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DUCCI'S FOUR NUMBER GAME WITH A MUTATION

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ABSTRACT. Ducci's Four-Number Game begins with a square labelled with four positive integers, one on each corner. The game proceeds by labelling the midpoints of each side with the positive difference of the side's two corners. We formalize this notion using points in $\mathbb{N}^4$ (including $\vec{0}$) to represent each set of numbers, and a map D such that $D([a,b,c,d]) = [|a-b|,|b-c|,|c-d|,|d-a|]$ to represent each turn of the game. This game serves as a fun activity for young kids. Naturally, these children make arithmetic mistakes, which begs the question: how do small mistakes impact the game? Inspired by this question, we introduce a variation to this game by allowing errors or "mutations" in the subtraction step. That is, every fixed number of turns, we use a different map D_n such that $D_n([a,b,c,d]) = [|a-b|,|b-c|,|c-d|,|d-a+n|]$ where the positive integer n is the size of the error and the mutation randomly occurs in one of the four spots.

In this paper, we show that if a mutation occurs every two, three, or four iterations, we have two cases. If n is even, any set of initial numbers will reach all 0s, like they would in the original game. If n is odd, no set of initial points reach all 0s. However, if there is a mutation five or more turns, every set of initial points reach all 0s, regardless of the parity of n. On the other hand, if there is a mutation every turn, no set of initial points reach all 0s, irrespective of n.

1. Introduction to Ducci's Four-Number Game

Ducci's Four-Number Game, introduced in [1], has a simple premise. We first start with a square and we label each corner with a natural number, as seen in Figure 1.

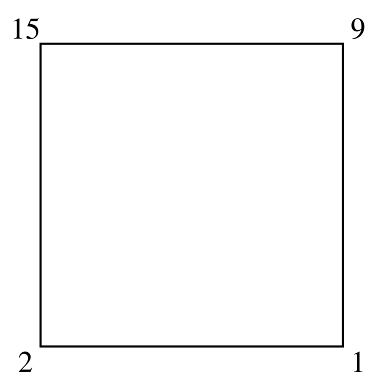

FIGURE 1. Start of game

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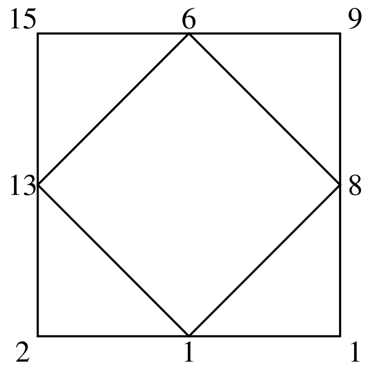

FIGURE 2. Game after one iteration

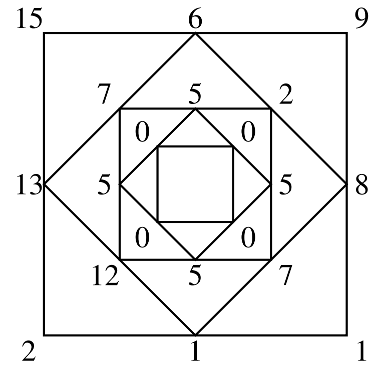

FIGURE 3. End of the game

We next mark each side's midpoint and label it with the absolute difference of its adjacent corners. This process is known as one *iteration*.

We then connect the midpoints, forming a new square, as seen in Figure 2.

If we continue iterating this process, we reach a set of values that are all 0, as seen in the nested squares in Figure 3.

As we can see, the game above ends with all 0's. Does this happen all the time? If not, for what initial conditions does it happen? In the first part of this paper, we explore this game further.

2. Characteristics of the Four-Number Game

The two properties of this game in which we will focus are the end behaviour, or whether the game converges, and the number of steps at which this happens.

2.1. ***Convergence.*** In order to investigate the Ducci game, we shall need some notation. The set of four points on a square can be represented as a four-dimensional vector in $\mathbb{N}^4$, We shall also use the convention that the first element of the vector represents one of the numbers and that the values proceed clockwise around the square. This means that, for our above example, $[15,9,1,2]$ and $[9,1,2,15]$ are both valid representations of the given square. Generally, *cyclic permutations* like these are considered to be equivalent. For the purposes of this paper, all vectors shall be considered as length four. We shall use both the names "vectors" and "points" (in $\mathbb{N}^4$) depending on context.

The process of performing one iteration as described in Section 1 is the *Ducci map*, D. In the example above, then, $D([15,9,1,2]) = [6,8,1,13]$, and the latter vector will be referred to as an *iteration* of $\vec{v}$.

