A third-order differential operator with dissipative boundary condition

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Abstract. This paper deals with a class of third-order dissipative differential operator generated by the general third-order symmetric regular differential expression and a certain boundary condition. By using a kind of quasi-derivative and some conditions we prove that this operator is dissipative, and in further the eigenvalue property and the completeness of the eigenfunctions and associated functions are given.

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1. Introduction

It is well known that the dissipative operators are important research topic in mathematics and physics, for example in the study of the Cauchy problems in partial differential equations such as the scattering theory and telegrapher's equation and in infinite dimensional dynamical systems [**16**, **18**].

Completeness of the root functions(eigenfunctions and generalized eigenfunctions) of a self-adjoint or non-self-adjoint operator is essential to the spectral theory of differential operators, especially for the expansion in root functions, Parseval equality as well as corresponding inverse spectral problems. Gohberg, Krein [**9**] and Keldysh [**14**] studied the spectrum and principal functions of non-self-adjoint differential operators and showed the completeness of the principal functions in the corresponding Hilbert function spaces of such problems.

Non-self-adjoint differential operators generated by symmetric differential expressions together with non-self-adjoint boundary conditions(BCs) have been investigated in many papers [**2**, **3**, **11**, **17**, **20**, **24**, **25**, **28**, **30**, **31**]. The determinant of perturbation connected with the dissipative operator L generated in $L^2[a,b)$ by the Sturm–Liouville differential expression in Weyl's limit circle case has been studied by Bairamov and Uğurlu in [**3**], they using the Livšic theorem, investigated the problem of completeness of the system of eigenfunctions and associated functions of L. They also studied the dissipative boundary value problems with transmission

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conditions and show the completeness of the root functions using Krein's theorem [**4**, **5**, **26**].

Even order dissipative operators such as dissipative Sturm–Liouville operators and dissipative fourth-order differential operators have been investigated by many authors, see [**2**–**5**, **11**, **17**, **20**, **24**–**26**, **28**, **30**, **31**] and their references. Odd order problems arise in physics and other areas of applied mathematics and have also been studied, e.g. in [**1**, **7**, **8**, **10**, **12**, **21**, **22**, **29**]. Especially, third-order differential equations often appear in many physical problems such as in modelling thin membrane flow of viscous liquid and elastic beam vibrations and so on [**6**, **10**, **19**], hence third-order differential equations have great significance in mathematical physics. Beside the Sturm–Liouville dissipative operators and fourth-order dissipative operators, there are few studies on third-order dissipative operators. Recently, the third-order dissipative differential operators generated by a maximal boundary conditions have been studied by Uğurlu [**21**]. The study of a special third-order dissipative differential operator can be found in [**27**]. However, for more complex BCs, there is no such results. This article will consider a third-order dissipative differential operator generated by general symmetric differential expression and a complex non-self-adjoint dissipative BC, and show the eigenvalue property and the completeness of the root function of it by Krein's theorem.

2. Third-Order Boundary Value Problems

Throughout the paper, we consider the following third-order differential expression

$$l(u) = \frac{1}{w}\left\{-i\left(q_0\left(q_0 u'\right)'\right)' - \left(p_0 u'\right)' + i\left[q_1 u' + \left(q_1 u\right)'\right] + p_1 u\right\}, \tag{2.1}$$

on the interval $[a, b]$, where $-\infty < a < b < \infty$, q_0, q_1, p_0, p_1 and w are continuous, real-valued functions on $[a, b]$, $q_0 \neq 0$ and $w > 0$ on $[a, b]$. Since q_0 is continuous on $[a, b]$ and different from zero at each point on the interval, we may consider that $q_0 > 0$ on $[a, b]$.

The quasi-derivatives of u is defined as [**13**, **22**]

$$u^{[0]} = u, \quad u^{[1]} = -\frac{1+i}{\sqrt{2}} q_0 u', \quad u^{[2]} = iq_0\left(q_0 u'\right)' + p_0 u' - iq_1 u. \tag{2.2}$$

As usual, let $L_w^2[a, b]$ be the Hilbert space consisting of all functions u such that

$$\int_a^b |u|^2 w dx < \infty$$

with the usual inner product

$$(u, v) = \int_a^b u\overline{v} w dx.$$

Now we shall consider a subspace Ω of $L_w^2[a, b]$,

$$\Omega = \left\{u \in L_w^2[a, b] : u, u^{[1]}, u^{[2]} \in AC_{loc}(a, b),\ l(u) \in L_w^2[a, b]\right\}.$$

For all $u, v \in \Omega$, we set

$$[u, v] := u\overline{v^{[2]}} - u^{[2]}\overline{v} + iu^{[1]}\overline{v^{[1]}}, \tag{2.3}$$

where the bar over a function denotes its complex conjugate.

