psi(s)ds,$$

which is a contradiction.
If $\eta_1 < \eta_0$, also suppose, without loss of generality, that:

$$v''(t) > 0 \text{ and } v''(\eta_0) \le v''(t) \le R = v''(\eta_1), \text{ for } t \in [\eta_1, \eta_0].$$

So,

$$|v'''(t)| = |g(t, v(t), v'(t), v''(t))| \le \psi(t)h(v''(t)), \text{ for } t \in [\eta_1, \eta_0].$$

From (6), we have that

$$\int_{v''(\eta_0)}^{v''(\eta_1)} \frac{1}{h(s)}ds \le \int_{\eta_1}^{\eta_0} \frac{|v'''(s)|}{h(v''(s))}ds = \int_{\eta_1}^{\eta_0} \frac{|g(t,v(s),v'(s),v''(s))|}{h(v''(s))}ds$$
$$\le \int_{\eta_1}^{\eta_0} \psi(s)ds \le \int_0^{+\infty} \psi(s)ds,$$

as $v''(\eta_0) \le \mu < R$, and

$$\int_{v''(\eta_0)}^{v''(\eta_1)} \frac{1}{h(s)}ds \ge \int_{\mu}^{v''(\eta_1)} \frac{1}{h(s)}ds > \int_0^{+\infty} \psi(s)ds,$$

which is a contradiction.
The other cases for $v''(\tau) < -R$, we use the same arguments to arrive at a contradiction. □

To apply a fixed-point theorem, the following lemma will be essential.
Lemma 3. *[2] A set $U \subset X$ is relatively compact if:*

1. *U is uniformly bounded;*
2. *U is equicontinuous on any compact interval of $[0,+\infty)$;*
3. *U is equiconvergent at infinity.*

The following definitions represent the functions considered to be the lower and upper solutions to the main problem.
Definition 3. A function $\alpha(t) \in X \cap PC^3[0,+\infty)$ is a lower solution of problem (1), (2), (3) if

$$\begin{cases} \alpha'''(t) + g(t,\alpha(t),\alpha'(t),\alpha''(t)) \ge 0, \ a.e., \ t \in [0,+\infty)\backslash\{t_1,t_2,\dots\}, \\ \triangle\alpha(t_k) \le I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)), \\ \triangle\alpha'(t_k) \ge I_{2k}(t_k,\alpha(t_k),\alpha'(t_k),\alpha''(t_k)), \\ \alpha''(t_k^+) \ge I_{3k}^+(t_k^+,\alpha(t_k^+),\alpha'(t_k^+),\alpha''(t_k^+)), \\ \alpha(0) \le A, \\ \alpha'(0) \le B, \\ \alpha''(+\infty) < C. \end{cases} \tag{10}$$

A function $\beta(t) \in X \cap PC^3[0,+\infty)$ and satisfies the opposite inequalities above, is an upper solution of problem (1), (2), (3).
Lemma 4. *[22, 24] For $y,z \in PC^1[0,+\infty)$ such that $y(t) \le z(t)$, for every $t \in [0,+\infty)$, define:*

$$q(t,x(t)) = \begin{cases} z(t), & x(t) > z(t) \\ x(t), & y(t) \le x(t) \le z(t) \\ y(t), & x(t) < y(t). \end{cases}$$

Then, for each $x \in PC^1[0,+\infty)$*, the next propertie holds:*

If $x, x_m \in PC^1[0,+\infty)$ *and* $\sup_{t\in[0,+\infty)} \left| \frac{x_m^{(i)}(t)-x^{(i)}(t)}{1+t^{1-i}} \right| \to 0$*, as* $m \to +\infty$*, for* $i = 0, 1$*, then:*

$$\frac{d}{dt} q\left(t, x_m(t)\right) \to \frac{d}{dt} q(t, x(t)) \text{ for a.e. } t \in [0,+\infty),$$

as $m \to +\infty$*.*

Presentation of the Schaefer Fixed Point Theorem

Theorem 1 ([20, Schaefer])**.** *Let* X *be a Banach space and* $T : X \to X$ *be a completely continuous operator. If the set*

$$\{x \in X : \; x = \lambda T x \quad \text{for } \lambda \in (0,1)\}$$

is bounded, then T *has at least one fixed point in* X*.*

3 Main result

In this section, we demonstrate that there is at least one solution to the problem (1), (2), (3).

Theorem 2. *Let* $g : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ *be a Carathéodory function, and* α, β *lower and upper solutions of* (1), (2), (3)*, respectively, such that*

$$\alpha'(t) \le \beta'(t), \quad \forall t \in [0,+\infty), \tag{11}$$

where $\sup_{k=0,1,2,\dots} \left\{ \left| \frac{\beta'(t_{k+1})-\alpha'(t_k)}{t_{k+1}-t_k} \right|, \left| \frac{\alpha'(t_{k+1})-\beta'(t_k)}{t_{k+1}-t_k} \right| \right\}$ *exists, with* $t_0 = 0$*.*

If g *satisfies Nagumo condition* (6)*, on the set*

$$E_* = \Big\{ (t, x, y, z) \in [0,+\infty) \times \mathbb{R}^3 : \alpha(t) \le x \le \beta(t), \; \alpha'(t) \le y \le \beta'(t) \Big\},$$

and

$$g(t, \alpha(t), y, z) \le g(t, x, y, z) \le g(t, \beta(t), y, z), \tag{12}$$

for (t, y, z) *fixed and* $\alpha(t) \le x \le \beta(t)$*. Moreover, if the impulsive functions satisfy*

$$I_{1k}(t_k, \alpha(t_k), \alpha'(t_k)) \le I_{1k}(t_k, x, y) \le I_{1k}(t_k, \beta(t_k), \beta'(t_k)), \tag{13}$$

for $\alpha(t_k) \le x \le \beta(t_k)$*,* $\alpha'(t_k) \le y \le \beta'(t_k)$*,* $k = 1, 2, \dots,$

$$I_{2k}(t_k, \alpha(t_k), y, z) \ge I_{2k}(t_k, x, y, z) \ge I_{2k}(t_k, \beta(t_k), y, z), \tag{14}$$

and

$$I_{3k}^+(t_k^+, \alpha(t_k^+), y, z) \ge I_{3k}^+(t_k^+, x, y, z) \ge I_{3k}^+(t_k^+, \beta(t_k^+), y, z), \tag{15}$$

for $\alpha(t_k) \leq x \leq \beta(t_k)$, $k = 1, 2, \ldots$, *and* (y, z) *fixed in* $\mathbb{R}^2$, *and if there exist positive* a_1, a_2 *and* a_3 *such that*