Definition 1. *For $\vec{v}$ in $\mathbb{N}^4$, $\vec{v}_1$ will denote the iteration of $\vec{v}$, so $\vec{v}_1 = D(\vec{v})$. Generally, $\vec{v}_i$ is the i-fold iteration of $\vec{v}$, so $\vec{v}_i = D^i(\vec{v})$. Similarly, $\vec{v}_{-i}$ represents an element of the i-fold inverse image of $\vec{v}$ under the Ducci map, so $D^i(\vec{v}_{-i}) = \vec{v}$. The set of all i-fold inverse images of $\vec{v}$ will be denoted V_{-i}.*

We can represent the function D as multiplication on the left by matrix P, as is done in [2], where

$$P = \begin{pmatrix} \pm1 & \mp1 & 0 & 0 \\ 0 & \pm1 & \mp1 & 0 \\ 0 & 0 & \pm1 & \mp1 \\ \mp1 & 0 & 0 & \pm1 \end{pmatrix}.$$

For example, if we start with the vector $\vec{v} = [15, 9, 2, 1]$ as in the introduction, we get

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} 15 \\ 9 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \\ 13 \end{bmatrix}$$

after one iteration. Note that we chose the signs of the 1's so that the resulting vector will not contain negative values. As we can tell, these are the same values we obtained from playing the game earlier.

It shall also be useful to introduce another definition.

Definition 2. *A fixed point is a vector $\vec{v}$ for which $D(\vec{v}) = \vec{v}$.*

With all of this notation established, let us start answering some of our earlier questions.

Theorem 1. *The only fixed point of the four numbers game is $\vec{0}$.*

Proof. We shall proceed by contradiction. Let's assume there exists a non-zero fixed point $\vec{v}$, so that $\vec{v}_1 = \vec{v}$. This can be rewritten as:

$$\begin{pmatrix} \pm1 & \mp1 & 0 & 0 \\ 0 & \pm1 & \mp1 & 0 \\ 0 & 0 & \pm1 & \mp1 \\ \mp1 & 0 & 0 & \pm1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

or

$$P\vec{v}^T = \vec{v}^T,$$

so that $\vec{v}^T$ is an eigenvector of P with eigenvalue 1. By a property of eigenvalues, $\det(P - I) = 0$, where I is the identity matrix, so

$$\begin{vmatrix} \pm1-\lambda & \mp1 & 0 & 0 \\ 0 & \pm1-\lambda & \mp1 & 0 \\ 0 & 0 & \pm1-\lambda & \mp1 \\ \mp1 & 0 & 0 & \pm1-\lambda \end{vmatrix} = 0.$$

This determinant simplifies to $(\pm1-\lambda)(\pm1-\lambda)(\pm1-\lambda)(\pm1-\lambda) \pm 1$. The key thing to note here is that at least one of the elements of $\vec{v}$ has to be the largest. For example, let this element be a. Then, the first diagonal of P is 1, since $|a-b| = a-b$. Regardless of what the largest element is, we see that one of the diagonal entries of P is 1 using this logic. This makes the entire first term of our determinant equal to 0, as $(1-1)$ is a factor of that term, and we are left with either 1 or -1 for the determinant. Since 1 is an eigenvalue, $\det(P-I)$ has to be 0, giving us a contradiction.

Therefore, $\vec{v}$ cannot be an eigenvector with eigenvalue 1, and $\vec{v}$ is not a fixed point.

The discussion above does not cover the three extra cases in which the output is a cyclic permutation of $\vec{v}$, but each of these can be shown in a similar way and shall be left as an exercise for the reader. □

In fact, it has been proven that any initial vector converges to $\vec{0}$ under the Ducci map[3, 4, 5, 6, 7].

2.2. ***Speed of Convergence.*** Having established the game always converges to $\vec{0}$, we next consider the number of steps this process takes. We broke down the initial vectors $\vec{v}$ into different cases based on the number and location of repeated values. Before we delve into these cases, we introduce notation to specify elements in a vector.

Definition 3. *Let $\vec{v} = [a,b,c,d]$. Then, $\vec{v}[0] = a, \vec{v}[1] = b, \vec{v}[2] = c, \vec{v}[3] = d$. Generally, $\vec{v}[k] = \vec{v}[k_0]$, where $k_0 \equiv k \mod 4$. In addition, let $|\vec{v}_{-1}[k] - \vec{v}_{-1}[k+1]| = \vec{v}[k]$.*

In order to visualize the above definition, refer to Figure 4.

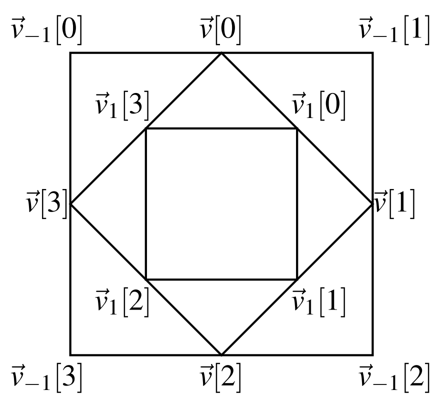

FIGURE 4. Notation used for each value of the game

In addition, let's formally define the topic of this section, which has also been explored in [8, 9, 10, 11].

Definition 4. *The speed of convergence of a vector $\vec{v}$ is the smallest natural number n such that $D^n(\vec{v}) = \vec{0}$.*

We are now ready to analyse each of the cases mentioned above. First, let us consider all of the cases in which all four values are not distinct. The speed of convergence of each of these can be checked by hand with a straightforward calculation; the results can be seen in Table 1. It is important to note that each of the vectors in this table are considered equivalent to their cyclical permutations.

