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ADMISSIBILITY FOR THE NONAUTONOMOUS DIFFERENTIAL EQUATIONS WITHOUT BOUNDED GROWTH

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ABSTRACT. In this paper, nonuniform exponential dichotomy (NED for short) is characterized for nonautonomous differential equations in terms of the admissibility of two classes of weighted bounded continuous functions. Sufficient conditions are obtained for the existence of NED on $\mathbb{R}_-, \mathbb{R}_+$, and $\mathbb{R}$. In a contrast to most previous literature charactering NED, neither Lyapunov norms nor bounded growth condition are used in this paper. Recently, Wu and Xia [PAMS, 2023] presented a discrete version of admissibility without Lyapunov norms or bounded growth condition for the difference equations. However, the discrete version of the admissibility for the difference equations can not cover our results (the continuous version). Since there are essential differences between the differential equations and the difference equations, novel proving techniques should be employed in this paper.

1. Introduction

Let J be one of $\mathbb{R}_-$, $\mathbb{R}_+$, $\mathbb{R}$ where $\mathbb{R}_- := (-\infty, 0]$, $\mathbb{R}_+ := [0, +\infty)$. We consider the following nonautonomous differential equation

$$x'(t) = A(t)x(t), \tag{1}$$

where $t \in J$, $x(t) \in \mathbb{R}^n$, and $A(t)$ is a matrix-valued function which is continuous on J. It is clear that for each $(t_0, x_0) \in J \times \mathbb{R}^n$, there exists a unique solution $x(t)$ of system (1) satisfying $x(t_0) = x_0$. Then we denote by $\Phi : J \times J \to \mathbb{R}^{n\times n}$ the fundamental matrix of system (1), i.e., the initial value problem (1) with $x(s) = \xi$ has a solution $\Phi(t,s)\xi$. Now we introduce some notations and concepts. Let $C(J,X)$ be the set of all continuous functions from J into X. Let $|\cdot|$ and $\|\cdot\|$ denote the Euclidean norm and operator norm, respectively. Let $\mathscr{R}(P)$ and $\mathscr{N}(P)$ be the range space and null space of matrix P, respectively.

An invariant projector of system (1) is a map $P : J \to \mathbb{R}^{n\times n}$ of projections $P(t)$, $t \in J$ satisfying

$$P(t)\Phi(t,s) = \Phi(t,s)P(s), \quad t,s \in J. \tag{2}$$

Besides, for $\omega \in \mathbb{R}$, let

$$C_\omega(J) := \{f \in C(J,\mathbb{R}^n) : \sup_{t\in J} |f(t)|e^{-\omega|t|} < +\infty\}, \qquad |f|_{\omega,J} := \sup_{t\in J} |f(t)|e^{-\omega|t|}.$$

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Clearly, $|\cdot|_{\omega,J}$ is a norm and $(C_\omega(J),|\cdot|_{\omega,J})$ is a Banach space. Set

$$\Omega_\omega(J)=\{x\in\mathbb{R}^n : |\Phi(t,0)x|_{\omega,J}<+\infty\}.$$

We say system (1) has a *nonuniform exponential dichotomy* (NED) if there exists an invariant projector P of system (1) such that

$$\begin{aligned} \|\Phi(t,s)P(s)\| &\le Ke^{-\alpha(t-s)+\varepsilon|s|}, \quad t\ge s,\\ \|\Phi(t,s)Q(s)\| &\le Ke^{-\alpha(s-t)+\varepsilon|s|}, \quad t\le s, \end{aligned} \tag{3}$$

for some constants $K\ge 1$, $\alpha>0$, $\varepsilon\ge 0$, where $Q(s)=I-P(s)$ and $t,s\in J$.

In particular, if we take $\varepsilon=0$ in (3), then the concept of NED mentioned above reduces to the notion of exponential dichotomy. Besides exponential dichotomy extends the hyperbolicity from autonomous systems to nonautonomous systems. For the theory of exponential dichotomy, one can refer to the books [24, 12, 13]. Exponential dichotomy plays an important role in the investigation of dynamical behaviors of dynamical systems, e.g. linearization [6, 27, 14, 8], invariant manifolds [17, 19, 10, 40], spectral theory [32, 33, 37, 38, 43], homoclinic orbits and heteroclinic orbits [27, 18, 44], reducibility [34, 9] and so on.

Studying the relation between dichotomy and admissibility is one of the most important topics in the study of dichotomy. As introduced in [24], we say that a pair of function classes (X,Y) is *(properly) admissible* for system (1) when the nonhomogeneous differential equation

$$x'(t)=A(t)x(t)+f(t), \quad t\in J,\ x(t)\in\mathbb{R}^n. \tag{4}$$

has a (unique) solution in Y for each function $f\in X$. The pioneering study is of Perron [28] in 1930, in which he proposed the notion of exponential dichotomy for ODEs and described it by the admissibility of the pair $(C_0(\mathbb{R}_+),C_0(\mathbb{R}_+))$. Later, Maĭzel' [22] proved that all the systems $x'(t)=A(t)x$ with $A(t)\in BC(\mathbb{R}_+,\mathbb{R}^{d\times d})$ are equivalent. In 1958, Massera and Schäffer [23] proved that the exponential dichotomy is equivalent to the admissibility on a Banach space with $J=\mathbb{R}_+$. In 1974, Dalec'kiĭ and Krein [13] proved the equivalence between that system (1) has an exponential dichotomy on $\mathbb{R}$ and that the pair $(C_0(\mathbb{R}),C_0(\mathbb{R}))$ is properly admissible under the assumption of local integrability. In the 1970s, Coppel [12] established the equivalence on $\mathbb{R}_+$ and $\mathbb{R}$ without the condition of local integrability.

