

## CIRCLES WITH FOUR RATIONAL POINTS IN GEOMETRIC PROGRESSION

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ABSTRACT. A set of rational points on a curve is said to be in geometric progression if either the abscissae or the ordinates of the points are in geometric progression. Examples of three points in geometric progression on a circle are already known. In this paper we obtain infinitely many examples of four points in geometric progression on circles with rational radii.

### 1. Introduction

Let  $C$  be a plane curve defined by an equation  $f(x, y) = 0$ . A set of  $n$  rational points on the curve  $C$  is said to be in arithmetic or geometric progression if the coordinates  $(x_i, y_i), i = 1, 2, \dots, n$ , of the  $n$  points are such that either the abscissae  $x_i, i = 1, 2, \dots, n$ , or the ordinates  $y_i, i = 1, 2, \dots, n$ , are in arithmetic or in geometric progression, respectively. Several authors have investigated the existence of sequences of arithmetic or geometric progressions on conics, elliptic curves and hyperelliptic curves ([1], [2], [3], [5], [6], [7], [8], [11], [12], [13], [15], [16]).

This paper is concerned with geometric progressions on a circle. Çelik, Sadek and Soydan [4] have proved that there exist infinitely many geometric progressions of length 3 on the unit circle  $x^2 + y^2 = 1$ . In this paper we obtain infinitely many geometric progressions of length 4 on circles with rational radii.

### 2. Geometric progressions of length 4 on a circle

We will obtain geometric progressions, with a positive common ratio, of length 4 on a circle whose centre is on the  $x$ -axis at  $(h, 0)$  and whose radius is  $a$ , so that the equation of the circle may be written as follows:

$$(2.1) \quad (x - h)^2 + y^2 = a^2.$$

If there exist four points  $P_i, i = 1, \dots, 4$ , with rational coordinates  $(x_i, y_i), i = 1, \dots, 4$ , respectively, on the circle (2.1) such that their abscissae  $x_i, i = 1, \dots, 4$ , are in geometric progression with common positive ratio  $r$ , and  $k$  is a nonzero rational number, then there exists another circle  $(x - kh)^2 + y^2 = (ka)^2$  on which there are four points  $(kx_i, ky_i), i = 1, \dots, 4$ , in geometric progression with the same common ratio  $r$ . Thus, by appropriate scaling, we can find a second circle on which there are four points, with integer coordinates, in geometric progression. Further, the abscissae of the four points  $P_i, i = 1, \dots, 4$ , written in the reverse order, form a geometric progression with common ratio  $1/r$ . There is, therefore, no loss of generality in assuming that  $r > 1$ . Since  $r$  is positive, all the four abscissae  $x_i$  must be of

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1 the same sign. If these four abscissae are all negative, we note that there is a corresponding circle  
 2  $(x+h)^2 + y^2 = a^2$  on which we have four points  $(-x_i, y_i), i = 1, \dots, 4$ , with positive abscissae and with  
 3 the same common ratio  $r$ . We may accordingly assume, without loss of generality, that the abscissae of  
 4 the four points  $P_i$  are all positive.

5 We take both  $h$  and  $a$  as rational so that the coordinates of an arbitrary rational point on the circle  
 6 (2.1) may be written as  $(h + 2au/(u^2 + 1), a(u^2 - 1)/(u^2 + 1))$  where  $u$  is some rational number.  
 7 Accordingly, we may take the coordinates of four rational points  $P_i, i = 1, 2, 3, 4$ , on the circle (2.1) as  
 8  $(x_i, y_i), i = 1, 2, 3, 4$ , where

$$9 \quad (2.2) \quad x_i = h + 2au_i/(u_i^2 + 1), i = 1, \dots, 4.$$

11 If the abscissae  $x_i$  of the four points  $P_i, i = 1, \dots, 4$ , are in geometric progression with common ratio  
 12  $r$ , we must have  $x_{i+1} = rx_i, i = 1, 2, 3$ , and we thus get the following three conditions, respectively:

$$13 \quad (2.3) \quad (u_1^2 + 1)(u_2^2 + 1)(r - 1)h + 2a(ru_1u_2^2 - u_1^2u_2 + ru_1 - u_2) = 0,$$

$$14 \quad (2.4) \quad (u_2^2 + 1)(u_3^2 + 1)(r - 1)h + 2a(ru_2u_3^2 - u_2^2u_3 + ru_2 - u_3) = 0,$$

$$15 \quad (2.5) \quad (u_3^2 + 1)(u_4^2 + 1)(r - 1)h + 2a(ru_3u_4^2 - u_3^2u_4 + ru_3 - u_4) = 0.$$