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We consider the boundary value problem consisting of the differential equation

$$-i\left(q_0\left(q_0 u'\right)'\right)' - \left(p_0 u'\right)' + i\left[q_1 u' + \left(q_1 u\right)'\right] + p_1 u = \lambda w u, \quad x \in [a,b], \tag{2.4}$$

and the boundary conditions(BCs):

$$l_1(u) = u(a) + \gamma_1 u^{[1]}(a) + \gamma_2 u^{[2]}(a) = 0, \tag{2.5}$$

$$l_2(u) = u^{[1]}(a) - i\overline{\gamma_1} u^{[2]}(a) + ire^{-2i\theta}\overline{\gamma_4} u(b) + re^{-2i\theta} u^{[1]}(b) = 0, \tag{2.6}$$

$$l_3(u) = \gamma_3 u(b) + \gamma_4 u^{[1]}(b) + u^{[2]}(b) = 0, \tag{2.7}$$

where λ is a complex parameter, r is a real number with $|r| \le 1$, $\theta \in (-\pi, \pi]$, $\gamma_j,\ j = 1,2,3,4$ are complex numbers with $2\Im\gamma_2 \ge -|\gamma_1|^2$ and $2\Im\gamma_3 \ge |\gamma_4|^2$.

In $L_w^2[a,b]$, define the operator L as $Lu = l(u)$ on $D(L)$, where the domain $D(L)$ of L is given by

$$D(L) = \{u \in \Omega :\ l_j(u) = 0,\ j = 1,2,3\}.$$

3. Dissipative Operators

The dissipative operators are defined as follows.

Definition 1. *A linear operator L, acting in the Hilbert space $L_w^2[a,b]$ and having domain $D(L)$, is said to be dissipative if $\Im(Lf, f) \ge 0,\ \forall f \in D(L)$.*

Theorem 1. *The operator L is dissipative in $L_w^2[a,b]$.*

Proof. For $u \in D(L)$, we have

$$2i\Im(Lu,u) = (Lu,u) - (u,Lu) = [u,u](b) - [u,u](a), \tag{3.1}$$

then, applying (2.3), it follows that

$$\begin{aligned} 2i\Im(Lu,u) =& u(b)\overline{u^{[2]}(b)} - u^{[2]}(b)\overline{u(b)} + iu^{[1]}(b)\overline{u^{[1]}(b)} \\ &- \left(u(a)\overline{u^{[2]}(a)} - u^{[2]}(a)\overline{u(a)} + iu^{[1]}(a)\overline{u^{[1]}(a)}\right). \end{aligned} \tag{3.2}$$

From (2.5)–(2.7), it has

$$u(a) = (-i\gamma_1\overline{\gamma_1} - \gamma_2)u^{[2]}(a) + ire^{-2i\theta}\gamma_1\overline{\gamma_4}u(b) + re^{-2i\theta}\gamma_1 u^{[1]}(b), \tag{3.3}$$

$$u^{[1]}(a) = i\overline{\gamma_1}u^{[2]}(a) - ire^{-2i\theta}\overline{\gamma_4}u(b) - re^{-2i\theta}u^{[1]}(b), \tag{3.4}$$

$$u^{[2]}(b) = -\gamma_3 u(b) - \gamma_4 u^{[1]}(b), \tag{3.5}$$

substituting (3.3)–(3.5) into (3.2) one obtains

$$2i\Im(Lu,u) = (Lu,u) - (u,Lu) = \left(\overline{u^{[2]}(a)}, \overline{u(b)}, \overline{u^{[1]}(b)}\right) \tag{3.6}$$

$$\begin{pmatrix} i\gamma_1\overline{\gamma_1} + \gamma_2 - \overline{\gamma_2} & 0 & 0 \\ 0 & -ir^2\gamma_4\overline{\gamma_4} + \gamma_3 - \overline{\gamma_3} & \gamma_4(1-r^2) \\ 0 & \overline{\gamma_4}(r^2-1) & i(1-r^2) \end{pmatrix} \begin{pmatrix} u^{[2]}(a) \\ u(b) \\ u^{[1]}(b) \end{pmatrix},$$

and hence

$$2\Im(Lu,u) = \left(\overline{u^{[2]}(a)}, \overline{u(b)}, \overline{u^{[1]}(b)}\right) \begin{pmatrix} s & 0 & 0 \\ 0 & c & f \\ 0 & \overline{f} & d \end{pmatrix} \begin{pmatrix} u^{[2]}(a) \\ u(b) \\ u^{[1]}(b) \end{pmatrix}, \tag{3.7}$$

where

$$s = 2\Im\gamma_2 + |\gamma_1|^2, \quad f = -i\gamma_4(1-r^2), \quad c = 2\Im\gamma_3 - r^2|\gamma_4|^2, \quad d = 1 - r^2.$$