Besides the works of characterizing exponential dichotomy mentioned above, efforts are also made to characterize NEDs by admissibility. In 2010, Barreira and Valls [3] used Lyapunov norms to characterize nonuniform exponential contractions by admissibility. Later, Barreira and Valls [7] extended the work in [3] to the case of NEDs under the assumption of bounded growth. In 2014, Barreira et al [5] studied the notation of a strong exponential dichotomy and characterized it in terms of the admissibility. Besides, they gave the robustness of strong exponential dichotomies. Later, these authors [4] characterized the dichotomy completely interms of the admisibility of bounded solutions. In 2017, Zhou et al [42] considered the admissibility with forward bounded growth. For the work of characterizing NEDs over linear skew-product semiflows, one can refer to Bătăran, C. Preda and Preda [1].

Note that the works mentioned above all use Lyapunov norm. For instance, in [7], the authors employed Lyapunov norm $\|\cdot\|_t'$ to define the function classes, where

$$\|v\|_t'=\sup_{s\ge t}\{\|T(s,t)P(t)v\|e^{-\alpha(s-t)}\}+\sup_{0\le s\le t}\{\|T(s,t)Q(t)v\|e^{-\alpha(t-s)}\},$$

and T is the evolution operator. Clearly, it is challenging to construct such Lyapunov norms before establishing the existence of a NED. For the research of characterizing NEDs for discrete systems without Lyapunov norms, one can refer to [41, 16].

Besides this line of research for ODEs, similar results can be obtained for difference equations (see e.g.[21, 11, 19, 35, 36, 29, 30, 31, 20, 25, 26]). In particular, Wu and Xia [39] presented a discrete version of admissibility without bounded growth condition or Lyapunov norms for the difference equations. However, there are essentially differences between the differential equations and the difference equations. Novel proving techniques should be employed. We characterize NEDs by two admissible pairs on certain weighted subspaces. We do not use either bounded growth condition or Lyapunov norms. However, the discrete version of the admissibility in Wu and Xia [39] for the difference equations can not cover our results (the continuous version).

The rest of this paper is organized as follows. Section 2 presents the main results of this paper. Section 3 gives the proofs of main results.

2. Main results

Theorem 2.1. *Let $J=\mathbb{R}_+$ or $\mathbb{R}_-$. Assume that for $i=1,2$, there are constants μ_i,ν_i satisfying $\mu_i\le\nu_i$, $\nu_1<0$ and $\nu_2>0$, such that the following conditions hold:*

(i) The pairs $(C_{\mu_i}(J),C_{\nu_i}(J)), i=1,2$ are both admissible for system (4);
(ii) $\Omega_{\nu_1}(J)=\Omega_{\nu_2}(J)$.

Then system (1) has a NED on J.

Theorem 2.2. *Assume that for $i=1,2$, there are constants μ_i,ν_i satisfying $\mu_i\le\nu_i$, $\nu_1<0$ and $\nu_2>0$, such that the following conditions hold:*

(i) The pairs $(C_{\mu_i}(\mathbb{R}),C_{\nu_i}(\mathbb{R})), i=1,2$ are properly admissible for system (4);
(ii) $\Omega_{\nu_1}(\mathbb{R}_+)=\Omega_{\nu_2}(\mathbb{R}_+)$ and $\Omega_{\nu_1}(\mathbb{R}_-)=\Omega_{\nu_2}(\mathbb{R}_-)$.

Then system (1) has a NED on $\mathbb{R}$.

Remark 2.1. *Recently, Wu and Xia [PAMS, 2023] presented a discrete version of admissibility without Lyapunov norms or bounded growth condition for the difference equations. However, the discrete version of the admissibility for the difference equations can not cover our results (the continuous version). Since there are essentially differences between the differential equations and the difference equations, novel proving techniques should be employed in this paper. For the detail, one can compare the proof of Theorem 2.2 and the proof of the results in [39].*

Remark 2.2. *In [15], the authors characterized the NEDs for differential equations on $\mathbb{R}$ by three properly admissible pairs. In a contrast to [15], we characterize NEDs on $\mathbb{R}$ by two properly admissible pairs. Finally, the function classes $C_\omega(J),\omega\in\mathbb{R}$ used in this paper are different from those in [15] and have a simpler structure. For instance, the set*

$$\mathscr{Y}_{\varepsilon,\beta|\cdot|} := \{y:\mathbb{R}\to X | y \text{ is locally integrable and } \sup_{t\in\mathbb{R}} e^{-\beta|t|}\int_t^{t+1} e^{\varepsilon|\tau|}|y(\tau)|d\tau<+\infty\}$$

is one of the function classes in [15], which is evidently different from the function classes in our paper.