18 The three equations (2.3), (2.4) and (2.5) may be solved to obtain three values of  $h$ , namely  $h_1, h_2, h_3$   
 19 which may be written as follows:

$$20 \quad (2.6) \quad \begin{aligned} 21 \quad h_1 &= -2a(ru_1u_2^2 - u_1^2u_2 + ru_1 - u_2)/((u_1^2 + 1)(u_2^2 + 1)(r - 1)), \\ 22 \quad h_2 &= -2a(ru_2u_3^2 - u_2^2u_3 + ru_2 - u_3)/((u_2^2 + 1)(u_3^2 + 1)(r - 1)), \\ 23 \quad h_3 &= -2a(ru_3u_4^2 - u_3^2u_4 + ru_3 - u_4)/((u_3^2 + 1)(u_4^2 + 1)(r - 1)). \end{aligned}$$

25 Naturally, if there exists a rational solution of the three simultaneous equations (2.3), (2.4) and (2.5),  
 26 the three values  $h_1, h_2, h_3$  must be identical, that is, we must have  $h_1 = h_2$  as well as  $h_2 = h_3$ . The  
 27 condition  $h_1 = h_2$  may be written as follows:

$$28 \quad (2.7) \quad (ru_2u_3^2 - u_2^2u_3 + u_2u_3^2 + ru_2 + u_2 - u_3)u_1^2 - (u_2^2 + 1)(u_3^2 + 1)ru_1 \\ 29 \quad \quad \quad + ru_2u_3^2 - u_2^2u_3 + u_2u_3^2 + ru_2 + u_2 - u_3 = 0.$$

32 Eq. (2.7) is a quadratic equation in  $u_1$ , it does not contain  $u_4$ , and it will have a rational solution for  $u_1$   
 33 if the discriminant  $\phi_1(r, u_2, u_3)$  of Eq. (2.7) with respect to  $u_1$  is a perfect square, where

$$34 \quad (2.8) \quad \phi_1(r, u_2, u_3) = (ru_2^2u_3^2 + 2ru_2u_3^2 + ru_2^2 + ru_3^2 - 2u_2^2u_3 + 2u_2u_3^2 + 2ru_2 + r + 2u_2 - 2u_3) \\ 35 \quad \quad \quad \times (ru_2^2u_3^2 - 2ru_2u_3^2 + ru_2^2 + ru_3^2 + 2u_2^2u_3 - 2u_2u_3^2 - 2ru_2 + r - 2u_2 + 2u_3).$$

38 Similarly, the condition  $h_2 = h_3$  yields a quadratic equation in  $u_4$  namely,

$$39 \quad (2.9) \quad (ru_2^2u_3 - ru_2u_3^2 + u_2^2u_3 - ru_2 + ru_3 + u_3)u_4^2 - (u_2^2 + 1)(u_3^2 + 1)u_4 \\ 40 \quad \quad \quad + ru_2^2u_3 - ru_2u_3^2 + u_2^2u_3 - ru_2 + ru_3 + u_3 = 0.$$

Eq. (2.9) does not contain  $u_1$  and it will have a rational solution for  $u_4$  if its discriminant  $\phi_2(r, u_2, u_3)$  with respect to  $u_4$  is a perfect square, where

$$(2.10) \quad \phi_2(r, u_2, u_3) = -(2ru_2^2u_3 - 2ru_2u_3^2 + u_2^2u_3^2 + 2u_2^2u_3 - 2ru_2 + 2ru_3 + u_2^2 + u_3^2 + 2u_3 + 1) \\ \times (2ru_2^2u_3 - 2ru_2u_3^2 - u_2^2u_3^2 + 2u_2^2u_3 - 2ru_2 + 2ru_3 - u_2^2 - u_3^2 + 2u_3 - 1).$$

We performed computer trials to find rational values of  $u_2, u_3$  and a positive rational value of  $r$  such that both the discriminants  $\phi_1(r, u_2, u_3)$  and  $\phi_2(r, u_2, u_3)$  are perfect squares. We took  $u_2 = p_1/p_2, u_3 = q_1/q_2, r = r_1/r_2$  where  $p_i, q_i, r_i, i = 1, 2$ , are integers such that  $r_1 > r_2 > 0$ , and performed trials over the range  $|p_1| + |p_2| + |q_1| + |q_2| + r_1 + r_2 \leq 100$ . This yielded 14 sets of numerical values of  $u_2, u_3, r$  such that both  $\phi_1(r, u_2, u_3)$  and  $\phi_2(r, u_2, u_3)$  are perfect squares, and we thus obtained 14 rational solutions of the simultaneous Eqs. (2.7) and (2.9) leading to 14 numerical examples of four points  $P_i, i = 1, \dots, 4$ , in geometric progression on the circle (2.1) with a rational radius. In Table 1 we have listed 6 of these numerical examples in which the abscissae of the points  $P_i, i = 1, \dots, 4$ , are all less than  $10^4$ .