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Note that the 3 by 3 matrix in (3.7) is Hermitian, its eigenvalues are

$$s, \quad \frac{c+d \pm \sqrt{(c-d)^2 + 4|f|^2}}{2}$$

and they are all non-negative if and only if

$$s \geq 0, \quad c+d \geq 0, \quad cd \geq |f|^2.$$

Since $|r| \leq 1$, $2\Im\gamma_2 \geq -|\gamma_1|^2$ and $2\Im\gamma_3 \geq |\gamma_4|^2$, we have

$$\Im(Lu, u) \geq 0, \ \forall u \in D(L).$$

Hence L is a dissipative operator in $L_w^2[a,b]$. □

THEOREM 2. *If $|r| < 1$, $2\Im\gamma_2 > -|\gamma_1|^2$ and $2\Im\gamma_3 > |\gamma_4|^2$, then the operator L has no real eigenvalue.*

PROOF. Suppose λ_0 is a real eigenvalue of L. Let $\phi_0(x) = \phi(x, \lambda_0) \neq 0$ be a corresponding eigenfunction. Since

$$\Im(L\phi_0, \phi_0) = \Im(\lambda_0 \|\phi_0\|^2) = 0,$$

from (3.7), it follows that

$$\Im(L\phi_0, \phi_0) = \frac{1}{2}\left(\overline{\phi_0^{[2]}(a)}, \overline{\phi_0(b)}, \overline{\phi_0^{[1]}(b)}\right)\begin{pmatrix} s & 0 & 0 \\ 0 & c & f \\ 0 & \overline{f} & d \end{pmatrix}\begin{pmatrix} \phi_0^{[2]}(a) \\ \phi_0(b) \\ \phi_0^{[1]}(b) \end{pmatrix} = 0,$$

since $|r| < 1$, $2\Im\gamma_2 > -|\gamma_1|^2$ and $2\Im\gamma_3 > |\gamma_4|^2$, the matrix

$$\begin{pmatrix} s & 0 & 0 \\ 0 & c & f \\ 0 & \overline{f} & d \end{pmatrix}$$

is positive definite. Hence $\phi_0^{[2]}(a) = 0$, $\phi_0(b) = 0$ and $\phi_0^{[1]}(b) = 0$, and by the boundary conditions (2.5)–(2.7), we obtain that $\phi_0^{[2]}(b) = 0$, $\phi_0(a) = 0$ and $\phi_0^{[1]}(a) = 0$, consequently, $\phi_0 \equiv 0$, this is contradictory to the fact that ϕ_0 is a eigenfunction of λ_0, hence the operator L has no real eigenvalue. □

4. Completeness Theorems

In this section we show the completeness theorems of the operator here.

Let $\psi_j(x, \lambda), \ j = 1,2,3$ represent a set of linearly independent solutions of the equation $l(u) = \lambda u$, where λ is a complex parameter, then by the well known theory of ordinary differential equations, for any $x \in [a,b]$, $\psi_j(x, \lambda), \ j = 1,2,3$ are entire functions of λ. Set $z_j(x) = \psi_j(x, 0), \ j = 1,2,3$, then the solutions $z_j(x), \ j = 1,2,3$ are linearly independent solutions of the equation $l(u) = 0$.

LEMMA 1. *For all $x \in [a,b]$, $\phi_{jk} = [\psi_k(\cdot, \lambda), z_j](x)$, $j,k = 1,2,3$, are entire functions of λ with growth order ≤ 1 and minimal type: for any $j,k = 1,2,3$ and $\varepsilon \geq 0$, there exists a positive constant $C_{j,k,\varepsilon}$ such that*

$$|\phi_{jk}| \leq C_{j,k,\varepsilon} e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}.$$

PROOF. See [**30**] and [**23**]. □

Set $\Phi = (\phi_{jk})_{3\times 3}$. Then, a complex number is an eigenvalue of L if and only if it is a zero of the entire function