$$\begin{cases} \sum_{k=1}^{\infty} \left| I_{1k}(t_k, x_k, y_k) \right| \leq \rho_1 a_1 < +\infty, \\ \sum_{k=1}^{\infty} \left| I_{2k}(t_k, x_k, y_k, z_k) \right| \leq \rho_1 a_2 < +\infty, \\ \sum_{k=1}^{\infty} \left| I_{3k}(t_k, x_k, y_k, z_k) \right| \leq \rho_1 a_3 < +\infty, \\ \textit{where } \rho_1 = \max\{R, \|\alpha\|_X, \|\beta\|_X\}, \ (R \textit{ is defined in Lemma 2}). \end{cases} \tag{16}$$

for $\alpha(t_k) \leq x_k \leq \beta(t_k)$, $\alpha'(t_k) \leq y_k \leq \beta'(t_k)$ *and* $-R \leq z_k \leq R$, $k = 1, 2, \ldots$, *with* $R > 0$. *Then problem* (1), (2), (3) *has at least one solution* $v \in X$ *and there exists* $N > 0$, *such that*

$$\alpha(t) \leq v(t) \leq \beta(t), \alpha'(t) \leq v'(t) \leq \beta'(t),$$
$$-N \leq v''(t) \leq N, \ t \in [0, +\infty).$$

Proof. Consider the j-modified equation for $j = 1, 2$ as follows:

$$\begin{aligned} &v'''(t) + g(t, \delta_0(t, v), \delta_1(t, v'), \delta_{2j}(t, v)) \\ &\quad - \frac{1}{1 + t^2} \frac{v'(t) - \delta_1(t, v')}{1 + |v'(t) - \delta_1(t, v')|} = 0, \ a.e., \ t \in J', \end{aligned} \tag{17}$$

such that the functions $\delta_i, \delta_{2j} : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, i = 0, 1, \ j = 1, 2$ are given by

$$\delta_i(t, x) = \begin{cases} \beta^{(i)}(t), & x > \beta^{(i)}(t) \\ x, & \alpha^{(i)}(t) \leq x \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & x < \alpha^{(i)}(t), \end{cases} \qquad i = 0, 1.$$

From Lemma 4, we can establish

$$\delta_{22}(t, v''(t)) = \frac{d\delta_1(t, v'(t))}{dt}, \ a.e \ t \in [0, +\infty) \text{ and}$$

$$\delta_{21}(t, v''(t)) = \begin{cases} N, & \delta_{22}(t, v''(t)) > N \\ \delta_{22}(t, v''(t)), & -N \leq \delta_{22}(t, v''(t)) \leq N \\ -N, & \delta_{22}(t, v''(t)) < -N, \end{cases}$$

$a.e,\ t\in[0,+\infty)$, where $N>\max\{\sup_{t\in[0,+\infty)}|\alpha''(t)|,\sup_{t\in[0,+\infty)}|\beta''(t)|\}$.
The j-impulsive conditions are

$$\begin{cases}\triangle v(t_k)=I_{1k}(t_k,\delta_0(t_k,v(t_k)),\delta_1(t_k,v'(t_k))),\\ \triangle v'(t_k)=I_{2k}(t_k,\delta_0(t_k,v(t_k)),\delta_1(t_k,v'(t_k)),\delta_{2j}(t_k,v''(t_k))),\\ \triangle v''(t_k^+)=I_{3k}(t_k,\delta_0(t_k,v(t_k)),\delta_1(t_k,v'(t_k)),\delta_{2j}(t_k,v''(t_k))),\\ v''(t_k^+)=I_{3k}^+(t_k^+,\delta_0(t_k^+,v(t_k^+)),\delta_1(t_k^+,v'(t_k^+)),\delta_{2j}(t_k^+,v''(t_k^+))).\end{cases}\tag{18}$$

For clarity, we do this proof in three steps.

Step 1: Every solution of (17), (2), (18) for $j=1,2$ satisfies $\alpha'(t)\le v_j'(t)\le\beta'(t)$ for all $t\in[0,+\infty)$. Let v_j be a solution of the j-modified problem (17), (2), (18). Suppose, by contradiction, that there is $t\in(0,+\infty)$ such that $\alpha'(t)>v_j'(t)$. Moreover,

$$\sup_{0\le t<+\infty}(v_j'(t)-\beta'(t))=v_j'(t_*)-\beta'(t_*)>0.$$

By (10) and (2), we have that $v_j'(0)-\beta'(0)\le 0$ and $v_j''(+\infty)-\beta''(+\infty)<0$, $\sup_{0\le t<+\infty}(v_j'(t)-\beta'(t))$ cannot be attained at 0 and $+\infty$.

If there is $t_*\in(0,+\infty)$ then, we can define

$$\sup_{0\le t<+\infty}(v_j'(t)-\beta'(t)):=v_j'(t_*)-\beta'(t_*)>0.$$

Let's consider the two cases.

Case 1: Assume there exists $k\in\mathbb{N}$ such that $t_*\in(t_k,t_{k+1})$, where $t_0=0$. Then there exists $(t_*,\bar t)\subset(t_k,t_{k+1})$ such that:

$$v_j'(t)-\beta'(t)>0,\ \ v_j''(t)-\beta''(t)\le 0 \text{ for all } t\in(t_*,\ \bar t)\ \ j=1,2.\tag{19}$$

Using (12) and Definition 3,

$$\begin{aligned}v_j'''(t)-\beta'''(t)&\ge -g(t,\delta_0(t,v_j(t)),\delta_1(t,v_j'(t)),\delta_{2j}(t,v_j''(t)))\\&\quad+\frac{1}{1+t^2}\frac{v_j'(t)-\delta_1(t,v_j'(t))}{1+|v_j'(t)-\delta_1(t,v_j'(t))|}+g(t,\beta(t),\beta'(t),\beta''(t))\\&=-g(t,\delta_0(t,v_j(t)),\beta'(t),\beta''(t))\\&\quad+\frac{1}{1+t^2}\frac{v_j'(t)-\delta_1(t,v_j'(t))}{1+|v_j'(t)-\delta_1(t,v_j'(t))|}+g(t,\beta(t),\beta'(t),\beta''(t))\\&\ge -g(t,\delta_0(t,v_j(t)),\beta'(t),\beta''(t))\\&\quad+\frac{1}{1+t^2}\frac{v_j'(t)-\beta'(t)}{1+|v_j'(t)-\beta'(t)|}+g(t,\beta(t),\beta'(t),\beta''(t))\\&\ge+\frac{1}{1+t^2}\frac{v_j'(t)-\beta'(t)}{1+|v_j'(t)-\beta'(t)|}>0,\ a.e.,\ \ t\in(t_*,\ \bar t).\end{aligned}$$

Then, $v_j'''(t) - \beta'''(t) > 0$, a.e., $t \in (t_*,\ \bar{t}) \subset (t_k, t_{k+1})$. The function $v_j''(t) - \beta''(t)$ is increasing for all $t \in (t_*,\ \bar{t})$, If $t \in (t_*,\ \bar{t})$,

$$0 = v_j''(t_*) - \beta''(t_*) < v_j''(t) - \beta''(t)$$

and $v_j''(t) - \beta''(t) > 0$. The function $v_j'(t) - \beta'(t)$ is increasing on $(t_*,\ \bar{t})$, which leads to a contradiction.

