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ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

AN EXPLICIT SECTION OF THE LAUDENBACH EXACT SEQUENCE OF THE MAPPING CLASS GROUP OF CONNECT SUMS OF $S^2 \times S^1$

ABSTRACT. Laudenbach proved that the mapping class group of the connect sum of n copies of $S^2 \times S^1$ is an extension of $Out(F_n)$ by a finite group. Brendle-Broaddus-Putman proved that this exact sequence splits. We provide an explicit section s of this split exact sequence.

1. Introduction

Let M_n be the connect sum of n copies of $S^2 \times S^1$ equipped with a basepoint x_0. $Mod(M_n)$ is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of M_n. We fix an isomorphism $\pi_1(M_n, x_0) \cong F_n$, where F_n is the free group of rank n. In [1,2], Laudenbach proved that that there exists a short exact sequence

$$1 \to Twist(M_n) \to Mod(M_n) \xrightarrow{\rho} Out(F_n) \to 1,$$

where $Twist(M_n) \cong (\mathbb{Z}/2)^n$ is generated by the sphere twists about the core spheres $S^2 \times *$. Brendle-Broaddus-Putman proved in [3] that this short exact sequence splits. In particular, they construct a crossed homomorphism $\mathfrak{T} : Mod(M_n) \to Twist(M_n)$ that restricts to the identity on $Twist(M_n)$. This determines a section $s : Out(F_n) \to Mod(M_n)$ of ρ , given by $s([\phi]) = \mathfrak{T}([f^{-1}])[f]$, where f is a diffeomorphism of M_n with $\rho([f]) = [\phi]$. The purpose of this paper is to provide a formula for the section s explicitly. In order to do that, we compute s for the Nielsen generators of $Out(F_n)$ given in [4]. We first describe explicit diffeomorphisms for each of the elements of the Nielsen generating set for $Out(F_n)$. Our computation shows that $\mathfrak{T}$ is trivial for these lifts. Our main result is the following:

Theorem 1.1. *The map $s : Out(F_n) \to Mod(M_n)$ that on the Nielsen generators $[R_{i,j}]$, and $[I_j]$, for $1 \le i, j \le n$ and $i \ne j$, given by:*

$$s([R_{i,j}]) = [F_{i,j}], \text{ and } s([I_j]) = [G_j],$$

is a section of ρ, where $F_{i,j}$, and G_j are diffeomorphisms of M_n defined in the section below.

2. Construction of the maps $F_{i,j}$, and G_j

For $1 \le i \le n$, choose loops a_i based at x_0 that generate the fundamental group of M_n. In [4, proposition 4.1], it is shown that $Out(F_n)$ is generated by the classes $[R_{i,j}]$, and $[I_j]$, for $1 \le i, j \le n$ and $i \ne j$, where:

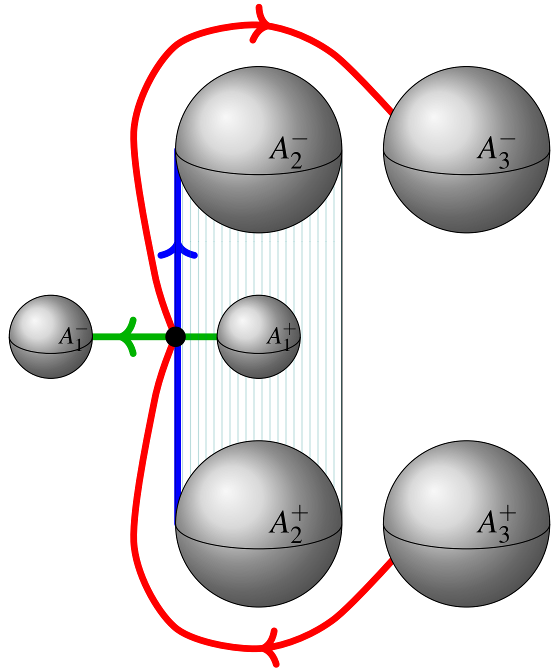

FIGURE 1. a_1, a_2, and a_3 are depicted in green, blue and red respectively. The neighborhood $N_{1,2}$ is depicted in light blue.