In order to discover the speed of convergence for vectors with all distinct values, we introduce the following theorem.

Number of Distinct Elements	Vector	Speed of Convergence	Requirements
1	$[a,a,a,a]$	1	$a \neq 0$
2	$[a,a,a,b]$	4	$a \neq b$
	$[a,a,b,b]$	3	$a \neq b$
	$[a,b,a,b]$	2	$a \neq b$
3	$[a,b,a,c]$	$2,4$	$a \neq b \neq c$
	$[a,a,b,c]$	$4,6$	$a \neq b \neq c$

TABLE 1. Table of the speed of convergence for points with at least two values in common.

Theorem 2. *The vector $\vec{v} = [a,b,c,d]$ has the same speed of convergence as the nonzero vector $\vec{w} = c_1\vec{v} + c_2I_{1\times 4} = c_1[a,b,c,d] + c_2[1,1,1,1]$, for $c_1, c_2 \in \mathbb{R}$ and both c_1 and $c_2 \neq 0$.*

Proof. Let us rewrite our above equation by introducing the variable c_3, where $c_3 = \frac{c_2}{c_1}$. With simplification, $\vec{w} = c_1(\vec{v} + c_3I_{1\times 4})$. After one iteration, we transform $\vec{w}$ by P, which results in $P\vec{w}$. We can substitute in our above simplification and expand to get $P(c_1(\vec{v}+c_3I_{1\times 4})) = c_1P(\vec{v}+c_3I_{1\times 4}) = c_1P\vec{v} + c_3PI_{1\times 4}$[1]. Notice that the last term equals $\vec{0}$, as we are iterating the identity vector. We now have that $P\vec{w} = c_1P\vec{v}$. Now, let us assume that $\vec{w}$ converges in n iterations and $\vec{v}$ converges in m iterations. Without loss of generality, let us assume that $m \geq n$. We can then transform both sides by P^{n-1} to get $P^n\vec{w} = c_1P^n\vec{v}$. Since we assumed that $\vec{w}$ converges in n iterations, the left hand side equals $\vec{0}$. Thus, the right hand side must also equal $\vec{0}$, since $c_1 \neq 0$. This affirms that $m = n$, meaning $\vec{v}$ and $\vec{w}$ converge in the same number of iterations. □

With this proof, we are now ready to handle our final case: a vector with four distinct values. Theorem 2 allows us to simplify our vector into one with only three unique non-zero values, by subtracting out the lowest element. Table 2 contains the speeds we are looking for, which can also be obtained through straightforward calculation.

Number of Distinct Elements	Vector	Speed of Convergence	Requirements
4	$[a,b,c,0]$	≤ 6	$b > c > a$ or $b > a > c$
		≤ 4	$a > c > b$ or $c > a > b$
		≥ 4	$a > b > c$ or $c > b > a$

TABLE 2. Table of the speed of convergence for points with values that are all distinct.

These results reveal to us the structure of this game and how straightforward it is to break it down. Now, let us move to something not so trivial.

[1]The P matrix is not always linear, as its values differ based on the vector applied to it. However, notice that adding $c_2I_{1\times 4}$ to $c_1\vec{v}$ preserves the order of $c_1\vec{v}$'s elements, so we can think of P being linear here.

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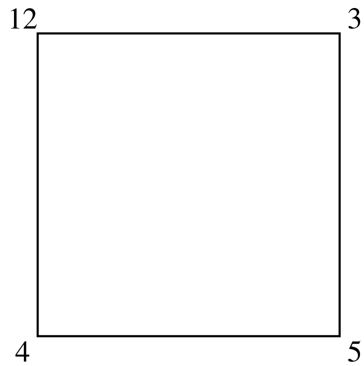

FIGURE 5. Start of a new game.

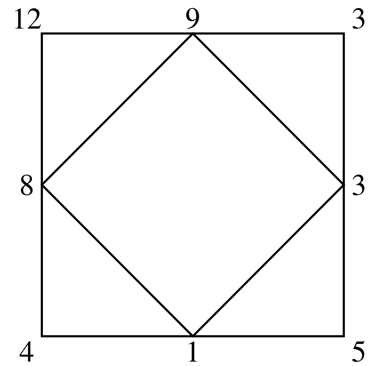

FIGURE 6. Game after one iteration.

3. The Game's Behaviour with a Mutation

Imagine we give this game to elementary school students. Like always, we begin with four random natural numbers as seen in Figure 5. We then iterate the game to reach a set of new values.

Taking a look at Figure 6, we discover that a child has made a mistake: The midpoint value on the right is off by 1. How does this seemingly innocent mistake, which we shall call a *mutation*, affect the overall game?

3.1. *Understanding the New Game.* As we did with the Four-Number game, we first try to decipher the behavior of the game. That is, does every vector converge if a single mutation occurs at every step? If not, under what circumstances do vectors converge and at what speed? Before we can tackle such questions, we introduce a definition involving the mutation.