Case 2: Suppose that $k \in \mathbb{N}^*$ such that

$$\sup_{t\in[0,+\infty)} (v_j'(t) - \beta'(t)) = \max_{t\in[0,+\infty)} (v_j'(t) - \beta'(t)) = v_j'(t_k) - \beta'(t_k) > 0. \tag{20}$$

If we have (20) and $\sup_{t\in[0,+\infty)}(v_j'(t) - \beta'(t)) = v_j'(t_k^+) - \beta'(t_k^+) > 0$, then

$$v_j''(t_k^+) - \beta''(t_k^+) \leq 0$$

and there is $\varepsilon > 0$ such that $v_j''(t) - \beta''(t) \leq 0$ and $v_j'(t) - \beta'(t) > 0\ \ \forall t \in (t_k, t_k + \varepsilon) \subset [t_k, t_{k+1}]$. Using Definition 3 and (12), we have

$$\begin{aligned}
v_j'''(t) - \beta'''(t) &\geq -g(t, \delta_0(t, v_j(t)), \delta_1(t, v_j'(t)), \delta_{2j}(t, v_j''(t))) \\
&\quad + \frac{1}{1+t^2} \frac{v_j'(t) - \delta_1(t, v_j'(t))}{1 + |v_j'(t) - \delta_1(t, v_j'(t))|} + g(t, \beta(t), \beta'(t), \beta''(t)) \\
&= -g(t, \delta_0(t, v_j(t)), \beta'(t), \beta''(t)) \\
&\quad + \frac{1}{1+t^2} \frac{v_j'(t) - \delta_1(t, v_j'(t))}{1 + |v_j'(t) - \delta_1(t, v_j'(t))|} + g(t, \beta(t), \beta'(t), \beta''(t)) \\
&\geq -g(t, \delta_0(t, v_j(t)), \beta'(t), \beta''(t)) \\
&\quad + \frac{1}{1+t^2} \frac{v_j'(t) - \beta'(t)}{1 + |v_j'(t) - \beta'(t)|} + g(t, \beta(t), \beta'(t), \beta''(t)) \\
&\geq \frac{1}{1+t^2} \frac{v_j'(t) - \beta'(t)}{1 + |v_j'(t) - \beta'(t)|} > 0,\ a.e.,\ t \in (t_k, t_k + \varepsilon).
\end{aligned}$$

Then, $v_j''(t) - \beta''(t)$ is increasing for all $t \in (t_k, t_k + \varepsilon)$. By integrating $v_j'''(t) - \beta'''(t)$ over $(t_k, t]$ (which is a subset of $(t_k, t_k + \varepsilon)$), we have

$$0 < v_j''(t) - \beta''(t) - v_j''(t_k^+) + \beta''(t_k^+) \leq -v_j''(t_k^+) + \beta''(t_k^+).$$

By using (10) and (15), we arrive at the following contradiction:

$$
\begin{aligned}
0 < -v_j''(t_k^+) + \beta''(t_k^+) &= -I_{3k}^+(t_k^+, \delta_0(t_k^+, v_j(t_k^+)), \delta_1(t_k^+, v_j'(t_k^+)), \delta_{2j}(t_k^+, v_j''(t_k^+))) + \beta''(t_k^+) \\
&= -I_{3k}^+(t_k^+, \delta_0(t_k^+, v_j(t_k^+)), \beta'(t_k^+), \beta''(t_k^+)) + \beta''(t_k^+) \\
&\le -I_{3k}^+(t_k^+, \delta_0(t_k^+, \beta''(t_k^+)), \beta'(t_k^+), \beta''(t_k^+)) \\
&\quad + I_{3k}^+(t_k^+, \beta(t_k^+), \beta'(t_k^+), \beta''(t_k^+)) \le 0.
\end{aligned}
$$

If we have (20) and $\sup_{t\in[0,+\infty)}(v_j'(t) - \beta'(t)) = v_j'(t_k^-) - \beta'(t_k^-) > 0$, then there exists $\varepsilon > 0$ such that $v_j'(t_k^-) - \beta'(t_k^-) > 0$, $v_j''(t_k^-) - \beta''(t_k^-) > 0$ and $v_j''(t) - \beta''(t) \ge 0$ for all $t \in (t_k^- - \varepsilon, t_k^-)$. Then, by (10) and (14), we encounter the following contradiction.

$$
\begin{aligned}
&0 > -v_j'(t_k^-) + \beta'(t_k^-) + v_j'(t_k^+) - \beta'(t_k^+) \\
&\quad > I_{2k}(t_k, \delta_0(t_k, v_j(t_k)), \delta_1(t_k, v_j'(t_k)), \delta_{2j}(t_k, v_j''(t_k)) - \triangle\beta'(t_k) \\
&\quad > I_{2k}(t_k, \delta_0(t_k, v_j(t_k)), \beta'(t_k), \beta''(t_k)) - I_{2k}(t_k, \beta(t_k), \beta'(t_k), \beta''(t_k)) \ge 0.
\end{aligned}
$$

From the two cases, we deduce that $v_j'(t) \le \beta'(t)$ for $t \in [0, +\infty)$. By similar arguments, the second inequality $\alpha'(t) \le v_j'(t)$ for $t \in [0, +\infty)$ can be proven. Therefore, we have:

$$
\alpha'(t) \le v_j'(t) \le \beta'(t) \text{ for } t \in [0, +\infty). \tag{21}
$$

By integrating (21) over $[0, t) \subset [0, t_1]$,

$$
\alpha(t) \le v_j(t) - v_j(0) + \alpha(0) \le v(t). \tag{22}
$$

Moreover,

$$
v_j(t_k^-) - \beta(t_k^-) \ge 0.
$$

By integrating (21) on $(t_1, t) \subset (t_1, t_2]$, using (13),

$$
\begin{aligned}
\alpha(t) &\le v_j(t) - v_j(t_1^+) + \alpha(t_1^+) \\
&\le v_j(t) - I_{11}(t_1, \delta_0(t_1, v_j), \delta_1(t_1, v_j')) - v_j(t_1^-) + I_{11}(t_1, \alpha(t_1), \alpha'(t_1)) + \alpha(t_1^-) \\
&\le v_j(t).
\end{aligned}
$$