(4.1)
$$\Delta(\lambda) = \begin{vmatrix} l_1(\psi_1(\cdot,\lambda)) & l_1(\psi_2(\cdot,\lambda)) & l_1(\psi_3(\cdot,\lambda)) \\ l_2(\psi_1(\cdot,\lambda)) & l_2(\psi_2(\cdot,\lambda)) & l_2(\psi_3(\cdot,\lambda)) \\ l_3(\psi_1(\cdot,\lambda)) & l_3(\psi_2(\cdot,\lambda)) & l_3(\psi_3(\cdot,\lambda)) \end{vmatrix} = \mathbf{det}(A\Phi(a,\lambda) + B\Phi(b,\lambda)).$$

Corollary 1. *The entire function $\Delta(\lambda)$ is also of growth order ≤ 1 and minimal type: for any $\varepsilon \geq 0$, there exists a positive constant C_ε such that*

$$|\Delta(\lambda)| \leq C_\varepsilon e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}, \tag{4.2}$$

and hence

$$\limsup_{|\lambda|\to\infty} \frac{\ln|\Delta(\lambda)|}{|\lambda|} \leq 0. \tag{4.3}$$

From Theorem 2 it follows that zero is not an eigenvalue of L, hence the operator L^{-1} exists. Now we show an analytical representation of L^{-1}.

Consider the non-homogeneous boundary value problem composed of the equation $l(u) = f(x)$ and the BCs (2.5)–(2.7), where $x \in I = [a,b]$, $f(x) \in L^2(I)$.

Let $u(x)$ be the solution of the above non-homogeneous boundary value problem, then

$$u(x) = C_1 z_1(x) + C_2 z_2(x) + C_3 z_3(x) + u^*(x),$$

where C_j, $j = 1,2,3$ are arbitrary constants and $u^*(x)$ is a special solution.

It can be obtained by the method of constant variation

$$u^*(x) = C_1(x)z_1(x) + C_2(x)z_2(x) + C_3(x)z_3(x),$$

where C_j, $j = 1,2,3$ satisfies

$$\begin{cases} C_1'(x)z_1(x) + C_2'(x)z_2(x) + C_3'(x)z_3(x) = 0, \\ C_1'(x)z_1'(x) + C_2'(x)z_2'(x) + C_3'(x)z_3'(x) = 0, \\ \frac{-iq_0^2}{w}(C_1'(x)z_1''(x) + C_2'(x)z_2''(x) + C_3'(x)z_3''(x)) = f(x). \end{cases}$$

Solve the equations above, one has

$$C_1'(x) = \frac{iw(x)f(x)}{q_0^2(x)D(x)} \begin{vmatrix} z_2(x) & z_3(x) \\ z_2'(x) & z_3'(x) \end{vmatrix}, \quad C_2'(x) = \frac{-iw(x)f(x)}{q_0^2(x)D(x)} \begin{vmatrix} z_1(x) & z_3(x) \\ z_1'(x) & z_3'(x) \end{vmatrix},$$

$$C_3'(x) = \frac{iw(x)f(x)}{q_0^2(x)D(x)} \begin{vmatrix} z_1(x) & z_2(x) \\ z_1'(x) & z_2'(x) \end{vmatrix},$$

where

$$D(x) = \begin{vmatrix} z_1(x) & z_2(x) & z_3(x) \\ z_1'(x) & z_2'(x) & z_3'(x) \\ z_1''(x) & z_2''(x) & z_3''(x) \end{vmatrix}.$$

By proper calculation, it can be obtained that

$$u^*(x) = \int_a^b K(x,\xi)f(\xi)d\xi,$$

where

$$K(x,\xi) = \begin{cases} \frac{iw(\xi)}{q_0^2(\xi)D(\xi)} \begin{vmatrix} z_1(\xi) & z_2(\xi) & z_3(\xi) \\ z_1'(\xi) & z_2'(\xi) & z_3'(\xi) \\ z_1(x) & z_2(x) & z_3(x) \end{vmatrix}, & a < \xi \leq x < b, \\ 0, & a < x \leq \xi < b. \end{cases} \tag{4.4}$$

then

$$u(x) = C_1 z_1(x) + C_2 z_2(x) + C_3 z_3(x) + \int_a^b K(x,\xi) f(\xi) d\xi,$$

substituting $u(x)$ into the BCs one obtains

$$C_j(x) = \frac{1}{\Delta(0)} \int_a^b F_j(\xi) f(\xi) d\xi, \quad j = 1,2,3,$$

where

$$F_1(\xi) = -\begin{vmatrix} l_1(K) & l_1(z_2) & l_1(z_3) \\ l_2(K) & l_2(z_2) & l_2(z_3) \\ l_3(K) & l_3(z_2) & l_3(z_3) \end{vmatrix}, \tag{4.5}$$