TABLE 1. Geometric progressions of length 4 on a circle

S. No.	$u_2$	$u_3$	$r$	$h$	$a$	Points $P_1, P_2, P_3, P_4$
1.	1/3	0	5/3	75	50	(27,14), (45,40), (75,50), (125,0)
2.	3/5	1/3	8/3	447	425	(27,65), (72,200), (192,340), (512,420)
3.	11/23	1/5	5/2	5075	4875	(512, 1716), (1280, 3060), (3200, 4500), (8000, 3900)
4.	11/23	4/7	5/3	-2125	3250	(243, 2226), (405, 2040), (675, 1650), (1125, 0)
5.	11/29	1/31	16/9	4825	3367	(1458, 0), (2592, 2520), (4608, 3360), (8192, 0)
6.	3/5	21/67	3	2559	2465	(128, 408), (384, 1160), (1152, 2024), (3456, 2296)

### 3. An infinitude of geometric progressions

We will now use the numerical results of Section 2 to obtain infinitely many circles with rational radii and having four rational points in geometric progression.

**3.1. Circles in which one or both points on the  $x$ -axis are included in the set of four points in geometric progression.** We observe from Table 1 that, in three of the geometric progressions listed at S. No. 1, 4 and 5, the point  $P_4$  lies on the  $x$ -axis, and the coordinates of  $P_4$  may be written as  $(h + a, 0)$ . On further investigation, we found that there are infinitely many such geometric progressions. While we can find them by solving the simultaneous equations (2.7) and (2.9), we obtain them below by starting *ab initio* since this is simpler.

If the point  $P_1$  on the circle (2.1) is defined as before, and  $P_4$  is taken as  $(h + a, 0)$ , and the common ratio of the geometric progression is  $r$ , then  $x_1 r^3 = h + a$ , that is,  $r^3(h + 2au_1/(u_1^2 + 1)) = h + a$ , hence

1 we get,

$$2 \quad (3.1) \quad h = a(u_1^2 - 2r^3u_1 + 1)/((u_1^2 + 1)(r^3 - 1)).$$

3 Since the points  $P_i, i = 1, \dots, 4$ , are in geometric progression, the abscissae of the points  $P_2$  and  $P_3$   
4 must be  $x_1r$  and  $x_2r$ , respectively, hence we get,

$$5 \quad (3.2) \quad x_2 = ar(u_1 - 1)^2/((u_1^2 + 1)(r^3 - 1)), \quad x_3 = ar^2(u_1 - 1)^2/((u_1^2 + 1)(r^3 - 1)).$$

6 Since both  $P_2$  and  $P_3$  lie on the circle (2.1), we must have  $a^2 - (x_2 - h)^2 = y_2^2$  and  $a^2 - (x_3 - h)^2 = y_3^2$ ,  
7 and on writing

$$8 \quad (3.3) \quad y_2 = v(u_1 - 1)a/((r^2 + r + 1)(u_1^2 + 1)), \quad y_3 = w(u_1 - 1)ra/((r^2 + r + 1)(u_1^2 + 1)),$$

9 we get the following two conditions, respectively:

$$10 \quad (3.4) \quad (u_1 + 1)^2r^4 + 2(u_1 + 1)^2r^3 + (3u_1^2 + 2u_1 + 3)r^2 + (2u_1^2 + 2)r = v^2,$$

$$11 \quad (3.5) \quad (u_1 + 1)^2r^2 + (2u_1^2 + 2)r + 2u_1^2 + 2 = w^2,$$

12 where  $v$  and  $w$  are some rational numbers.

13 In the next subsection we obtain solutions of the simultaneous diophantine equations (3.4) and (3.5)  
14 that lead to geometric progressions containing both points of the circle (2.1) on the  $x$ -axis, while in the  
15 following subsection, we obtain solutions that yield geometric progressions in which only one point of  
16 the circle on the  $x$ -axis is included.