By recursion, we have

$$
\alpha(t) \le v_j(t), \quad \forall t \in (t_k, t_{k+1}], \text{ for } k = 1, 2, \ldots,
$$

thus $\alpha(t) \le v_j(t), \quad \forall t \in [0, +\infty)$. By similar arguments, it can be proven that

$$
v_j(t) \le \beta(t), \quad \forall t \in [0, +\infty).
$$

Therefore,

$$
\alpha(t) \le v_j(t) \le \beta(t), \quad \forall t \in [0, +\infty).
$$

Step 2: Applying Lemma 2, if v is a solution of the 2-modified problem (17), (2), (18), then

$$\|v''\|_2 < R.$$

Let $N = N_1$ such that

$$N_1 > \max\{2R, \sup_{t\in[0,+\infty)} |\alpha''(t)|, \sup_{t\in[0,+\infty)} |\beta''(t)|\}.$$

If the 1-modified IBVP (17), (2), (18) has a solution v, then v is a solution of problem (1), (2), (3) where

$$\|v''\|_2 < R < \frac{N_1}{2} < N_1.$$

Step 3: The 1-modified IBVP (17), (2), (18) has at least one solution. Let us define the operator $T : X \to X$ for $k \in \mathbb{N}^*$ by

$$\begin{aligned} Tv(t) =& A + Bt + \frac{C}{2}t^2 + \sum_{t_k<t} I^*_{1k}(t_k, v(t_k), v'(t_k)) \\ &+ \sum_{t_k<t} I^*_{2k}(t_k, v(t_k), v'(t_k), v''(t_k))(t - t_k) \\ &+ \int_0^t (t-s)\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I^*_{3k}(t_k, v(t_k), v'(t_k), v''(t_k))\right) ds, \end{aligned}$$

where

$$\begin{aligned} G_v(s) :=& \, g(s, \delta_0(s,v), \delta_1(s,v'), \delta_{21}(s,v'')) \\ &- \frac{1}{1+s^2}\frac{v'(s) - \delta_1(s,v')}{1 + |v'(s) - \delta_1(s,v')|}, \end{aligned}$$

$$\begin{cases} I^*_{1k}(t_k, v(t_k), v'(t_k)) = I_{1k}(t_k, \delta_0(t_k, v(t_k)), \delta_1(t_k, v'(t_k))), \\ I^*_{2k}(t_k, v(t_k), v'(t_k), v''(t_k)) = I_{2k}(t_k, \delta_0(t_k, v(t_k)), \delta_1(t_k, v'(t_k)), \delta_{21}(t_k, v''(t_k))), \\ I^*_{3k}(t_k, v(t_k), v'(t_k), v''(t_k)) = I_{3k}(t_k, \delta_0(t_k, v(t_k)), \delta_1(t_k, v'(t_k)), \delta_{21}(t_k, v''(t_k))). \end{cases}$$

By Lemma 1, the fixed points of T are solutions to 1-modified IBVP (17), (2), (18). It is sufficient to prove that T has a fixed point.

(i) $T : X \to X$ is well defined. Let $v \in X$. Taking

$$\rho_1 = \max\{N_1, \|\alpha\|_X, \|\beta\|_X\}.$$

T is well defined on X,

$$\int_0^{+\infty} |G_v(s)|ds \leq \int_0^{+\infty} \psi(s)h(|\delta_{21}(s,v'')|) + \Big|\frac{1}{1+s^2}\frac{v'(s) - \delta_1(s,v')}{1 + |v'(s) - \delta_1(s,v')|}\Big|ds$$

$$\leq \int_0^{+\infty} \psi(s) \max_{z\in[0,2R_1]} h(z) + \frac{1}{1+s^2} ds \leq M_\psi < +\infty,$$

$$\begin{aligned}
\lim_{t\to+\infty}(Tv)''(t) &= \lim_{t\to+\infty}\left(C + \int_t^{+\infty} G_v(\mu)d\mu - \sum_{t_k>t} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right)\\
&= \lim_{t\to+\infty}\left(C - \sum_{t_k>t} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right) = C\\
&= \lim_{t\to+\infty}\frac{(Tv)'(t)}{1+t} = 2\lim_{t\to+\infty}\frac{Tv(t)}{1+t^2}.
\end{aligned}$$

So, $Tv \in X$.

(ii) T is continuous. Let $v_n \to v$ in X, there exists $\rho > 0$ such that $\sup_n \|v_n\|_X < \rho$, we obtain that

$$\begin{aligned}
&\sup_{0\leq t<+\infty} |(Tv_n)''(t) - (Tv)''(t)|\\
&= \sup_{0\leq t<+\infty}\Big|\int_t^{+\infty} G_{v_n}(\mu)d\mu - \sum_{t_k>t} I_{3k}^*(t_k, v_n(t_k), v_n'(t_k), v_n''(t_k))\\
&- \int_t^{+\infty} G_v(\mu)d\mu + \sum_{t_k>t} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\Big|\\
&\leq \sup_{0\leq t<+\infty}\int_t^{+\infty}\Big|G_{v_n}(\mu) - G_v(\mu)\Big|d\mu\\
&+ \sup_{0\leq t<+\infty}\sum_{t_k>t}\Big|I_{3k}^*(t_k, v_n(t_k), v_n'(t_k), v_n''(t_k)) - I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\Big|\\
&\leq \int_0^{+\infty}\Big|G_{v_n}(\mu) - G_v(\mu)\Big|d\mu\\
&+ \sum_{k=1,2,\ldots}\Big|I_{3k}^*(t_k, v_n(t_k), v_n'(t_k), v_n''(t_k)) - I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\Big|.
\end{aligned}$$

By using (16) and Lebesgue Dominated Convergence Theorem, we have

$$\int_0^{+\infty}\Big|G_{v_n}(\mu) - G_v(\mu)\Big|d\mu \to 0,$$

as $n \to +\infty$. Moreover,

$$\begin{aligned}
&\sum_{k=1,2,\ldots}\Big|I_{3k}^*(t_k, v_n(t_k), v_n'(t_k), v_n''(t_k)) - I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\Big|\\
&\leq 2\rho_1 a_3 < +\infty
\end{aligned}$$