3. Proofs of main results

Lemma 3.1. *Assume that there are $\mu,\nu\in\mathbb{R}$ such that $(C_\mu(\mathbb{R}_+),C_\nu(\mathbb{R}_+))$ is admissible for system* (4) *with $J=\mathbb{R}_+$. Then for any subspace $\Omega_\nu^c(\mathbb{R}_+)$ complemented to $\Omega_\nu(\mathbb{R}_+)$, there exist an invariant projector P of system* (1) *and a bounded linear operator $T_{\mu,\nu}:C_\mu(\mathbb{R}_+)\to C_\nu(\mathbb{R}_+)$ such that*

(i) $\mathscr{R}(P(0))=\Omega_\nu(\mathbb{R}_+)$, $\mathscr{N}(P(0))=\Omega_\nu^c(\mathbb{R}_+)$, and
(ii) $(T_{\mu,\nu}f)(0)\in\mathscr{N}(P(0))$.

Proof. It can be easily seen that there exists a projection $\Pi:J\to\mathbb{R}^{n\times n}$ such that $\mathscr{R}(\Pi)=\Omega_\nu(\mathbb{R}_+)$, $\mathscr{N}(\Pi)=\Omega_\nu^c(\mathbb{R}_+)$. Let $P:J\to\mathbb{R}^{n\times n}$ be a map with $P(t)=\Phi(t,0)\Pi\Phi(0,t)$, $t\in\mathbb{R}_+$. Obviously, P is an invariant projector with $P(0)=\Pi$. Therefore, assertion (i) is true.

Now we turn to prove assertion (ii). By the definition of admissibility, we know that for any $f\in C_\mu(\mathbb{R}_+)$, system (4) has a solution $x_f\in C_\nu(\mathbb{R}_+)$. Since the initial value $x_f(0)$ can be uniquely written as $x_f(0)=x_f^R(0)+x_f^N(0)$ with $x_f^R(0)\in\mathscr{R}(P(0))$ and $x_f^N(0)\in\mathscr{N}(P(0))$. Let $x_f^*(t)=x_f(t)-\Phi(t,0)x_f^R(0)$. Then x_f^* is a solution of nonhomogeneous equation (4) with its initial value $x_f^*(0)=x_f^N(0)\in\mathscr{N}(P(0))$. Suppose that $y_f(t)$ is a solution of system (4) in $C_\nu(\mathbb{R}_+)$ with $y(0)\in\mathscr{N}(P(0))$. Then $y_f(t)-x_f^*(t)$ is a solution of (1) with its initial value $y_f(0)-x_f^*(0)\in\mathscr{N}(P(0))$, because $\mathscr{N}(P(0))$ is a linear subspace. On the other hand, noticing that $x_f^*,y_f\in C_\nu(\mathbb{R}_+)$, we obtain that $y_f-x_f^*$ is a solution of system (1) with $y_f-x_f^*\in C_\nu(\mathbb{R}_+)$. By the definition of $\Omega_\nu(\mathbb{R}_+)$, we get $y_f(0)-x_f^*(0)=(y_f-x_f^*)(0)\in\Omega_\nu(\mathbb{R}_+)=\mathscr{R}(P(0))$. Hence, $y_f(0)-x_f^*(0)\in\mathscr{R}(P(0))\cap\mathscr{N}(P(0))=\{0\}$, which implies that $y_f(t)\equiv x_f^*(t)$. Therefore, we define $T_{\mu,\nu}:C_\mu(\mathbb{R}_+)\to C_\nu(\mathbb{R}_+)$ by $T_{\mu,\nu}(f)=x_f^*$. According to the above discussion, the map $T_{\mu,\nu}$ is well defined and $(T_{\mu,\nu}f)(0)=x_f^*(0)\in\mathscr{N}(P(0))$.

Now we show that the map T is bounded and linear. For any $f,g\in C_\mu(\mathbb{R}_+)$ and $\alpha,\beta\in\mathbb{R}$, the function $\alpha(T_{\mu,\nu}f)(t)+\beta(T_{\mu,\nu}g)(t)=(\alpha T_{\mu,\nu}f+\beta T_{\mu,\nu}g)(t)$ is a solution of the system

$$x'(t)=A(t)x(t)+\alpha f(t)+\beta g(t) \tag{5}$$

with $\alpha T_{\mu,\nu}f+\beta T_{\mu,\nu}g\in C_\nu(\mathbb{R}_+)$ and $(\alpha T_{\mu,\nu}f+\beta T_{\mu,\nu}g)(0)\in\mathscr{R}(P(0))$. Note that $\alpha f+\beta g\in C_\mu(\mathbb{R}_+)$ and then we get $T_{\mu,\nu}(\alpha f+\beta g)=\alpha T_{\mu,\nu}f+\beta T_{\mu,\nu}g$. Hence, $T_{\mu,\nu}$ is linear. On the other hand, suppose that there is a sequence of function $\{f_i\}_{i=1}^{+\infty}\subset C_\mu(\mathbb{R}_+)$ such that

$$f_i\overset{|\cdot|_{\mu,\mathbb{R}_+}}{\longrightarrow}f\quad\text{and}\quad T_{\mu,\nu}f_i\overset{|\cdot|_{\nu,\mathbb{R}_+}}{\longrightarrow}\varphi(t)$$

as $i\to+\infty$. Note that $|Tf_i-\varphi|_{\nu,\mathbb{R}_+}\ge|(Tf_i)(0)-\varphi(0)|$, which implies that