17 **3.1.1. Geometric progressions containing both points  $(h - a, 0)$  and  $(h + a, 0)$ .** When the geometric  
18 progression begins with the point  $P_1 = (h - a, 0)$ , we have  $u_1 = -1$ , and now the two conditions (3.4)  
19 and (3.5) reduce to  $4r(r + 1) = v^2$  and  $4(r + 1) = w^2$ , respectively. Thus, both  $r$  and  $r + 1$  must be  
20 perfect squares, hence we take  $r = (t^2 - 1)^2/(4t^2)$ , where  $t$  is an arbitrary rational parameter. This  
21 leads to the circle (2.1) where

$$22 \quad (3.6) \quad h = (t^8 - 8t^6 + 30t^4 - 8t^2 + 1)(t^2 + 1)^2, \quad a = t^{12} - 6t^{10} + 15t^8 - 84t^6 + 15t^4 - 6t^2 + 1,$$

23 with the following four points in geometric progression:

$$24 \quad (3.7) \quad P_1 = (128t^6, 0), \quad P_2 = (32(t^2 - 1)^2t^4, 8t^2(t^8 - 6t^6 + 6t^2 - 1)),$$

$$25 \quad P_3 = (8(t^2 - 1)^4t^2, 4t(t^{10} - 7t^8 + 6t^6 + 6t^4 - 7t^2 + 1)), \quad P_4 = (2(t^2 - 1)^6, 0).$$

26 Taking  $t = 3$  yields, on appropriate scaling, the numerical example listed at S. No. 5 in Table I.

27 **3.1.2. Geometric progressions with only one point on the  $x$ -axis.** We will now solve the simultaneous  
28 diophantine equations (3.4) and (3.5) to obtain geometric progressions that begin with a point on the  
29 circle (2.1) that is not on the  $x$ -axis and end with the point  $(h + a, 0)$  which lies on the  $x$ -axis.

30 We write

$$31 \quad (3.8) \quad u_1 = (1 + t)/(1 - t), \quad v = 2p/(t - 1), \quad w = 2q/(t - 1),$$

32 when Eqs. (3.4) and (3.5) may be written as

$$33 \quad (3.9) \quad r(r + 1)(r^2 + t^2 + r + 1) = p^2,$$

$$34 \quad (3.10) \quad (r + 1)t^2 + r^2 + r + 1 = q^2,$$



1 reduces Eq. (3.18) to the Weierstrass form of the elliptic curve given by

$$2 \quad (3.21) \quad y^2 = x^3 - 14700x + 286000.$$

3  
4 A reference to Cremona's well-known database of elliptic curves [9] shows that the rank of the  
5 curve (3.21) is 2 with the two generators of the Mordell-Weil group being  $R_1 = (-40, 900)$  and  
6  $R_2 = (14, 288)$ . We can, therefore, find infinitely many rational points on the elliptic curve (3.21) using  
7 the group law, and then find the corresponding point on the quartic curve (3.18) using the relations  
8 (3.19) and thus obtain the four points  $P_i$  in geometric progression on the circle (2.1).

9 As a numerical example, the point  $R_1$  on the curve (3.21) yields the four points

$$10 \quad (4563, 29666), (7605, 27560), (12675, 22750), (21125, 0),$$

11 which are in geometric progression with common ratio  $5/3$ , on the circle

$$12 \quad (3.22) \quad (x + 13725)^2 + y^2 = 34850^2.$$

13 Similarly, the point  $R_2$  on the curve (3.21) yields the four points

$$14 \quad (88356987, 171988866), (147261645, 198917640), (245436075, 201974850), (409060125, 0)$$

15 which are in geometric progression, with the same common ratio  $5/3$ , on the circle

$$16 \quad (3.23) \quad (x - 202590875)^2 + y^2 = 206469250^2.$$

17 Finally we note that there also exist circles (2.1) which have geometric progressions of four points  
18 beginning with the point  $(h - a, 0)$  which lies on the  $x$ -axis while the fourth point does not lie on the  
19  $x$ -axis. Such examples may be found by following a method similar to that used above for finding  
20 geometric progressions ending with the point  $(h + a, 0)$ . Accordingly, we restrict ourselves to giving  
21 only a numerical example. The circle

$$22 \quad (3.24) \quad (x - 966123)^2 + y^2 = 963235^2$$

23 has the two points  $(2888, 0)$  and  $(1929358, 0)$  on the  $x$ -axis, and it has the following four points in  
24 geometric progression with common ratio 8, and beginning with the point  $(2888, 0)$ :

$$25 \quad (2888, 0), (23104, 196308), (184832, 563388), (1478656, 815556).$$

26  
27 **3.2. Circles in which the points on the  $x$ -axis are not included in the geometric progression.** We  
28 will now obtain infinitely many examples of geometric progressions of points  $P_1, \dots, P_4$ , on the circle  
29 (2.1) such that the points of intersection of the circle with the  $x$ -axis are not included in the geometric  
30 progression.