From the dominated convergence theorem for series, we have

$$\sum_{k=1,2,\ldots}\Big|I^*_{3k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\to 0,$$

as $n\to+\infty$. Therefore,

$$\|(Tv_n)''-(Tv)''\|_2\to 0,$$

as $n\to+\infty$. Also,

$$\begin{aligned}
&\sup_{0\le t<+\infty}\Big|\frac{(Tv_n)'(t)}{1+t}-\frac{(Tv)'(t)}{1+t}\Big|\\
&=\sup_{0\le t<+\infty}\frac{1}{1+t}\Big|\sum_{t_k<t}I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-\sum_{t_k<t}I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\\
&+\int_0^t\left(\int_s^{+\infty}G_{v_n}(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))\right)\\
&-\left(\int_s^{+\infty}G_v(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)ds\Big|\\
&\le\sup_{0\le t<+\infty}\frac{1}{1+t}\Big|\sum_{t_k<t}I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-\sum_{t_k<t}I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\frac{1}{1+t}\int_0^t\Big|\left(\int_s^{+\infty}G_{v_n}(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))\right)\\
&-\left(\int_s^{+\infty}G_v(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|ds\\
&\le\sup_{0\le t<+\infty}\frac{1}{1+t}\sum_{k=1,2,\cdots}\Big|I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\frac{2}{1+t}\int_0^t\|(Tv_n)''-(Tv)''\|_2ds.
\end{aligned}$$

By the dominated convergence theorem for series, we get that

$$\sum_{k=1,2,\cdots}\Big|I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\to 0,$$

as $n\to+\infty$, and

$$\sup_{0\le t<+\infty}\frac{2}{1+t}\int_0^t\|(Tv_n)''-(Tv)''\|_2ds\le 2\|(Tv_n)''-(Tv)''\|_2\to 0,$$

as $n \to +\infty$.
Also, we can demonstrate that $\|Tv_n - Tv\|_0 \to 0$ in a similar manner,

$$
\begin{aligned}
&\sup_{0\le t<+\infty}\Big|\frac{(Tv_n)(t)}{1+t^2}-\frac{(Tv)(t)}{1+t^2}\Big|\\
&\le \sup_{0\le t<+\infty}\sum_{t_k<t}\frac{1}{1+t^2}\Big|I^*_{1k}(t_k,v_n(t_k),v'_n(t_k))-I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\sum_{t_k<t}\frac{t-t_k}{1+t^2}\Big|I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\int_0^t\frac{t-s}{1+t^2}\Big|\left(\int_s^{+\infty}G_{v_n}(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))\right)\\
&-\left(\int_s^{+\infty}G_v(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|ds\\
&\le \sup_{0\le t<+\infty}\sum_{k=1,2,\cdots}\frac{1}{1+t^2}\Big|I^*_{1k}(t_k,v_n(t_k),v'_n(t_k))-I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\sum_{k=1,2,\cdots}\frac{t-t_k}{1+t^2}\Big|I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+2\sup_{0\le t<+\infty}\int_0^t\frac{t-s}{1+t^2}\|(Tv_n)''-(Tv)''\|_2ds\\
&\le \sum_{k=1,2,\cdots}\Big|I^*_{1k}(t_k,v_n(t_k),v'_n(t_k))-I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|\\
&+\sum_{k=1,2,\cdots}\Big|I^*_{2k}(t_k,v_n(t_k),v'_n(t_k),v''_n(t_k))-I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+2\|(Tv_n)''-(Tv)''\|_2\to 0,
\end{aligned}
$$

as $n \to +\infty$, by using the dominated convergence theorem for series.

In the following, we assume that

$$
L_{\rho_1}=\Big|(M_\psi+\rho_1 a_3)\Big|.
$$

(iii) T is compact. Assuming each bounded subset $U\subset X$, there exists $r>0$ such that $\|v\|_X<r$ for all $v\in U$. Let $v\in U$, then one has:

$$
\begin{aligned}
\|Tv\|_0=\sup_{0\le t<+\infty}\frac{|Tv(t)|}{1+t^2}\le |A|+|B|+|C|+\sup_{0\le t<+\infty}\Big|\sum_{t_k<t}I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|\\
+\sup_{0\le t<+\infty}\Big|\sum_{t_k<t}\frac{t-t_k}{1+t^2}I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_{0\le t<+\infty}\int_0^t \frac{t-s}{1+t^2}\Big|\left(\int_s^{+\infty} G_v(\mu)d\mu-\sum_{t_k>s} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|ds\\
&\le |A|+|B|+|C|+\Big|\sum_{t_k<t} I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|+\Big|\sum_{t_k<t} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\int_0^t \frac{t-s}{1+t^2}L_{\rho_1}ds\le |A|+|B|+\rho_1 a_1+\rho_1 a_2+\frac{L_{\rho_1}}{2}<+\infty.
\end{aligned}
$$

$$
\begin{aligned}
\|(Tv)'\|_1&=\sup_{0\le t<+\infty}\frac{|(Tv)'(t)|}{1+t}\le |B|+|C|+\max_{0\le t<+\infty}\Big|\frac{1}{1+t}\sum_{t_k<t} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\\
&+\sup_{0\le t<+\infty}\int_0^t \frac{1}{1+t}\Big|\left(\int_s^{+\infty} G_v(\mu)d\mu-\sum_{t_k>s} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|ds\\
&\le |B|+|C|+\Big|\sum_{t_k<t} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|+\sup_{0\le t<+\infty}\int_0^t \frac{1}{1+t}L_{\rho_1}ds\\
&\le |B|+|C|+\rho a_2+L_{\rho_1}<+\infty.
\end{aligned}
$$

Moreover,

$$
\|(Tv)''\|_2=\frac{1}{2}\sup_{0\le t<+\infty}\Big|\left(\int_t^{+\infty} G_v(\mu)d\mu-\sum_{t_k>t} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|\le L_{\rho_1}<+\infty,
$$

$\|(Tv)''\|_2<+\infty$. Additionally,

$$
\|Tv\|_X=\max\{\|Tv\|_0,\|(Tv)'\|_1,\|(Tv)''\|_2\}<+\infty,
$$

that is, TU is uniformly bounded.

TU is equicontinuous because for $L>0$ and $r_1,r_2\in[0,L]\cap J_k$ for $k\in\mathbb{N}$, $r_1<r_2$, we have