Definition 5. *Let D_n represent a variation of the Ducci map in which a single mutation of size n occurs. That is, for a vector $\vec{v}$, one element of $D_n(\vec{v})$ will satisfy $D_n(\vec{v})[k] = |\vec{v}[k] - \vec{v}[k+1] + n|$.*

Now we can begin to explore the characteristics of this game and answer our questions above. An initial question is: if there is a mutation every iteration, can any vector converge to the zero vector? If so, for what size of error[2]? That is, for what values of n and vector $\vec{v}$ does $D_n^k(\vec{v}) = \vec{0}$ for some k? This is answered by our first theorem.

Theorem 3. *No vector can converge to the zero vector if a mutation occurs every iteration of the game. That is, $D_n^k(\vec{v}) \neq \vec{0}$ for every vector $\vec{v}$ and for all positive natural numbers k and n.*

Proof. First, let a positive integer n be the size of the mutation.

Let's start with $\vec{0}$ and work backwards. Now, we take a look at $D^{-1}(\vec{0})$. For the case where there is no mutation, we know that every vector in V_{-1} is of the form $xI_{4\times 1}$, where $x \in \mathbb{N}$, and $I_{4\times 1}$ is the identity vector. However, with a mutation occurring in this step, we cannot assume this.

Let $[a,b,c,d] \in \mathbb{N}^4$ be a vector in $V_{-1} = D_n^{-1}(\vec{0})$. Without loss of generality, let's assume that the error occurs on the calculation between c and d. Thus, in order for $D_n(\vec{v}) = \vec{0}$, $a = b = c$. In addition,

[2] Speed of convergence is ill-defined here, due to its dependency on mutations.

d must equal a for their absolute difference to equal 0. However, because of the mutation, the absolute difference between c and d is $|(d-c)+n| = |(a-c)+n| = |(c-c)+n| = n$. In order for this side to equal 0, $n=0$, which is not a value that n can take.

Therefore, V_{-1} is the empty set. This means that no initial vector converges to $\vec{0}$ when each iteration has an error. □

Notice that this proof also helps us reach a similar, yet unique conclusion.

Corollary 1. *For any vector* $\vec{v}, D_n(\vec{v}) \neq \vec{0}$.

In Theorem 3, we answered the question of what happens if one makes a mistake every iteration of the game. However, what if one consistently makes a subtracting error every few iterations? Will the game converge in this case[3]?

3.2. *Mutation Every r Iterations.* We now reach the main portion of this paper: to answer the general question of what happens if a mistake is made every r iterations. We have just shown in Theorem 3 that the game never converges when $r=1$. Before we move on to the case where $r=2$, let us prove a few fundamental results that will significantly simplify our work.

Lemma 1. *Let* $\vec{v}[k] = x$ *for some natural number* k. *If* $\vec{v}_{-1}[k] = a \geq x$, *then* $\vec{v}[k+1] = a \pm x$. *If* $a < x$, *then* $\vec{v}[k+1] = a + x$.

Proof. By definition of the game, $x = |a-b|$, where b is $\vec{v}_{-1}[k+1]$. Thus, $b = a \pm x$. However, in the case of $x > a$, b has to be equal to $a+x$, as $b = x-a < 0$ is not a possible state of the game. □

We will primarily be working backwards when proving our results, so this lemma will help divide our work into more simple cases. Another theorem that will simplify our work is stated below.

Theorem 4. *Let* $\vec{v} = [a,b,c,d]$ *be a vector in* $\mathbb{N}$. *For all* $e \in \mathbb{Q}^+$, *we define the vector* $\vec{w} = \frac{1}{e}\vec{v} = [\alpha,\beta,\gamma,\lambda]$ *such that* $a = e\alpha, b = e\beta, c = e\gamma,$ *and* $d = e\lambda$. *Then, for an arbitrary vector* $\vec{s} \in V_{-1}$, *there exists a vector* $\vec{t} \in W_{-1}$ *such that* $\vec{s} = e\vec{t}$. *Similarly, for an arbitrary vector* $\vec{t} \in W_{-1}$, *there exists a vector* $\vec{s} \in V_{-1}$ *such that* $\vec{s} = e\vec{t}$.

Proof. Let us begin by iterating backwards on vector $\vec{v}$. Let $\vec{v}_{-1}[1] = x$. Thus, $\vec{v}_{-1}[2] = x \pm b$ and $\vec{v}_{-1}[0]$ is $x \pm a$ by Lemma 1. The final value $\vec{v}_{-1}[3]$ can be written as $x \pm b \pm c$, where the b's must share the same sign for both $\vec{v}_{-1}[2]$ and $\vec{v}_{-1}[3]$. Thus, $D^{-1}(\vec{v}) = \vec{v}_{-1} = [x \pm a, x, x \pm b, x \pm b \pm c]$.

Now, let us work backwards on $\vec{w}$. Let us choose $\vec{w}_{-1}[1] = \frac{x}{e}$. We shall denote this as y, and note that y is involved in the calculations of both α and β. After calculating the other values, we end up with the vector $D^{-1}(\vec{w}) = \vec{w}_{-1} = [y \pm \alpha, y, y \pm \beta, y \pm \beta \pm \gamma]$.