31 **3.2.1. Geometric progressions in which the points  $P_1$  and  $P_4$  are symmetrically situated on either side**  
32 **of the diameter parallel to the  $y$ -axis.** The existence of geometric progressions beginning with the  
33 point  $(h - a, 0)$  and ending with  $(h + a, 0)$  suggests the possibility of geometric progressions in which  
34 the two points  $P_1$  and  $P_4$  are symmetrically situated on either side of the vertical diameter of the circle.  
35 Accordingly, we take the abscissae of the points  $P_1$  and  $P_4$  as  $h - 2au/(u^2 + 1)$  and  $h + 2au/(u^2 + 1)$ .  
36 Since the common ratio is  $r$ , we have  $(h - 2au/(u^2 + 1))r^3 = h + 2au/(u^2 + 1)$ , and hence we get

$$37 \quad (3.25) \quad h = 2au(r^3 + 1)/((u^2 + 1)(r^3 - 1)).$$

1 Since the abscissae of the points  $P_i, i = 1, \dots, 4$ , are in geometric progression the abscissae of the  
2 points  $P_2$  and  $P_3$  are given by

$$3 \quad (3.26) \quad x_2 = (h - 2au/(u^2 + 1))r, \quad \text{and} \quad x_3 = (h - 2au/(u^2 + 1))r^2,$$

4 respectively.

5 Since  $P_2$  and  $P_3$  are rational points on the circle (2.1), the following two conditions must be satisfied:

$$6 \quad (3.27) \quad a^2 - (x_2 - h)^2 = y_2^2, \quad a^2 - (x_3 - h)^2 = y_3^2,$$

7 where  $y_2$  and  $y_3$  are some rational numbers. Using the values of  $h, x_2$  and  $x_3$  given by (3.25), and (3.26),  
8 respectively, and taking

$$9 \quad (3.28) \quad y_2 = az_2/((r^2 + r + 1)(u^2 + 1)), \quad y_3 = az_3/((r^2 + r + 1)(u^2 + 1)),$$

10 the conditions (3.27) may be written as follows:

$$11 \quad (3.29) \quad (u^2 - 1)^2 r^4 + 2(u^2 - 1)^2 r^3 + (u^2 + 3)(3u^2 + 1)r^2 + 2(u^4 + 6u^2 + 1)r + (u^2 - 1)^2 = z_2^2,$$

$$12 \quad (3.30) \quad (u^2 - 1)^2 r^4 + 2(u^4 + 6u^2 + 1)r^3 + (u^2 + 3)(3u^2 + 1)r^2 + 2(u^2 - 1)^2 r + (u^2 - 1)^2 = z_3^2,$$

13 When  $u = 3$  the discriminants, with respect to  $r$ , of each of the two equations (3.29) and (3.30)  
14 vanish, and these two equations may now be written as follows:

$$15 \quad (3.31) \quad 16(r^2 + r + 4)(2r + 1)^2 = z_2^2,$$

$$16 \quad (3.32) \quad 16(4r^2 + r + 1)(r + 2)^2 = z_3^2.$$

17 By choosing  $r$  such that  $r^2 + r + 4$  becomes a perfect square, we readily obtain the following solution  
18 of Eq. (3.31):

$$19 \quad (3.33) \quad r = (p^2 - 4q^2)/(q(2p + q)), \quad z_2 = 4(2p^2 + 2pq - 7q^2)(p^2 + pq + 4q^2)/(q^2(2p + q)^2),$$

20 where  $p$  and  $q$  are arbitrary parameters.

21 Now, on using the value of  $r$  given by (3.33), and writing

$$22 \quad (3.34) \quad p = Xq, \quad z_3 = 4(X^2 + 4X - 2)Y/(2X + 1)^2,$$

23 Eq. (3.32) may be written as follows:

$$24 \quad (3.35) \quad Y^2 = 4X^4 + 2X^3 - 27X^2 - 4X + 61.$$

25 Eq. (3.35) represents the quartic model of an elliptic curve, and the birational transformation defined  
26 by

$$27 \quad (3.36) \quad X = -(x - 2y - 7)/(2(4x - 73)),$$

$$28 \quad Y = (8x^3 - 219x^2 - 4y^2 + 45y + 24187)/(2(4x - 73)^2),$$

29 and

$$30 \quad (3.37) \quad x = 8X^2 + 2X + 4Y - 9, \quad y = 32X^3 + 12X^2 + 16XY - 108X + 2Y - 8,$$

31 reduces the quartic curve (3.35) to the Weierstrass form of the elliptic curve given by the cubic equation,

$$32 \quad (3.38) \quad y^2 = x^3 - 1227x + 16346.$$







1 Now Eq. (3.44) represents the quartic model of an elliptic curve, and the birational transformation  
2 defined by

$$\begin{aligned} 3 \quad u_1 &= (68677x - 2616678847 + 294y)/(252y + 10671x + 1337348919), \\ 4 \quad (3.45) \quad v &= (3148740x^3 - 701722434420x^2 + 24429873322170120x \\ 5 &\quad - 53230085194000y + 316866956396335870560)/(84y + 3557x + 445782973)^2, \\ 6 \end{aligned}$$