$$
\begin{aligned}
&\left|\frac{Tv(r_2)}{1+r_2^2}-\frac{Tv(r_1)}{1+r_1^2}\right|\\
&=\Big|\frac{A+Br_2+\frac{C}{2}r_2^2+\sum_{t_k<r_2} I^*_{1k}(t_k,v(t_k),v'(t_k))+\sum_{t_k<r_2} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))(r_2-t_k)}{1+r_2^2}\\
&-\frac{A+Br_1+\frac{C}{2}r_1^2+\sum_{t_k<r_1} I^*_{1k}(t_k,v(t_k),v'(t_k))+\sum_{t_k<r_1} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))(r_1-t_k)}{1+r_1^2}\\
&+\int_0^{r_2}\frac{r_2-s}{1+r_2^2}\left(\int_s^{+\infty} G_v(\mu)d\mu-\sum_{t_k>s} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)ds\\
&-\int_0^{r_1}\frac{r_1-s}{1+r_1^2}\left(\int_s^{+\infty} G_v(\mu)d\mu-\sum_{t_k>s} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big|A+\sum_{t_k<r_1} I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|\Big|\frac{1}{1+r_2^2}-\frac{1}{1+r_1^2}\Big| + \frac{1}{1+r_2^2}\sum_{r_1<t_k<r_2}\Big|I^*_{1k}(t_k,v(t_k),v'(t_k))\Big| \\
&\quad + \Big|\frac{r_2}{1+r_2^2}-\frac{r_1}{1+r_1^2}\Big|\Big|B+\sum_{t_k<r_1} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big| \\
&\quad + \frac{r_2}{1+r_2^2}\sum_{r_1<t_k<r_2}\Big|I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big| + \Big|\frac{r_2^2}{1+r_2^2}-\frac{r_1^2}{1+r_1^2}\Big|\Big|\frac{C}{2}\Big| \\
&\quad + \Big|\frac{1}{1+r_2^2}-\frac{1}{1+r_1^2}\Big|\Big|\sum_{t_k<r_1} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))t_k\Big| \\
&\quad + \Big|\frac{1}{1+r_2^2}-\frac{1}{1+r_1^2}\Big|\Big|\sum_{r_1<t_k<r_2} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))t_k\Big| \\
&\quad + \int_0^{r_1}\Big|\frac{r_2-s}{1+r_2^2}-\frac{r_1-s}{1+r_1^2}\Big|\Big|\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|ds \\
&\quad + \int_{r_1}^{r_2}\Big|\frac{r_2-s}{1+r_2^2}\Big|\Big|\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)\Big|ds \\
&\leq \Big|A+\sum_{t_k<r_1} I^*_{1k}(t_k,v(t_k),v'(t_k))\Big|\Big|\frac{1}{1+r_2^2}-\frac{1}{1+r_1^2}\Big| + \frac{1}{1+r_2^2}\sum_{r_1<t_k<r_2}\Big|I^*_{1k}(t_k,v(t_k),v'(t_k))\Big| \\
&\quad + \frac{r_2}{1+r_2^2}\sum_{r_1<t_k<r_2}\Big|I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big| + \Big|\frac{r_2^2}{1+r_2^2}-\frac{r_1^2}{1+r_1^2}\Big|\Big|\frac{C}{2}\Big| \\
&+ \Big|\frac{r_2}{1+r_2^2}-\frac{r_1}{1+r_1^2}\Big|\Big|B+\sum_{t_k<r_1} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big| \\
&\quad + \frac{r_2}{1+r_2^2}\sum_{r_1<t_k<r_2}\Big|I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big| + \Big|\frac{r_2^2}{1+r_2^2}-\frac{r_1^2}{1+r_1^2}\Big|\Big|\frac{C}{2}\Big| \\
&\quad + \Big|\frac{1}{1+r_2^2}-\frac{1}{1+r_1^2}\Big|\Big|\sum_{t_k<r_1} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))t_k\Big| \\
&\quad + \Big|\frac{1}{1+r_2^2}-\frac{1}{1+r_1^2}\Big|\Big|\sum_{r_1<t_k<r_2} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))t_k\Big| \\
&+ \int_0^{r_1}\Big|\frac{r_2-s}{1+r_2^2}-\frac{r_1-s}{1+r_1^2}\Big|L_{\rho_1}ds + \int_{r_1}^{r_2}\Big|\frac{r_2-s}{1+r_2^2}\Big|L_{\rho_1}ds \to 0,
\end{aligned}
$$

as $r_1 \to r_2$. Utilizing the same technique, we obtain

$$
\begin{aligned}
&\left|\frac{(Tv)'(r_2)}{1+r_2}-\frac{(Tv)'(r_1)}{1+r_1}\right| \\
&= \Big|\frac{B+Cr_2+\sum_{t_k<r_2} I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))}{1+r_2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{B + Cr_1 + \sum_{t_k<r_1} I_{2k}^*(t_k, v(t_k), v'(t_k), v''(t_k))}{1+r_1} \\
&+ \int_0^{r_2} \frac{1}{1+r_2}\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right)ds \\
&- \int_0^{r_1} \frac{1}{1+r_1}\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right)ds\Big| \\
\le &\left|\frac{1}{1+r_2} - \frac{1}{1+r_1}\right|\left|B + \sum_{t_k<r_1} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k))\right| \\
&+ \frac{1}{1+r_2}\sum_{r_1<t_k<r_2}\left|I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k))\right| + \left|\frac{r_2}{1+r_2} - \frac{r_1}{1+r_1}\right|\left|C\right| \\
&+ \int_0^{r_1}\left|\frac{1}{1+r_2} - \frac{1}{1+r_1}\right|\left|\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right)\right|ds \\
&+ \int_{r_1}^{r_2}\left|\frac{1}{1+r_2}\right|\left|\left(\int_s^{+\infty} G_v(\mu)d\mu - \sum_{t_k>s} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right)\right|ds \\
\le &\left|\frac{1}{1+r_2} - \frac{1}{1+r_1}\right|\left|B + \sum_{t_k<r_1} I_{2k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right| \\
&+ \frac{1}{1+r_2}\sum_{r_1<t_k<r_2}\left|I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k))\right| + \left|\frac{r_2}{1+r_2} - \frac{r_1}{1+r_1}\right|\left|C\right| \\
&+ \int_0^{r_1}\left|\frac{1}{1+r_2} - \frac{1}{1+r_1}\right|L_{\rho_1}ds + \int_{r_1}^{r_2}\left|\frac{1}{1+r_2}\right|L_{\rho_1}ds \to 0,
\end{aligned}
$$

as $r_1 \to r_2$. Also,

$$
\begin{aligned}
&\left|(Tv)''(r_2) - (Tv)''(r_1)\right| \\
&= \int_{r_2}^{+\infty} G_v(\mu)d\mu - \sum_{t_k>r_2} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k)) \\
&\quad - \int_{r_1}^{+\infty} G_v(\mu)d\mu + \sum_{t_k>r_1} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k)) \\
&\le \left| - \int_{r_1}^{r_2} G_v(\mu)d\mu + \sum_{r_2>t_k>r_1} I_{3k}^*(t_k, v(t_k), v'(t_k), v''(t_k))\right| \to 0,
\end{aligned}
$$

as $r_1 \to r_2$, then $\left|(Tv)''(r_2) - (Tv)''(r_1)\right| \to 0$, as $r_1 \to r_2$.
In a similar way, we can see that TU is equiconvergent at each t_k^+ for $k \in \mathbb{N}^*$.