Now, let us rewrite both $\vec{v}_{-1}$ and $\vec{w}_{-1}$ as $\vec{v}_{-1} = xI_{4\times 1} + e[\pm\alpha, 0, \pm\beta, \pm\beta \pm \gamma]$ and $\vec{w}_{-1} = yI_{4\times 1} + [\pm\alpha, 0, \pm\beta, \pm\beta \pm \gamma]$. Note that when $\vec{v}_{-1}$ is divided by e, we get $\vec{w}_{-1}$. Similarly, if we multiply $\vec{w}_{-1}$ by e, we get $\vec{v}_{-1}$. Since these vectors represent arbitrary elements of V_{-1} and W_{-1}, we prove our claim. □

[3]In this paper, we will only consider the fixed point of $\vec{0}$. The other fixed points (of which there are infinitely many), are only a result of the mutation and so, are not effective for comparing it to the original game.

This theorem brings an immediate question to mind: if e is a positive rational number, wouldn't our $\vec{w}_{-1}$ vector contain non-natural numbers? We utilize rational numbers in order to take advantage of the fact that natural numbers are a subset of rational numbers. That is, if we can prove that $\vec{w}_{-1} \notin \mathbb{Q}^{+4}$, then we know that $\vec{w}_{-1} \notin \mathbb{N}^{+4}$ since $\mathbb{N}^4 \in \mathbb{Q}^{+4}$.

The previous theorem does not account for an occurrence of an error, so we address this in the following corollary.

Corollary 2. *Let $\vec{v}_{i+1} = D_n(\vec{v}_i)$. Then, $\vec{w}_{i+1} = D_{\frac{n}{e}}(\vec{w}_i)$, where $\vec{v}_i = e\vec{w}_i$ and $i \in \mathbb{Z}$, assuming the error occurs between elements k and $k+1$ of both vectors.*

Proof. Let $\vec{v}_i = [a,b,c,d]$ and $\vec{w}_i = [\alpha,\beta,\gamma,\lambda]$. Without loss of generality, we shall assume the error happens between the second and third values for both vectors. Since we assume that $\vec{v}_i = e\vec{w}_i$, we know that $|b-c+n| = e|\beta-\gamma+k|$, where k is the mutation value for $\vec{w}_i$. Since $e > 0$, we can distribute it into the absolute value, and we get that $n = ek$. □

Finally, we prove one more supporting result.

Lemma 2. *If there is a vector $\vec{v}_i$ such that $\vec{v}_{i+1} = D_n(\vec{v}_i) = [a,1,b,1]$, where $a,b \in \mathbb{Q}^+$ and the error value is $+n$, then $\pm a \pm b = \pm 1 \pm 1 - n$.*

Proof. Since the vector is not a multiple of the identity vector, we need to break this up into two cases, based on where the mutation occurs. We shall first start with the case where the mutation results in either a or b in $\vec{v}_{i+1}$. Let $w = \vec{v}_i[0]$ be a value that does not occur in a calculation with a mutation. Then, the values adjacent to it are $w \pm 1$ and $w \pm a$. The final value is $w \pm a \pm 1$. By the game's definition, we know that $|(w \pm a \pm 1) - (w \pm 1) + n| = b$. With some simplification, we get $\pm a \pm b = \pm 1 \pm 1 - n$. For the second case, we now assume the mutation produces a 1 in the game. In a very similar approach, let $w = \vec{v}_i[0]$ be a value that does not occur in a calculation with a mutation. The rest of the values turn out to be the same as before, where we now get the constraint $|(w \pm a \pm 1) - (w \pm b) + n| = 1$, which results in the exact same expression from earlier, proving the theorem. □

With these new results in mind, we continue to address our questions about the game more efficiently. We shall start with the setting in which a mutation occurs every second iteration.

Theorem 5. *If $r = 2$ and n is odd, then no vector will converge to $\vec{0}$. That is, if a mutation of odd size occurs every two iterations, then the game never reaches $\vec{0}$.*

Proof. Let us work backwards in order to prove this. Let $\vec{v} = mI_{4\times1}$, where $m \in \mathbb{N}$ and $m \neq 0$, be the previous iteration of the zero vector. Note that $\vec{v}$ could not have contained a mutation by Corollary 1. By Theorem 4, we can simplify our case to $\vec{w} = I_{4\times1}$.

Thus, there is an error between two elements of $\vec{w}_{-1}$, since $i = 2$. Now, let $\vec{w}_{-1}[k] = a$ such that the calculations a is involved in do not contain the mutation. Without loss of generality, let $k = 0$. $\vec{w}_{-1}[\pm 1]$ must equal $a \pm 1$ by Lemma 1. Now, let the final value equal x. Thus, $\vec{w}_{-1} = [a, a\pm 1, x, a\pm 1]$. All of this can be seen in Figure 7.

By Theorem 4, there must exist a vector $\vec{v}_{-1} \in V_{-1}$ such that $\vec{v}_{-1} = m\vec{w}_{-1}$. We then know that one calculation yields the equation $|mx - (ma \pm m)| = m$ and the other, $|mx - (ma \pm m) + n| = m$. Simple algebra results in $\pm m + n = \pm m$, meaning either 0 or m can only possibly equal $\pm\frac{n}{2}$. However, n is odd, which means that equality is never true. We are done. □

Notice that this theorem does not restrict a game converging when the size of the mutation is even. Figure 8 illustrates such a game.