7 and

$$\begin{aligned} 8 \quad x &= -(10439127u_1^2 - 21636858u_1 - 16065v + 3865183)/(8(6u_1 - 7)^2), \\ 9 \quad (3.46) \quad y &= -765(17040807u_1^3 + 49967001u_1^2 + 10671u_1v - 171887163u_1 \\ 10 &\quad - 68677v + 80372691)/(16(6u_1 - 7)^3), \\ 11 \end{aligned}$$

12 reduces Eq. (3.44) to the Weierstrass form of the elliptic curve given by

$$13 \quad (3.47) \quad y^2 = x^3 + x^2 - 7758767360x + 237867647099508.$$

14 The conductor of the elliptic curve (3.47) is 43368331440 and the curve is not included in the  
15 databases of Cremona and of Stein on elliptic curves. Accordingly, we used the software APECS  
16 (a package written in MAPLE) to determine that the rank of the elliptic curve (3.47) is 3, and three  
17 independent rational points  $R_1, R_2, R_3$  on it are as follows:  
18  
19

$$20 \quad R_1 = (32760151/441, 78279537050/9261), \quad R_2 = (64588, 2486862), \quad R_3 = (5836, 13884750).$$

21 We can now obtain infinitely many rational points on the elliptic curve (3.47) using the group law  
22 and corresponding to each such rational point, we can obtain a rational solution of the simultaneous  
23 equations (3.40) and (3.41). All such solutions will not yield the desired examples of geometric  
24 progressions on a circle. For instance, while the point  $R_1$  yields the known example of a geometric  
25 progression listed at S. No. 2 of Table 1, the point  $R_2$  leads to a negative value of  $r$ , and hence we do  
26 not get a geometric progression with a positive common ratio. The point  $R_3$ , however, yields a new  
27 example of a geometric progression on the circle  
28

$$29 \quad (3.48) \quad (x - 14942502807)^2 + y^2 = 13813414685^2,$$

30 on which the four points,  
31

$$\begin{aligned} 32 \quad &(632132844180584652/554537833, 303396613510115972/554537833), \\ 33 \quad &(2754195732, 6500430440), \quad (6654453996, 11050731748), \\ 34 \quad &(1230048832913343556/76505437, 1053225199474661304/76505437), \\ 35 \end{aligned}$$

36 have their abscissae in geometric progression with common ratio  $554537833/229516311$ .

37 While all rational points on the elliptic curve (3.47) do not yield examples of geometric progressions  
38 with a positive common ratio, the aforementioned theorem of Poincaré and Hurwitz ensures the  
39 existence of infinitely many rational points on the curve (3.47) in the neighbourhood of the points  $R_1$   
40 and  $R_3$  and these rational points will yield infinitely many examples of geometric progressions on a  
41 circle with a positive common ratio and such that the terms of the geometric progression do not contain  
42 the points of intersection of the circle with the  $x$ -axis.

1 We can similarly use the other solutions listed in Table 1 to obtain further examples of geometric  
 2 progressions on a circle such that the terms of the geometric progression do not contain the points of  
 3 intersection of the circle with the  $x$ -axis.

4 **3.2.3. Parametric solutions.** We will now obtain parametric solutions of our problem by solving the  
 5 simultaneous diophantine equations (2.7) and (2.9). It follows from the discussion in Section 2 that  
 6 such solutions can be obtained by solving the simultaneous equations,  
 7

$$8 \quad (3.49) \quad \phi_1(r, u_2, u_3) = v^2,$$

$$9 \quad (3.50) \quad \phi_2(r, u_2, u_3) = w^2,$$

10  
 11 where  $\phi_1(r, u_2, u_3)$  and  $\phi_2(r, u_2, u_3)$  are defined by (2.8) and (2.10), respectively, while  $v$  and  $w$  are  
 12 arbitrary rational parameters.