Moreover, $TU \subset X$ is equiconvergent at infinity, we use (16) and the Carathéodory function G. for all $v \in U$, $\lim_{t\to+\infty}(Tv)''(t) = C$, then, $\lim_{t\to+\infty}\frac{(Tv)'(t)}{1+t} = C$ and

$\lim_{t\to+\infty}\frac{(Tv)(t)}{1+t^2}=\frac{C}{2}$

$$\begin{aligned}&|(Tv)''(t)-C|\\&=\Big|\int_t^{+\infty}G_v(\mu)d\mu-\sum_{t_k>t}I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\Big|\to 0,\end{aligned}$$

as $t\to+\infty$. For the first derivatives, we obtain

$$\begin{aligned}&\lim_{t\to+\infty}\Big|\frac{(Tv)'(t)}{1+t}-C\Big|\\&=\lim_{t\to+\infty}\Big|\frac{1}{1+t}\left(B+\sum_{t_k<t}I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))+Ct\right)\\&\quad+\frac{1}{1+t}\int_0^t\left(\int_s^{+\infty}G_v(\mu)d\mu-\sum_{t_k>s}I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)ds-C\Big|\\&=\lim_{t\to+\infty}\Big|\frac{1}{1+t}\left(B+\sum_{t_k<t}I^*_{2k}(t_k,v(t_k),v'(t_k),v''(t_k))+Ct\right)\\&\quad+\left(\int_t^{+\infty}G_v(\mu)d\mu-\sum_{t_k>t}I^*_{3k}(t_k,v(t_k),v'(t_k),v''(t_k))\right)-C\Big|=0.\end{aligned}$$

In the same line, we get that

$$\left|\frac{Tv(t)}{1+t^2}-\frac{C}{2}\right|\to 0,$$

as $t\to+\infty$.

From Lemma 3, the set TU is relatively compact. Based to the Schaefer Fixed Point Theorem 1, T is completely continuous such that the set

$$\{v\in X:\ v=\lambda Tv,\ \lambda\in(0,1)\}$$

is bounded, satisfying

$$\|\lambda Tv\|_X\le|A|+|B|+|C|+\rho_1a_1+\rho_1a_2+L_{\rho_1}<+\infty,\ \forall\lambda\in(0,1).$$

Therefore, T has at least one fixed point $v\in X$, such that

$$\alpha\le v\le\beta,\ \alpha'\le v'\le\beta',\ -2R\le v''\le 2R.$$

□

4 Example

Consider the following impulsive boundary value problem:

$$v'''(t) + g(t, v(t), v'(t), v''(t)) = 0, \; a.e., \; t \in [0, +\infty) \backslash \{27, 81, \ldots, t_k, 3t_k, \ldots\}, \qquad (23)$$

where

$$g(t,x,y,z) = e^{-t}|\sin(z)| \left(\frac{x + t^2 + 10t + 10 - \frac{(t)^3}{10^{(t_k)^3}}}{(1+t^2)} \right) \left(\frac{\left| y - 2t - 10 + \frac{3(t)^2}{10^{(t_k)^3}} \right|}{(1+t)} \right),$$

$t \in J_k,\; k \in \mathbb{N}\backslash\{0\}$, such that $J_k = (t_{k-1}, t_k]$ for $k \in \mathbb{N}\backslash\{0,1\}$ and $J_1 = [t_0, t_1]$ for $k = 1$, where $t_0 = 0$. The boundary conditions are

$$v(0) = 0, \; v'(0) = 0, \quad -2 < v''(+\infty) = 0 < 2, \qquad (24)$$

and the impulsive conditions satisfy (16), where

$$\begin{cases} I_{1k}(t_k, v(t_k), v'(t_k)) = \frac{(t_k)^3}{3(10)^{(t_k)^3}} \left(t_k^2 + 12t_k + 20 - \frac{(t_k)^3}{10^{(t_k)^3}} - \frac{3(t_k)^2}{10^{(t_k)^3}} \right)^{-1} \\ \qquad \times \left(v(t_k) + v'(t_k) \right), \\ I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k)) = -\frac{3(t_k)^2}{10^{(t_k)^3}} \left(3(t_k)^2 + 10t_k - 10 - \frac{5(t_k)^3}{10^{(t_k)^3}} \right)^{-1} \\ \qquad \times \left(v(t_k) - 2t_k v'(t_k) \right) \left((\sin v''(t_k))^2 + 1 \right), \\ I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k)) = \frac{1}{(100)^{t_k}} \left(\frac{v(t_k)}{1+t_k^2}, \frac{v'(t_k)}{1+t_k}, v''(t_k) \right), \\ I_{3k}^{+}(t_k^{+}, v(t_k^{+}), v'(t_k^{+}), v''(t_k^{+})) = -\left(\frac{v(t_k^{+})}{t_k^{+}} - 5t_k^{+} v'(t_k^{+}) \right) \left((\cos v''(t_k^{+}))^2 + 1 \right). \end{cases} \qquad (25)$$

The functions α, β are lower and upper solutions for this IBVP, satisfying (13), (14), and (15), where $\alpha(t) = -t^2 - 10t - 10 + \frac{(t)^3}{10^{(t_k)^3}}$ and $\beta(t) = t^2 + 10t + 10 - \frac{(t)^3}{10^{(t_k)^3}}$ for $t \in J_k,\; k \in \mathbb{N}\backslash\{0\}$. We have that

$$\alpha(t) \le \beta(t), \; \alpha'(t) \le \beta'(t), \quad \forall t \in [0, +\infty),$$

such that $\left| \frac{\beta'(t_{k+1}) - \alpha'(t_k)}{t_{k+1} - t_k} \right| < 14,\; , \left| \frac{\alpha'(t_{k+1}) - \beta'(t_k)}{t_{k+1} - t_k} \right| < 14$, then

$$\sup_{k=1,2,\ldots} \left\{ \left| \frac{\beta'(t_{k+1}) - \alpha'(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\alpha'(t_{k+1}) - \beta'(t_k)}{t_{k+1} - t_k} \right| \right\} \text{ exists.}$$

g is a Carathéodory function that satisfies the Nagumo conditions (6) and (12), where

$$g(t,x,y,z) = e^{-t} z \left(\frac{x + t^2 + 10t + 10 - \frac{(t)^3}{10^{(t_k)^3}}}{(1+t^2)} \right) \left(\frac{y - 2t - 10 + \frac{3(t)^2}{10^{(t_k)^3}}}{(1+t)} \right)$$

$$\leq 1500e^{-t}|z|.$$

The positive continuous functions ψ, h are $\psi(t) = 1500e^{-t}$ and $h(z) = 1$, such that $\int_0^{+\infty} 1500e^{-s}ds < +\infty$, $\int_\mu^{+\infty} \frac{1}{s+1}ds = +\infty$. The impulsive conditions (25) satisfy the condition (16), for all $v \in X$, such that $\alpha(t_k) \leq x(t_k) \leq \beta(t_k)$, $\alpha'(t_k) \leq y(t_k) \leq \beta'(t_k)$ and $-R \leq z(t_k) \leq R$ for each positive R, we have