$$\varphi(0)=\lim_{i\to+\infty}(Tf_i)(0)\in\mathscr{N}(P(0)). \tag{6}$$

In order to prove $T_{\mu,\nu}$ is bounded, by the well-known Closed Graph Theorem, it is sufficient to prove that $\varphi=T_{\mu,\nu}f$. For any fixed $t\in\mathbb{R}_+$ and $x\in\mathbb{R}^n$, it follows that $\sup_{s\in[0,t]}\{|\Phi(t,s)x|\}$ is finite. Then by Banach-Steinhaus Theorem, $\sup_{s\in[0,t]}\|\Phi(t,s)\|$ is finite. In fact, we have known that

$$(T_{\mu,\nu}f_i)(t)-\Phi(t,0)(T_{\mu,\nu}f_i)(0)=\int_0^t\Phi(t,s)f_i(s)\mathrm{d}s. \tag{7}$$

Then for any fixed $t \in \mathbb{R}_+$, we obtain that

$$\begin{aligned} &|(T_{\mu,\nu}f_i)(t) - \Phi(t,0)(T_{\mu,\nu}f_i)(0) - \varphi(t) + \Phi(t,0)\varphi(0)| \\ \le&|(T_{\mu,\nu}f_i)(t) - \varphi(t)| + |\Phi(t,0)(T_{\mu,\nu}f_i)(0) - \Phi(t,0)\varphi(0)| \\ \le&e^{\nu t}|T_{\mu,\nu}f_i - \varphi|_{\nu,\mathbb{R}_+} + \|\Phi(t,0)\| \cdot |T_{\mu,\nu}f_i)(0) - \varphi(0)| \to 0 \end{aligned} \tag{8}$$

as $i \to +\infty$ and

$$\begin{aligned} \left| \int_0^t \Phi(t,s)f_i(s)\mathrm{d}s - \int_0^t \Phi(t,s)f(s)\mathrm{d}s \right| &\le \int_0^t \|\Phi(t,s)\| \cdot |f_i(s) - f(s)|\mathrm{d}s \\ &\le \sup_{s\in[0,t]} \|\Phi(t,s)| \cdot |f_i - f|_{\mu,\mathbb{R}_+} \cdot \int_0^t e^{\mu s}\mathrm{d}s \to 0 \end{aligned} \tag{9}$$

as $i \to +\infty$. From (7)-(9), we get

$$\varphi(t) = \Phi(t,s)\varphi(0) + \int_0^t \Phi(t,s)f(s)\mathrm{d}s, \quad t \in \mathbb{R}. \tag{10}$$

Then, by (6) and (10), we have that $T_{\mu,\nu}f = \varphi$. □

Using essentially the same arguments as above, we get the following lemma.

Lemma 3.2. *Assume that there are $\mu, \nu \in \mathbb{R}$ such that $(C_\mu(\mathbb{R}_-), C_\nu(\mathbb{R}_-))$ is admissible for system (4) with $J = \mathbb{R}_-$. Then for any subspace $\Omega_\nu^c(\mathbb{R}_-)$ complemented to $\Omega_\nu(\mathbb{R}_-)$, there exist an invariant projector P of system (1) and a bounded linear operator $T_{\mu,\nu} : C_\mu(\mathbb{R}_-) \to C_\nu(\mathbb{R}_-)$ such that*

(i) $\mathscr{N}(P(0)) = \Omega_\nu(\mathbb{R}_-)$, $\mathscr{R}(P(0)) = \Omega_\nu^c(\mathbb{R}_-)$, and
(ii) $(T_{\mu,\nu}f)(0) \in \mathscr{R}(P(0))$.

Lemma 3.3. *Assume that there are $\mu, \nu \in \mathbb{R}$ such that $(C_\mu(\mathbb{R}_+), C_\nu(\mathbb{R}_+))$ is admissible for system (4) with $J = \mathbb{R}_+$. Then there exist an invariant projector P of system (1) and a constant $M_{\mu,\nu} \ge 1$ such that*

$$\begin{aligned} \|\Phi(t,s)P(s)\| \le M_{\mu,\nu}e^{\nu t - \mu s}, \quad t \ge s \ge 0, \\ \|\Phi(t,s)Q(s)\| \le M_{\mu,\nu}e^{\nu t - \mu s}, \quad 0 \le t \le s, \end{aligned} \tag{11}$$

where $Q(s) = I - P(s)$.

Proof. For any $s > 0, 0 < h < \min\{s,1\}$ and $x \in \mathbb{R}^n$, let

$$j(t) = \begin{cases} \frac{h-|t-s|}{h}, & t \in [s-h, s+h], \\ 0, & t \in \mathbb{R}_+ \backslash [s-h, s+h], \end{cases} \quad \text{and} \quad f(t) = \begin{cases} e^{\mu t}j(t)x, & t \in [s-h, s+h], \\ 0, & t \in \mathbb{R}_+ \backslash [s-h, s+h]. \end{cases}$$

By Lemma 3.1, there exists an invariant projector P satisfying *(i)* of Lemma 3.1. Now we construct a function x_h as follows

$$x_h(t) = \int_0^t \Phi(t,\tau)P(\tau)f(\tau)\mathrm{d}\tau - \int_t^{+\infty} \Phi(t,\tau)Q(\tau)f(\tau)\mathrm{d}\tau. \tag{12}$$