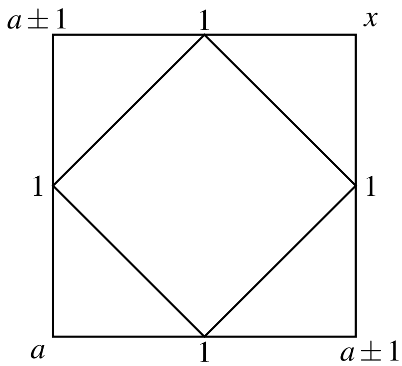

FIGURE 7. Vectors $\vec{w}$ and $\vec{w}_{-1}$ from Theorem 5 put in Ducci's Four-Number game.

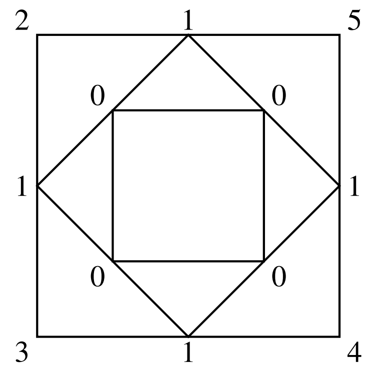

FIGURE 8. A game converging with $n = 2$. Here, $|2-5+2|$ produces the mutated value 1 at the top.

Now, let's move on to the case where $r = 3$.

Theorem 6. *If $r = 3$ and n is odd, then no initial vector will converge to $\vec{0}$. That is, if a mutation of odd value occurs every three iterations, then the game never reaches $\vec{0}$.*

Proof. As we did in the proof for the $r = 2$ case, let us move backwards. Theorem 6 and Lemma 1 tell us that the mutation cannot occur on the previous two steps. Thus, $D_n(\vec{v_{-3}}) = \vec{v}_{-2}$.

Working backwards from $\vec{0}$, we reach the vector $\vec{v}_{-1} = [x,x,x,x]$, where $x \in \mathbb{N} - \{0\}$. By Theorem 4, we can work with the vector $\vec{w}_{-1} = I_{4\times1}$, with $\vec{v}_{-1} = x\vec{w}_{-1}$. This means that for every vector in W_{-2}, there exists a vector in V_{-2} such that they are multiples of each other. Let $\vec{w}_{-2}[0] = a$. By Lemma 1, we know that $\vec{w}_{-2}[\pm1] = a\pm1$. Now, let the final value be labelled as b, where $|b-(a\pm1)| = 1$.

Now, we break up our game into three cases and simplify. We get:

- Case 1: $\vec{w}_{-2} = [a,a+1,b,a+1] = [a,a+1,a+k,a+1]$, where $k = 0,2$.
- Case 2: $\vec{w}_{-2} = [a,a-1,b,a+1] = [a,a-1,a,a+1]$.
- Case 3: $\vec{w}_{-2} = [a,a-1,b,a-1] = [a,a-1,a-k,a-1]$, where $k = 0,2$.

Let us focus on Case 1. By Theorem 4, let us define $\vec{u}_{-2} = [\frac{a}{a+1}, 1, \frac{a+k}{a+1}, 1]$, where $\vec{w}_{-2} = (a+1)\vec{u}_{-2}$. By Lemma 2, we know that $|\frac{a}{a+1} \pm \frac{a+k}{a+1}| = \pm1\pm1-m$. Note that by Corollary 1, $mx(a+1) = n$, which is the error we are working with for our original vector. By plugging in and making sure our values are non-negative, we get that the only possible solution is $n = 2xa$ or $2x(a+1)$. Clearly, these are all even numbers, so games with a odd mutation value n will not converge to $\vec{0}$.

For readers who want to visualize the proof and vectors created for Case 1, please refer to Figure 9. The other cases are very similar and shall be left as an exercise for the readers. □

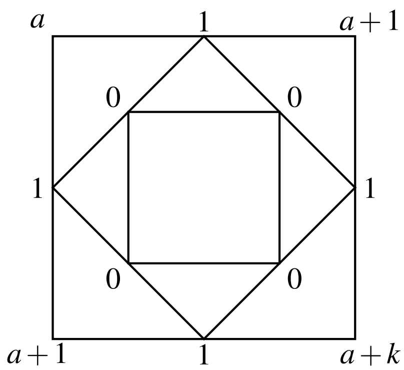

FIGURE 9. Vectors $\vec{w}$, $\vec{w}_{-1}$, and $\vec{0}$ from Theorem 6 put in Ducci's Four-Number Game.

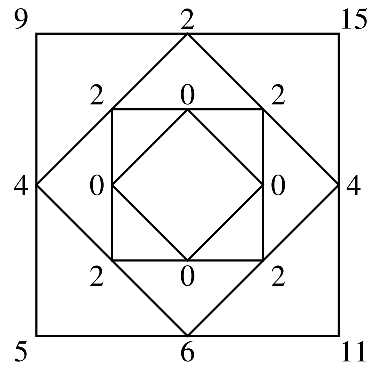

FIGURE 10. A game converging with $n=4$. Here, $|9-15+4|$ produces the mutated value 2 at the top.