13 It was pointed out by the referee that the simultaneous equations (3.49) and (3.50) represent the  
 14 intersection of two quadrics over the function field  $\mathbb{Q}(u_2, u_3)$  and so define a curve of genus 1. There  
 15 are four rational points defined over  $\mathbb{Q}(u_2, u_3)$  on the intersection given by

$$16 \quad (3.51) \quad (r, v, w) = (-1, \pm(u_2^2 + 1)(u_3^2 - 1), \pm(u_2^2 - 1)(u_3^2 + 1)),$$

17  
 18 and so the curve is elliptic. The referee transformed the elliptic curve to the cubic model of an elliptic  
 19 curve, and corresponding to one of the four points (3.51), found a rational point  $Q$  of infinite order on  
 20 the cubic model of the elliptic curve, and pulling back multiples of  $Q$  gives points on the intersection  
 21 of the quadrics (3.49) and (3.50) with  $r \neq -1$ , and in this manner it is possible to construct infinitely  
 22 many two-parameter quartets of rational points in geometric progression on a circle. The referee  
 23 actually computed the pullback of the point  $2Q$  and thus obtained a monstrous parametrization of four  
 24 rational points in geometric progression, with the common ratio being given by a rational function of  
 25 two bivariate polynomials of degree 40, on the circle (2.1) in which the values of  $h$  and  $a$  are given by  
 26 bivariate polynomials of degrees exceeding 120!

27 As desired by the referee, we studied the simultaneous equations (3.49) and (3.50) more closely,  
 28 and we will now describe two methods of obtaining parametric solutions of these equations. We begin  
 29 by writing

$$30 \quad (3.52) \quad u_2 = (1 + s)/(1 - s), \quad u_3 = (1 + t)/(1 - t),$$

31 and

$$32 \quad (3.53) \quad v = 8v_1/((1 - s)^2(1 - t)^2), \quad w = 8w_1/((1 - s)^2(1 - t)^2),$$

33  
 34 when Eqs. (3.49) and (3.50) reduce to the following two equations, respectively:

$$35 \quad (3.54) \quad (rs^2t^2 + rs^2 + s^2 - t^2)(rt^2 - s^2 + t^2 + r) = v_1^2,$$

$$36 \quad (3.55) \quad (rs^2 - rt^2 + s^2 + 1)(s^2t^2 - rs^2 + rt^2 + t^2) = w_1^2.$$

37  
 38 It is readily found that the left-hand side of Eq. (3.54) becomes a perfect square on taking  
 39

$$40 \quad (3.56) \quad r = (s - t)(s + t)(m^2 + 1)/((m - s)(m + s)(t^2 + 1)),$$

1 and, with  $r$  defined by (3.56), and taking  $w_1 = w_2/((m-s)(m+s)(t^2+1))$ , Eq. (3.55) reduces to

$$\begin{aligned} 2 & (3.57) \quad (s^4 - s^2t^2 + t^4 + s^2 + t^2 + 1)(s^2t^4 - s^4 + 3s^2t^2 + t^2)m^4 \\ 3 & \\ 4 & \quad - (s^8t^4 + s^4t^8 + 4s^6t^4 + 4s^4t^6 + s^8 - 4s^6t^2 + 18s^4t^4 - 4s^2t^6 + t^8 + 4s^4t^2 + 4s^2t^4 + s^4 + t^4)m^2 \\ 5 & \quad + (s^4t^2 + 3s^2t^2 - t^4 + s^2)(s^4t^4 + s^4t^2 + s^2t^4 + s^4 - s^2t^2 + t^4) = w_2^2. \end{aligned}$$

6 We will now obtain two parametric solutions of Eq. (3.57) by using two different methods.

7 We first note that the coefficient of  $m^4$  in the polynomial on the left-hand side of Eq. (3.57) vanishes  
8 if we take

$$9 \quad (3.58) \quad s = -(q^4 - 1)/(4q^2), \quad t = (q^2 - 1)^2/(2q(q^2 + 1)),$$

11 and the left-hand side of Eq. (3.57) reduces to

$$12 \quad (3.59) \quad -(q^2 - 1)^4(q^8 + 14q^4 + 1)^6(2mq + q^2 - 1)(2mq - q^2 + 1)/(2048q^{11}(q^2 + 1)^4)^2,$$

14 and it becomes a perfect square on taking

$$15 \quad (3.60) \quad m = (p^2q^2 - p^2 - q^2 + 1)/(2q(p^2 + 1)),$$

17 where  $p$  is an arbitrary parameter.