Obviously, $x_h(t)$ is a solution of system (4), which can be simplified as

$$x_h(t) = \begin{cases} \int_{s-h}^{s+h} \Phi(t,\tau)P(\tau)e^{\mu\tau}j(\tau)x\mathrm{d}\tau, & t > s+h, \\ \int_{s-h}^{t} \Phi(t,\tau)P(\tau)e^{\mu\tau}j(\tau)x\mathrm{d}\tau - \int_t^{s+h} \Phi(t,\tau)Q(\tau)e^{\mu\tau}j(\tau)x\mathrm{d}\tau, & s-h \le t \le s+h, \\ -\int_{s-h}^{s+h} \Phi(t,\tau)Q(\tau)e^{\mu\tau}j(\tau)x\mathrm{d}\tau, & 0 \le t < s-h. \end{cases} \tag{13}$$

We claim that $x_h = T_{\mu,\nu} f$. In fact, it can be easily verified that $f \in C_\mu(\mathbb{R}_+)$, $|f|_{\mu,\mathbb{R}_+} \le |x|$ and

$$\begin{aligned} x_h(0) &= -\int_{s-h}^{s+h} \Phi(0,\tau)Q(\tau)e^{\mu\tau} j(\tau)x\,\mathrm{d}\tau \\ &= -Q(0)\int_{s-h}^{s+h} \Phi(0,\tau)e^{\mu\tau} j(\tau)x\,\mathrm{d}\tau \in \mathscr{R}(Q(0)) = \mathscr{N}(P(0)). \end{aligned} \tag{14}$$

Furthermore, let

$$\varphi(t) = \int_{s-h}^{s+h} \Phi(t,\tau)P(\tau)e^{\mu\tau} j(\tau)x\,\mathrm{d}\tau = \Phi(t,0)P(0)\int_{s-h}^{s+h} \Phi(0,\tau)e^{\mu\tau} j(\tau)x\,\mathrm{d}\tau,$$

which is a solution of (1) with $\varphi(0) \in \mathscr{R}(P(0)) = \Omega_\nu(\mathbb{R}_+)$. Then, by definition of $\Omega_\nu(\mathbb{R}_+)$, we have that $\varphi \in C_\nu(\mathbb{R}_+)$. Noticing that $\varphi(t) = x_h(t)$ for $t > s+h$, we have that $x_h \in C_\nu(\mathbb{R}_+)$. Hence, $T_{\mu,\nu} f = x_h$.

Since $T_{\mu,\nu}$ is bounded, it follows that $|T_{\mu,\nu} f|_{\nu,\mathbb{R}_+} \le \|T_{\mu,\nu}\| \cdot |f|_{\mu,\mathbb{R}_+}$, implying that $e^{-\nu t}|x_h(t)| \le \|T_{\mu,\nu}\| \cdot |x|$ for any $h > 0$. Combined with equation (13), we get that for $t > s > 0$,

$$\begin{aligned} |\Phi(t,s)P(s)x| &= e^{-\mu s}\left|\Phi(t,s)P(s)\lim_{h\to 0^+}\int_{s-h}^{s+h} \Phi(s,\tau)e^{\mu\tau} j(\tau)x\,\mathrm{d}\tau\right| \\ &= e^{\nu t-\mu s}\lim_{h\to 0^+} e^{-\nu t}|x_h(t)| \\ &\le \|T_{\mu,\nu}\| e^{\nu t-\mu s}|x|. \end{aligned}$$

Since $\Phi(t,s)P(s)x : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous, we obtain that

$$|\Phi(t,s)P(s)x| \le \|T_{\mu,\nu}\| e^{\nu t-\mu s}|x|, \quad t \ge s \ge 0. \tag{15}$$

Combined with equation (13), we get that for $0 \le t < s$,

$$\begin{aligned} |\Phi(t,s)Q(s)x| &= e^{-\mu s}\left|\Phi(t,s)Q(s)\lim_{h\to 0^+}\int_{s-h}^{s+h} \Phi(s,\tau)e^{\mu\tau} j(\tau)x\,\mathrm{d}\tau\right| \\ &= e^{\nu t-\mu s}\lim_{h\to 0^+} e^{-\nu t}|x_h(t)| \\ &\le \|T_{\mu,\nu}\| e^{\nu t-\mu s}|x|. \end{aligned}$$

Since $\Phi(t,s)P(s)x : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous, we obtain that

$$|\Phi(t,s)Q(s)x| \le \|T_{\mu,\nu}\| e^{\nu t-\mu s}|x|, \quad 0 \le t \le s. \tag{16}$$

Let $M_{\mu,\nu} = \max\{1, \|T_{\mu,\nu}\|\}$. Then (11) follows by (15) and (16). □

Similarly, we have the following lemma.

Lemma 3.4. *Assume that there are $\mu, \nu \in \mathbb{R}$ such that $(C_\mu(\mathbb{R}_-), C_\nu(\mathbb{R}_-))$ is admissible for system (4) with $J = \mathbb{R}_-$. Then there exist an invariant projection $P(s)$ and a constant $M_{\mu,\nu} \ge 1$ such that*

$$\begin{aligned} \|\Phi(t,s)P(s)\| &\le M_{\mu,\nu} e^{\nu t-\mu s}, \quad 0 \ge t \ge s, \\ \|\Phi(t,s)Q(s)\| &\le M_{\mu,\nu} e^{\nu t-\mu s}, \quad t \le s \le 0, \end{aligned}$$

where $Q(s) = I - P(s)$.