Interestingly enough, we reach the same conclusion for $r=2$ and $r=3$: even mutations pose no harm to the game's convergence, but odd mutations do. An example of an even mutation is given in Figure 10.

Moving on, we reach the next value of r.

Theorem 7. *If $r=4$ and $n=1$, then no initial vector will converge to $\vec{0}$. That is, if a mutation of value 1 occurs every four iterations, then the game never reaches $\vec{0}$.*

Proof. Based on Theorems 6 and 7 and Lemma 1, $D_n(\vec{v}_{-4}) = \vec{v}_{-3}$. Instead of iterating backwards to find V_{-3}, let us use the material we covered in Section 2 of this paper. According to Tables 1 and 2, vectors in 0_{-3} are of the form:

- Case 1: $[a,a,b,b] = a[1,1,x,x]$
- Case 2: $[a,|a-b|,b,0] = a[1,1-x,x,0]$
- Case 3: $[a,a+b,b,0] = a[1,1+x,x,0]$

where we have assumed that $a>b$ without loss of generality and rewrote our vectors with respect to $x=\frac{b}{a}$. It is important to note that we used Theorem 4 to simplify our cases.

Let's start with Case 1. Now, let us introduce $\vec{w}_{-3} = \frac{1}{a}\vec{v}_{-3} = [1,1,x,x]$. Now, we begin constructing $\vec{w}_{-4}$ with $\vec{w}_{-4}[1] = u$. Let us further break down this case into sub-cases, where in one sub-case the mutation produces an x and in the other, produces 1.

In the first sub-case, $\vec{w}_{-4}[1\pm 1] = u\pm 1$. The final value is $u\pm x\pm 1$. Thus, $|\pm x\pm 1\pm 1+\frac{n}{a}| = x$. With simplification, $\frac{1}{a}(\pm b\pm b+n) = 2,-2$, or 0. The only values of n that solve this equation are $2a, 2(a\pm b)$, and $2b$, which are all obviously even. In the second sub-case, the other values turn out to be $u\pm 1, u\pm 1\pm x$, and $u\pm 1\pm x\pm x$. This means that $|\pm x\pm x\pm 1+\frac{n}{a}| = 1$. This simplifies down to the first sub-case. Thus, by Theorem 4, there exists no vector $\vec{v}_{-4}$ such that $D_n(\vec{v}_{-4}) = [a,a,b,b]$ for any odd positive integer n.

For our second case, let us similarly construct a vector $\vec{w}_{-3} = \frac{1}{a}\vec{v}_{-3} =$ such that $\vec{w}_{-3} = [1,1-x,x,0]$. The mutation can produce any value of $\vec{w}_{-3}$. Let us work with one such value: $1-x$ (the others follow nearly identically). Now, let $\vec{w}_{-4}[0] = u$. Lemma 1 tells us that $\vec{w}_{-4}[1] = u \pm 1, \vec{w}_{-4}[-1] = u$, and $\vec{w}_{-4}[2] = u \pm x$. Thus, $|\pm x \pm 1 + \frac{n}{a}| = 1 - x$. This simplifies down to $\pm x \pm x + n = \pm 1 \pm 1$, so $\frac{1}{a}(\pm b \pm b + 1) = 2,\ -2,$ or 0. This is the exact same equation as Case 1. Thus, there exists no vector $\vec{v}_{-4}$ such that $D_n(\vec{v}_{-4}) = [a, |a-b|, b, 0]$ for any odd positive integer n.

We approach our third and final case similar to Case 2. Let $\vec{w}_{-3} = \frac{1}{a}\vec{v}_{-3} = [1, 1+x, x, 0]$. Let us suppose that the mutation produces $1+x$. If $\vec{w}_{-4}[0] = u$, then $\vec{w}_{-4}[1] = u \pm 1, \vec{w}_{-4}[-1] = u$, and $\vec{w}_{-4}[2] = u \pm x$, which we obtained when working with Case 2. This means that $|\pm x \pm 1 + \frac{n}{a}| = x + 1$, which becomes $\pm x \pm x + \frac{n}{a} = \pm 1 \pm 1$, which was the exact same expression from Case 1 and 2. Thus, there exists no vector $\vec{v}_{-4}$ such that $D_n(\vec{v}_{-4}) = [a, a+b, b, 0]$ for any odd positive integer n.

Since we have considered all possible cases, we have therefore proved that no vector can converge to the zero vector if a mutation of odd parity occurs every four iterations. □

Like its predecessors, the $r = 4$ case demands an even-sized error to converge. An example of such a game can be seen in Figure 11.

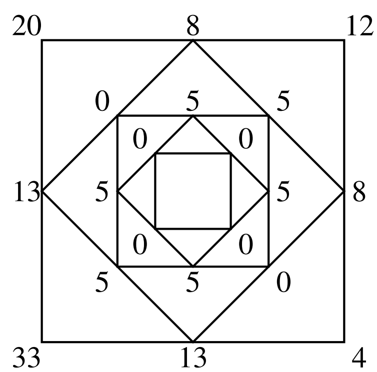

FIGURE 11. A game converging with $n = 16$. Here, $|4 - 33 + 16|$ produces the mutated value 13 at the bottom.