$$\begin{cases} \sum_{k=1}^{\infty} \left|I_{1k}(t_k, x(t_k), , y(t_k))\right| \leq \frac{1}{12}\rho_1 < +\infty, \\ \sum_{k=1}^{\infty} \left|I_{2k}(t_k, x(t_k), y(t_k), z(t_k))\right| \leq \frac{2}{4}\rho_1 < +\infty, \\ \sum_{k=1}^{\infty} \left|I_{3k}(t_k, x(t_k), y(t_k), z(t_k))\right| \leq \frac{1}{99}\rho_1 < +\infty, \\ \text{where } \rho_1 = \max\{R, \|\alpha\|_X, \|\beta\|_X\}. \end{cases}$$

Then, IBVP (23), (24), (25) has at least one solution $v \in X$, and there is $N > 0$ such that

$$\alpha(t) \leq v(t) \leq \beta(t), \ \alpha'(t) \leq v'(t) \leq \beta'(t), \ -N \leq v''(t) \leq N, \ t \in [0, +\infty).$$

Conflict of interest The author declares that there is no conflict of interest regarding the publication of this paper.

Data Availability Statement: This manuscript has no associated data.

References

[1] Abd El-Salam, F. A., Sabbagh, El-A. A., Zaki, Z. A.: *The numerical solution of linear third order boundary value problems using nonpolynomial spline technique*, Journal of American Science 6.12 (2010): 303-309.

[2] Agarwal, R. P., O'Regan, D.: *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publisher, Glasgow 2001.

[3] Al-Said, E. A., Noor, M. A.: *Cubic splines method for a system of third-order boundary value problems*, Applied Mathematics and Computation 142.2-3 (2003): 195-204.

[4] Belhannache, F., Hamani, S., Henderson, J.: *Upper and lower solutions methods for impulsive Caputo-Hadamard fractional differential inclusions*, Electr. J. Diff Equat, (2019), No. 22, pp. 1-13.

[5] Benbaziz Z., Djebali, S.: *Positive solutions for nonlinear ϕ-Laplacian third-order impulsive differntial equations on infinite intervals*, J.Nonlinear Funct. Anal. 2018(2018), Article ID 32.

1 May 2024 15:31:48 PDT
231121-Zerki Version 2 - Submitted to Rocky Mountain J. Math.

[6] Cabada, A., Lois, S.: *Existence of solution for discontinuous third order boundary value problems*, Journal of computational and applied mathematics 110.1 (1999): 105-114.

[7] Castillo, S., Pinto, M., Torres, R.: *Asymptotic formulae for solutions to impulsive differential equations with piecewise constant argument of generalized type*, Electron. J. Differential Equations, Vol. 2019 (2019), No. 40, pp. 1-22.

[8] Coronel, A., Friz, L., Huancas, F., Pinto, M.: *L^p-solutions of a nonlinear third order differential equation and the Poincaré–Perron problem*, Journal of Fixed Point Theory and Applications 21 (2019): 1-22.

[9] El-Dib, Y. O.: *The simplest approach to solving the cubic nonlinear jerk oscillator with the non-perturbative method*, Mathematical Methods in the Applied Sciences 45.9 (2022): 5165-5183.

[10] Guo, Y., Shu, X. B., Yin, Q.: *Existence of solutions for first-order Hamiltonian random impulsive differential equations with Dirichlet boundary conditions*, Discrete and Continuous Dynamical Systems-B 27.8 (2022): 4455.

[11] Infante, G., Pietramala, P.: *A third order boundary value problem subject to nonlinear boundary conditions*, Mathematica Bohemica 135.2 (2010): 113-121.

[12] Kengne, L. K., Kengne, J., Pone, J. R. M.: *Coexisting bubbles, multiple attractors, and control of multistability in a simple jerk system under the influence of a constant excitation force*, Pramana 94.1 (2020): 81.

[13] Li, Z., Shu, X. B., Miao, T.: *The existence of solutions for Sturm–Liouville differential equation with random impulses and boundary value problems*, Boundary Value Problems 2021 (2021): 1-23.

[14] Li, Z., Shu, X., Xu, F.: *The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem*, AIMS Mathematics 5.6 (2020): 6189-6210.

[15] Liu, K., Wang, J., O'Regan, D., Fečkan, M.: *A new class of (w, c)-periodic non-instantaneous impulsive differential equations*, Mediterranean Journal of Mathematics 17.5 (2020): 155.

[16] Minhós, F., Carapinha, R.: *Half-linear impulsive problems for classical and singular ϕ-Laplacian with generalized impulsive conditions*, Journal of Fixed Point Theory and Applications, 2018, vol. 20, no 3, p. 117.

[17] Minhós, F., Carapinha, R.: *Third-order generalized discontinuous impulsive problems on the half-line*, Mathematical Modelling and Analysis 26.4 (2021): 548-565

[18] Pandey, P. K.: *A numerical method for the solution of general third order boundary value problem in ordinary differential equations*, Bull. Int. Math. Virtual Inst. 7, 129–138 (2017)

[19] Pang, H., Lu, M., Cai, C.: *The method of upper and lower solutions to impulsive differential equations with integral boundary conditions*, Advances in Difference Equations 2014 (2014): 1-11.

[20] Smart, D. R.: *Fixed point theorems*, Cambridge University Press, London-New York, 1974.

[21] Tang, Y., Zhang, G.: *Existence of positive solutions for periodic boundary value problems of second-order impulsive differential equation with derivative in the nonlinearity*, Journal of Fixed Point Theory and Applications 23 (2021): 1-18.

[22] Wang, M.X., Cabada, A. Nieto, J.J.: *Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions*, Ann, Polon, Math. 1993, 58, 221–235.

[23] Wang, J. R., Ren, L., Zhou, Y.: (w, c)*-Periodic solutions for time varying impulsive differential equations*, Advances in Difference Equations 2019.1 (2019): 259.

[24] Zerki, A.: *Existence and localization of solutions for second-order nonlinear impulsive bvps on infinite interval*, under review.

[25] Zerki, A., K. Bachouche, Ait-Mahiout, K.: *Existence of solutions for higher order* ϕ*-Laplacian BVPs on the half-line using a one-sided Nagumo condition with nonordered upper and lower solutions*, Cubo (Temuco) 25.2 (2023): 173-193.

1 May 2024 15:31:48 PDT
231121-Zerki Version 2 - Submitted to Rocky Mountain J. Math.