Proof of Theorem 2.1. Suppose that $J=\mathbb{R}_+$. From assumption (ii), let $\mathscr{F}=\Omega_{\nu_1}(\mathbb{R}_+)=\Omega_{\nu_2}(\mathbb{R}_+)$ and $\mathscr{E}$ be a subspace complemented to $\mathscr{F}$, i.e., $\mathbb{R}_n=\mathscr{F}\oplus\mathscr{E}$. From Lemma 3.1, there exist invariant projectors P_1,P_2 of system (1) such that $\mathscr{R}(P_1(0))=\mathscr{R}(P_2(0))=\mathscr{F}$ and $\mathscr{N}(P_1(0))=\mathscr{N}(P_2(0))=\mathscr{E}$. Therefore, we have that $P_1(0)=P_2(0)$. Moreover, it can be seen that

$$P_1(t)=\Phi(t,0)P_1(0)\Phi(0,t)=\Phi(t,0)P_2(0)\Phi(0,t)=P_2(t).$$

Let $P(t)=P_1(t)=P_2(t)$. Obviously, P is an invariant projector. Then by Lemma 3.3, there exist constants $L_{\mu_1,\nu_2},L_{\mu_2,\nu_2}\geq 1$ such that

$$\begin{aligned}\|\Phi(t,s)P(s)\|&\leq L_{\mu_1,\nu_1}e^{\nu_1 t-\mu_1 s}, \quad t\geq s\geq 0,\\ \|\Phi(t,s)Q(s)\|&\leq L_{\mu_2,\nu_2}e^{\nu_2 t-\mu_2 s}, \quad 0\leq t\leq s,\end{aligned}$$

where $Q(s)=I-P(s)$. Let $K=\max\{L_{\mu_1,\nu_2},L_{\mu_2,\nu_2}\}$, $\alpha=\min\{-\nu_1,\nu_2\}$, $\varepsilon=\max\{\nu_1-\mu_1,\nu_2-\mu_2\}$. It can be easily verified that $K\geq 1$, $\alpha>0$, $\varepsilon\geq 0$ and (3) holds for $J=\mathbb{R}_+$. Therefore, system (1) has a NED on J. In the case of $J=\mathbb{R}_-$, by Lemma 3.2 and Lemma 3.4, the conclusion of this theorem can be proved in a similar way.

Proof of Theorem 2.2. By assumption (ii), let $\mathscr{F}=\Omega_{\nu_1}(\mathbb{R}_+)=\Omega_{\nu_2}(\mathbb{R}_+)$ and $\mathscr{E}=\Omega_{\nu_1}(\mathbb{R}_-)=\Omega_{\nu_2}(\mathbb{R}_-)$. We claim that $\mathbb{R}^n=\mathscr{F}\oplus\mathscr{E}$. In fact, if $\xi\in\mathscr{F}\cap\mathscr{E}$, then $x_1(t)=\Phi(t,0)\xi$ and $x_2(t)\equiv 0$ are both solutions of system (1) in $C_{\nu_1}(\mathbb{R})$. Then they are solutions of system (15) for $f(t)=0$. It follows from assumption (i) and the definition of proper admissibility that $x_1(t)=x_2(t)=0$, implying that $\xi=0$. Hence, $\mathscr{F}\cap\mathscr{E}=\{0\}$. Now we show that $\mathbb{R}^n=\mathscr{F}+\mathscr{E}$. For any $x\in\mathbb{R}^n$, let

$$\varphi(t)=\begin{cases}1-|t|, & |t|\leq 1,\\ 0, & |t|>1,\end{cases} \quad \text{and} \quad f(t)=\varphi(t)\Phi(t,0)x.$$

Obviously, $f\in C_{\mu_1}(\mathbb{R})$. Let

$$x_f(t)=\Phi(t,0)x\int_0^t\varphi(\tau)\mathrm{d}\tau.$$

Clearly, $x(t)$ is a solution of (4) and it can be rewritten as

$$x_f(t)=\begin{cases}\frac{1}{2}\Phi(t,0)x, & t\geq 1,\\ \Phi(t,0)x\int_0^t\varphi(\tau)\mathrm{d}\tau, & |t|<1,\\ -\frac{1}{2}\Phi(t,0)x, & t\leq -1.\end{cases} \tag{17}$$

By the definition of admissibility, there is a unique solution $\widetilde{x}(t)$ of system (4) satisfying $\widetilde{x}\in C_{\nu_1}(\mathbb{R})$. Therefore, $x_1(t)=\widetilde{x}(t)-x_f(t)+\frac{1}{2}\Phi(t,0)x$ and $x_2(t)=\widetilde{x}(t)-x_f(t)-\frac{1}{2}\Phi(t,0)x$ are solutions of system (1). Note that $x_1(t)=\widetilde{x}(t)$ for $t\geq 1$ and $x_2(t)=\widetilde{x}(t)$ for $t\leq -1$. Then we have that $x_1\in C_{\nu_1}(\mathbb{R}_+)$ and $x_2\in C_{\nu_1}(\mathbb{R}_-)$, which implies that $x_1(0)\in\Omega_{\nu_1}(\mathbb{R}_+)=\mathscr{F}$ and $x_2(0)\in\Omega_{\nu_1}(\mathbb{R}_-)=\mathscr{E}$. It can be seen that $x_1(0)-x_2(0)=\Phi(0,0)x=x$. Therefore, we get $\mathbb{R}^n\subset\mathscr{F}+\mathscr{E}$. It follows from $\mathscr{F}+\mathscr{E}\subset\mathbb{R}^n$ and $\mathscr{F}\cap\mathscr{E}=\{0\}$ that $\mathbb{R}^n=\mathscr{F}\oplus\mathscr{E}$.