18 Using the values of  $s, t$  given by (3.58) and the value of  $m$  given by (3.60), we get, from the relation  
19 (3.56),

$$20 \quad (3.61) \quad r = -f(q, p)/f(p, q),$$

22 where  $f(p, q)$  is a polynomial defined by

$$23 \quad (3.62) \quad f(p, q) = (p^2(q-1)^2 + (q+1)^2)(p^2(q+1)^2 + (q-1)^2).$$

24 We now have a two-parameter solution of the simultaneous equations (3.54) and (3.55) which yields  
25 an example of four rational points in geometric progression on the circle (2.1) with

$$\begin{aligned} 26 & h = (q^{12} - 4q^{10} - q^8 - 24q^6 - q^4 - 4q^2 + 1)p^4 + 4(q^8 - 8q^6 - 34q^4 - 8q^2 + 1)p^2q^2 \\ 27 & \\ 28 & (3.63) \quad + q^{12} - 4q^{10} - q^8 - 24q^6 - q^4 - 4q^2 + 1, \\ 29 & \\ 30 & a = (p^4q^4 + p^4 + 12p^2q^2 + q^4 + 1)(q^8 + 14q^4 + 1), \end{aligned}$$

31 where  $p$  and  $q$  are arbitrary parameters. The four rational points  $P_i, i = 1, \dots, 4$ , in geometric progression,  
32 may be written as

$$\begin{aligned} 33 & P_1 = (4q^2(q^2 - 1)^2 f^2(p, q)/f(q, p), 4q(q^2 - 1)(q^8 + 14q^4 + 1)(p^4 - 1)g(p, q)/f(q, p)), \\ 34 & \\ 35 & (3.64) \quad P_2 = -4q^2(q^2 - 1)^2 f(p, q), 8q^2(q^4 - 1)g(p, q)), \\ 36 & P_3 = 4q^2(q^2 - 1)^2 f(q, p), 4q(q^2 + 1)(q^2 - 1)^2 g(p, q)), \\ 37 & \\ 38 & P_4 = -4q^2(q^2 - 1)^2 f^2(q, p)/f(p, q), 8pq(q^2 - 1)(q^8 + 14q^4 + 1)(p^2 + 1)g(p, q)/f(p, q), \end{aligned}$$

39 where  $g(p, q) = p^4q^4 + p^4 + 12p^2q^2 + q^4 + 1$  while the common ratio  $r$  of the geometric progression  
40 is given by (3.61). We note that  $r$  is necessarily negative.

41 To obtain a second parametric solution, we note that Eq. (3.57), considered as a quartic equation  
42 in  $m$  and  $w_2$ , has a solution  $(m, w_2) = (t, s(s^2 - t^2)(t^2 + 1)^2)$ . Thus, we may consider Eq. (3.57) as a

1 quartic model of an elliptic curve defined over the function field  $\mathbb{Q}(s, t)$ , reduce the curve to the cubic  
 2 model, and find a point of infinite order on it, and following an argument similar to the one used by  
 3 the referee, show the existence of infinitely many parametric solutions of the simultaneous equations  
 4 (3.54) and (3.55). The resulting parametrizations are, however, too cumbersome to write.

5 A simpler way of obtaining a solution of Eq. (3.57) is to apply a method, described by Fermat  
 6 (as quoted by Dickson [10, p. 639]), of making a quartic function a perfect square using the known  
 7 solution with  $m = t$ . This gives the following value of  $m$  which makes the left-hand side of Eq. (3.57)  
 8 a perfect square:

$$\begin{aligned}
 9 \quad (3.65) \quad m = & t(s^{10}t^4 - 3s^8t^6 - s^6t^8 - s^4t^{10} - 12s^6t^6 - 4s^4t^8 - 2s^8t^2 - 10s^6t^4 - 10s^4t^6 - 2s^2t^8 - 12s^6t^2 \\
 10 & + 4s^4t^4 - 8s^2t^6 - 3s^6 + s^4t^2 - s^2t^4 - t^6)/(s^{10}t^4 + s^8t^6 - s^6t^8 + 3s^4t^{10} + 8s^8t^4 - 4s^6t^6 \\
 11 & + 12s^4t^8 + 2s^8t^2 + 10s^6t^4 + 10s^4t^6 + 2s^2t^8 + 4s^6t^2 + 12s^4t^4 + s^6 + s^4t^2 + 3s^2t^4 - t^6). \\
 12 \\
 13
 \end{aligned}$$

14 This yields a two-parameter solution of four rational points on the circle (2.1) with the values of  $h$   
 15 and  $a$  being given by bivariate polynomials of degree 32, while the common ratio of the geometric  
 16 progression is given by a rational function of two bivariate polynomials of degree 28. While this  
 17 solution is relatively much simpler, it is still too cumbersome to write, so we do not give it explicitly.

18 We found a couple of other two-parameter solutions of four rational points in geometric progression  
 19 on a circle but we could not find a single instance in which numerical values of the parameters yield  
 20 a geometric progression with a positive common ratio. This detracts from the aesthetic appeal of  
 21 such solutions. We, however, cannot rule out the existence of two-parameter solutions that may yield  
 22 examples of four rational points on a circle in geometric progression with a positive common ratio.

23 Acknowledgment. I am grateful to the referee for his/ her comments which have led to improvements  
 24 in the paper.

## 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42

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