$$F_2(\xi) = -\begin{vmatrix} l_1(z_1) & l_1(K) & l_1(z_3) \\ l_2(z_1) & l_2(K) & l_2(z_3) \\ l_3(z_1) & l_3(K) & l_3(z_3) \end{vmatrix}, \tag{4.6}$$

$$F_3(\xi) = -\begin{vmatrix} l_1(z_1) & l_1(z_2) & l_1(K) \\ l_2(z_1) & l_2(z_2) & l_2(K) \\ l_3(z_1) & l_3(z_2) & l_3(K) \end{vmatrix}, \tag{4.7}$$

then

$$u(x) = \int_a^b \frac{1}{\Delta(0)} [F_1(\xi) z_1(x) + F_2(\xi) z_2(x) + F_3(\xi) z_3(x) + K(x,\xi)\Delta(0)] f(\xi) d\xi,$$

let

$$G(x,\xi) = -\frac{1}{\Delta(0)} \begin{vmatrix} z_1(x) & z_2(x) & z_3(x) & K(x,\xi) \\ l_1(z_1) & l_1(z_2) & l_1(z_3) & l_1(K) \\ l_2(z_1) & l_2(z_2) & l_2(z_3) & l_2(K) \\ l_3(z_1) & l_3(z_2) & l_3(z_3) & l_3(K) \end{vmatrix}, \tag{4.8}$$

then

$$u(x) = \int_a^b G(x,\xi) f(\xi) d\xi.$$

Define the operator T as

$$Tu = \int_a^b G(x,\xi) u(\xi) d\xi, \quad \forall u \in L^2(I), \tag{4.9}$$

then T is an integral operator and $T = L^{-1}$, this implies that the root vectors of the operators T and L coincide, since $z_j(x) \in L^2(I)$, $j = 1,2,3$, then

$$\int_a^b \int_a^b |G(x,\xi)|^2 dx d\xi < +\infty, \tag{4.10}$$

hence the integral operator T is a Hilbert–Schmidt operator.

The next theorem is known as Krein's Theorem.

THEOREM 3. *Let S be a compact dissipative operator in $L^2(I)$ with nuclear imaginary part $\Im S$. The system of all root vectors of S is complete in $L^2(I)$ so long as at least one of the following two conditions is fulfilled:*

$$\lim_{m\to\infty} \frac{n_+(m, \Re S)}{m} = 0, \quad \lim_{m\to\infty} \frac{n_-(m, \Re S)}{m} = 0, \tag{4.11}$$

where $n_+(m, \Re S)$ and $n_-(m, \Re S)$ denote the number of characteristic values of the real component $\Re S$ of S in the intervals $[0, m]$ and $[-m, 0]$, respectively.

Proof. See [**9**]. $\square$

Theorem 4. *If an entire function $h(\mu)$ is of order ≤ 1 and minimal type, then*

$$\lim_{\rho\to\infty} \frac{n_+(\rho, h)}{\rho} = 0, \quad \lim_{\rho\to\infty} \frac{n_-(\rho, h)}{\rho} = 0, \tag{4.12}$$

where $n_+(\rho, h)$ and $n_-(\rho, h)$ denote the number of the zeros of the function $h(\mu)$ in the intervals $[0, \rho]$ and $[-\rho, 0]$, respectively.

Proof. See [**15**]. $\square$

The operator T can be written as $T = T_1 + iT_2$, where $T_1 = \Re T$ and $T_2 = \Im T$, T and T_1 are Hilbert–Schmidt operators, T_1 is a self-adjoint operator in $L^2(I)$, and T_2 is a nuclear operator (since it is a finite dimensional operator) [**9**]. It is easy to verify that T_1 is the inverse of the real part L_1 of the operator L.

Since the operator L is dissipative, it follows that the operator $-T$ is dissipative. Consider the operator $-T = -T_1 - iT_2$, the root vectors of the operator $-T_1$ and L_1 coincide. Since the characteristic function of L_1 is an entire function, therefore using Theorem 4 and Krein's Theorem we arrive at the following results.

Theorem 5. *The system of all root vectors of the operator $-T$ (also of T) is complete in $L^2(I)$.*

Theorem 6. *The system of all eigenvectors and associated vectors of the dissipative operator L is complete in $L^2(I)$.*

Remark 1. *In this paper, a third-order dissipative differential operator generated by general symmetric differential expression and a complex dissipative BC is studied. However, what kind of BCs can cause the third-order differential operators generated by symmetric differential expressions to be dissipative? Namely that what about the analytical representation of the dissipative BCs of this problem? It is still unknown. It is an interesting problem and we plan to study it in our future studies.*

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