Let Π be a projection on $\mathbb{R}^n$ such that $\mathscr{R}(\Pi)=\mathscr{F}$ and $\mathscr{N}(\Pi)=\mathscr{E}$. Let $P:\mathbb{R}\to\mathbb{R}^{n\times n}$ is such that $P(t)=\Phi(t,0)\Pi\Phi(0,t)$. Clearly, P is an invariant projector. From Lemma 3.1, Lemma 3.2 and Theorem 2.1, we have that system (1) has NEDs on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections $P(t)$, $t\geq 0$ and $P(t)$,

$t \leq 0$, respectively. Hence, there exist constants $K_i \geq 1, \alpha_i \geq 0$ and $\varepsilon_i \geq 0$ for $i = 1,2$, such that

$$\begin{aligned} \|\Phi(t,s)P(s)\| &\leq K_1 e^{-\alpha_1(t-s)+\varepsilon_1|s|}, \quad t \geq s \geq 0, \\ \|\Phi(t,s)Q(s)\| &\leq K_1 e^{-\alpha_1(s-t)+\varepsilon_1|s|}, \quad 0 \leq t \leq s, \\ \|\Phi(t,s)P(s)\| &\leq K_2 e^{-\alpha_2(t-s)+\varepsilon_2|s|}, \quad 0 \geq t \geq s, \\ \|\Phi(t,s)Q(s)\| &\leq K_2 e^{-\alpha_2(s-t)+\varepsilon_2|s|}, \quad t \leq s \leq 0, \end{aligned} \tag{18}$$

where $Q(s) = I - P(s)$. Then, for $t \geq 0 \geq s$,

$$\begin{aligned} \|\Phi(t,s)P(s)\| =& \|\Phi(t,0)P(0)\Phi(0,s)P(s)\| \\ \leq& \|\Phi(t,0)P(0)\| \cdot \|\Phi(0,s)P(s)\| \\ \leq& K_1 e^{-\alpha_1 t} K_2 e^{\alpha_2 s+\varepsilon_2|s|}. \end{aligned} \tag{19}$$

Similarly, we have that for $t \leq 0 \leq s$,

$$\begin{aligned} \|\Phi(t,s)P(s)\| =& \|\Phi(t,0)P(0)\| \cdot \|\Phi(0,s)P(s)\| \\ \leq& K_2 e^{\alpha_2 t} K_1 e^{-\alpha_1 s+\varepsilon_2|s|}. \end{aligned} \tag{20}$$

Let $K = K_1K_2$, $\alpha = \min\{\alpha_1, \alpha_2\}$, $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Then, by (18), (19) and (20), the inequities in (3) hold for $t, s \in \mathbb{R}$. Therefore, system (1) has a NED on $\mathbb{R}$.

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References

[1] F. Bătăran, C. Preda and P. Preda, Extending some results of L. Barreira and C. Valls to the case of linear skew-product semiflows, *Result Math.*, 72 (2017) 965–978.

[2] L. Barreira, D. Dragičević and C. Valls, *Admissibility and hyperbolicity*, SpringerBriefs in Mathematics, Springer, Cham, 2018.

[3] L. Barreira, C. Valls, Admissibility for nonuniform exponential contractions, *J. Differ. Equ.*, 249 (2010) 2889–2904.

[4] L. Barreira, D. Dragičević, C. Valls, Admissibility on the half line for evolution families, *J. Anal. Math.*, 132 (2017) 157–176.

[5] L. Barreira, D. Dragičević, C. Valls, Strong and weak (L^p, L^q)-admissibility, *Bull. Sci. Math.,* 138 (2014) 721–741.

[6] L. Barreira, C. Valls, A Grobman-Hartman theorem for nonuniformly hyperbolic dynamics, *J. Differential Equations*, 228 (2006) 285–310.

[7] L. Barreira, C. Valls, Nonuniform exponential dichotomies and admissibility, *Discrete Contin. Dyn. Syst.*, 30 (2011) 39–53.

[8] A. Belacel, A. Bougoutaia, P. Rueda, Summability of multilinear operators and their linearizations on Banach lattices, *J. Math. Anal. Appl.*, 527 (2023), 127459.

[9] B.F. Bylov, Almost reducible system of differential equations, Sibirsk. Mat. Zh., 3 (1962), 333– 359.

[10] S.N. Chow, K. Lu, Invariant manifolds for flows in Banach spaces, *J. Differ. Equ.,* 74 (1988) 285-317.

[11] C.V. Coffman, J.J. Schäffer, Dichotomies for linear difference equations, *Math. Ann,* 172 (1967) 139–166.

[12] W.A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Math., vol. 629, Springer, Berlin, 1978.