We now reach the final theorem of this section that concludes this discussion. It will be helpful to first state the following lemma.

Lemma 3. *Given any vector $\vec{v}$, its fourth iteration (assuming no mutation), has all even entries. That is, $\vec{v}_4[k] \equiv 0 \bmod 2$, for $k = 0, 1, 2, 3$.*

Proof. This can easily be checked by cases by considering all combinations of even and odd entries. For a more detailed discussion, see [12]. □

Theorem 8. *If $r \geq 5$, then every vector will converge to $\vec{0}$. That is, if a mutation of value 1 occurs every five or more iterations, then the game always reaches $\vec{0}$.*

Proof. Since this game deals with absolute differences, the maximum element of $\vec{v}_i$ is greater than or equal to the maximum element of $\vec{v}_{i+4}$. To make our argument more clear, let $\max(\vec{v})$ be the value of the maximum element of $\vec{v}$.

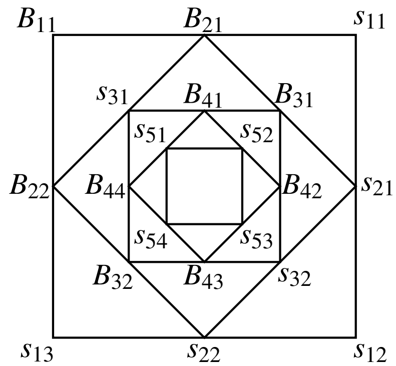

FIGURE 12. Effect of a mutation on the game. The vector $\vec{x}_{n+i}$ is $[B_{11}, s_{11}, s_{12}, s_{13}]$, where B_{11} represents a big value, while the others are small values. After B_{11} is produced through a mutation, we can easily see that the game returns to small values after four iterations.

We now start the game with a vector $\vec{v}$. Let's introduce the vector $\vec{v}_k$ such that $D_n(\vec{v}_{k-1}) = \vec{v}_k$. Let m be the smallest value such that $\max(\vec{v}_k) \leq 2^m$. Also, recall by Lemma 3 that the elements of $\vec{v}_{k+4}$ are even since $r \geq 5$.

With the help of Theorem 2, we can divide $\vec{v}_{k+4}$ by 2 to get another vector $\vec{x}_{k+4}$ with the same speed of convergence as $\vec{v}_{k+4}$. Note that $\max(\vec{x}_{k+4}) \leq 2^{m-1}$. Now, we move onto the next occurrence of the mutation which produces the vector $\vec{x}_{k+i}$. It is easy to see that $\max(\vec{x}_{k+i}) \leq 2^{m-1}$. Firstly, if the mutation makes an element smaller than what it would have been before, then clearly this is true. However, the argument for the other case is more subtle: if n is relatively larger than the elements of $\vec{x}_{k+i+4}$, then we can illustrate the mutation's effect for the next four iterations, as seen in Figure 12.

In this figure, we see that as we iterate, the mutation 'spreads' to the other elements, but after four iterations, the mutation is ejected from the vector. Due to the subtracting nature of the game, we can easily assume that $B_{ij} \geq B_{kj}$ and $s_{ij} \geq s_{lj}$ for $i \in [0, \cdots, 5], l, k \in [i, \cdots, 5]$, and $j \in [0, \cdots, 4]$. That is, after an iteration, the values that are either designated as 'small' or 'big' either decrease or stay the same. This indicates that $\vec{x}_{k+i+4} = [s_{51}, s_{52}, s_{53}, s_{54}]$ is also bounded by 2^{m-1}. We know that this vector only consists of even numbers (by Lemma 3), so we divide by 2 again to get $\vec{y}_{k+i+4}$. And so, $\max(\vec{y}_{k+i+4}) \leq 2^{m-2}$. Since every value in this game is a natural number including 0, this means that $\max(\vec{y}_{k+i+4}) \leq 2^{m-2}$. As this process continues, this generalises to $\max(\vec{w}_{g(i+4)+n}) \leq 2^{m-g-1}$. As g gets larger, the right hand side of the inequality approaches 0. If the maximum element of a vector is zero, then the vector is the zero vector.

Thus, all vectors converge to $\vec{0}$ even if there is a mutation every 5 or more iterations. □

4. Conclusion

The behaviour of the Ducci game, though well studied in the literature for decades, turns out to have interesting and novel properties when we modify the rules slightly, by introducing errors (mutations). We find that no point converges to $\vec{0}$ if there is a mutation every iteration. This is also the case for when a mutation occurs every two, three, or four iterations and the mutation value is odd. However, there is no restriction for even sized errors. In addition, all points converge to $\vec{0}$ if a mutation occurs every five or more iterations.

It may be interesting to consider other mathematical functions that make use of iteration and to consider the ways in which their behaviour, too, would change if occasional mutations occur. We also note that we have explored the case in which mutations in the Ducci game occur not at fixed intervals, but at random ones. This work is under review.

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