[13] J.L. Dalec'kiĭ, M.G. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Transl. Math. Monogr., vol. 43, American Mathematical Soc., Providence, RI, 1974.

[14] D. Dragičević, W. Zhang, W. Zhang, Smooth linearization of nonautonomous differential equations with a nonuniform dichotomy, *Proc. Lond. Math. Soc.*, 121 (2020) 32–50.

[15] D. Dragičević, W. Zhang, L. Zhou, Admissibility and nonuniform exponential dichotomies, *J. Differential Equations,* 326 (2022) 201–226.

[16] D. Dragičević, L. Zhou, W. Zhang, Description of tempered exponential dichotomies by admissibility with no Lyapunov norms, preprint. arXiv:2206.06778

[17] J. Hale, *Ordinary Differential Equations*, Wiley, New York, 1969.

[18] J. Hale, X. Lin, Heteroclinic orbits for retarded functional differential equations, *J. Differ. Equ.*, 65 (1986) 175–202.

[19] D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Lecture Notes in Math., vol. 840, Springer, Berlin, 1981.

[20] N.T. Huy, N.V. Minh, Exponential dichotomy of difference equations and applications to evolution equations on the half-line, *Comput. Math. Appl.,* 42 (2001) 301–311.

[21] T. Li, Die stabilitätsfrage bei differenzengleichungen, *Acta Mathematica*, 63 (1934) 99–141.

[22] A. Maĭzel', On stability of solutions of systems of differential equations, *Ural. Politehn. Inst. Trudy,* 51 (1954) 20–50.

[23] J.L. Massera, J.J. Schäffer, Linear differential equations and functional analysis I, *Ann. Math.* 67 (1958) 517–573.

[24] J.L. Massera, J.J. Schäffer, *Linear Differential Equations and Function Spaces,* Academic Press, New York, 1966.

[25] M. Megan, A.L. Sasu, B. Sasu, Discrete admissibility and exponential dichotomy for evolution families, *Discrete Contin. Dyn. Syst.*, 9 (2003) 383–397.

[26] M. Megan, A.L. Sasu, B. Sasu, Uniform exponential dichotomy and admissibility for linear skew-product semiflows, *Recent Advances in Operator Theory, Operator Algebras, and their Applications,* 185–195, Springer, 2004.

[27] K.J. Palmer, A generalization of Hartman's linearization theorem, *J. Math. Anal. Appl.*, 41 (1973) 753–758.

[28] O. Perron, Die stabilitätsfrage bei differenzengleichungen, *Math. Z*, 32 (1930) 703–728.

[29] C. Preda, A discrete Perron–Ta Li type theorem for the dichotomy of evolution operators, *J. Math. Anal. Appl.,* 332 (2007) 727–734.

[30] C. Preda, $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$-admissibility and exponential dichotomies of cocycles, *J. Differential Equations*, 249 (2010) 578–598.

[31] P. Preda, A. Pogan, C. Preda, Schäffer spaces and exponential dichotomy for evolutionary processes, *J. Differential Equations,* 230 (2006) 378–391.

[32] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, *J. Differential Equations*, 27 (1978) 320–358.

[33] S. Siegmund, Dichotomy spectrum for nonautonomous differential equations, *J. Dynam. Differential Equations*, 14 (2002) 243–258.

[34] S. Siegmund, Reducibility of nonautonomous linear differential equations, *J. London Math. Soc.*, 65 (2002) 397–410.

[35] A.L. Sasu, B. Sasu, Discrete admissibility, ℓ^p-spaces and exponential dichotomy on the real line, *Dyn. Contin. Discrete Impul. Syst.*, 13 (2006) 551–561.

[36] B. Sasu, A.L. Sasu, Exponential dichotomy and (ℓ^p, ℓ^q)-admissibility on the half-line, *J. Math. Anal. Appl.,* 316 (2006) 397–408.

[37] Y. Xia, Y. Bai and D. O'Regan, A new method to prove the nonuniform dichotomy spectrum theorem in R^n, *Proc. Amer. Math. Soc.*, 149:7(2019), 3905-3917.

[38] M. Wu, Y. Xia, Is the Sacker-Sell type spectrum equal to the contractible set? *Proc. Royal Soc. Edinburgh Sect. A: Math.*, S0308210523000100

[39] M. Wu, Y. Xia, Admissibility and nonuniform exponential dichotomies for difference equations without bounded growth or Lyapunov norms, *Proc. Amer. Math. Soc.*, 151 (2023), 4389-4403.

[40] W. Zhang, Generalized exponential dichotomies and invariant manifolds for differential equations, *Adv. Math.* (China), 22 (1993) 1–45.

[41] L. Zhou, W. Zhang, Admissibility and roughness of nonuniform exponential dichotomies for difference equations, *J. Funct. Anal.*, 271 (2016) 1087–1129.

[42] L. Zhou, K. Lu, W. Zhang, Equivalences between nonuniform exponential dichotomy and admissibility, *J. Differential Equations*, 262 (2017) 682–747.

[43] H. Zhu, Z. Li, Nonuniform dichotomy spectrum intervals: theorem and computation, *J. Appl. Anal. Comput.*, 9 (2019) 1102–1119.

[44] C. Zhu, W. Zhang, Linearly independent homoclinic bifurcations parameterized by a small function, *J. Differ. Equ.*, 240 (2007) 38–57.

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