$$R_{i,j}(a_k) = \begin{cases} a_k a_j & \text{if } k = i \\ a_k & \text{if } k \neq i \end{cases}, \text{ and } I_j(a_k) = \begin{cases} a_k^{-1} & \text{if } k = j \\ a_k & \text{if } k \neq j. \end{cases}$$

We want to obtain diffeomorphisms $F_{i,j}$, and G_j of M_n such that $\rho([F_{i,j}]) = [R_{i,j}]$ and $\rho([G_j]) = [I_j]$.

M_n can be described by removing $2n$ open balls of S^3, and then gluing the boundary spheres of these balls in pairs. The resulting boundary spheres correspond to the core spheres of the n summands of $S^2 \times S^1$ in M_n. Let A_i denote the core sphere of the ith summand $S^2 \times S^1$ of M_n. Let A_i^- and A_i^+ denote the two boundary spheres that were identified in S^3 minus $2n$ open balls that give rise to the sphere A_i in M_n. Define a_i as the equivalence class of the curve starting at the base point that reaches A_i^- and comes back through A_i^+, and then reaches the base point, without intersecting the other boundary spheres. Then, $\{a_1, \ldots, a_n\}$ forms a basis for $\pi_1(M_n, x_0)$. Choose a subset $N_{i,j}$ of M_n diffeomorphic to $D^2 \times S^1$ minus an open ball with boundary A_i, which is contained in $\{(x,y) | x^2 + y^2 < 1/9\} \times S^1$, where $* \times S^1$ is freely homotopic to a_j, as depicted in Figure 1 for the case $n = 3, i = 1$, and $j = 2$.

For the case of I_j, choose a subset P_j' of S^3 minus $2n$-open balls diffeomorphic to $B^3 = \{(x,y,z) | x^2 + y^2 + z^2 \leq 1\}$ minus the boundary spheres A_j^+ and A_j^-, such that these spheres are contained in $\{(x,y,z) | x^2 + y^2 + z^2 < 1/9\}$, and they are symmetric with respect to a rotation by π around the $z-$axis. Denote by P_j the subset of M_n corresponding to P_j' with A_j^+ and A_j^- being identified.

2020 *Mathematics Subject Classification.* 57M07, 20F65, 20E36.

Key words and phrases. Laudenbach exact sequence, Outer automorphism group of a free group.

Parametrize a_j in such a way that $a_j(t) \in \{(x,y,z)|x^2+y^2+z^2 < 1/9\}$ for $1/3 \leq t \leq 2/3$, and $a_j(t) \in \{(x,y,z)|1/9 < x^2+y^2+z^2 < 1\}$ for $0 \leq t \leq 1/3$ and $2/3 \leq t \leq 1$. We also homotope a_j so that $a_j(t)$ lives in the $z-$axis for t as in the last case. Figure 2 depicts this for the case $j=1$.

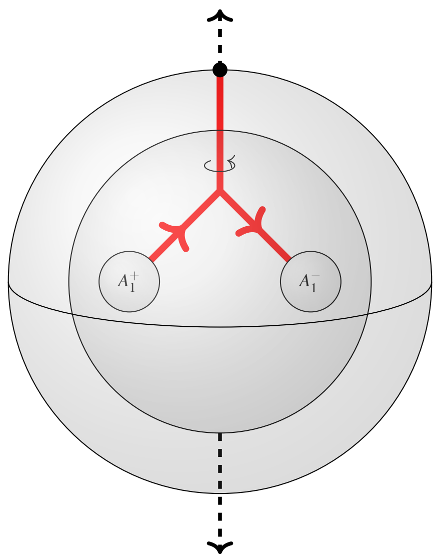

FIGURE 2. a_1 is depicted in red

Construct a smooth function $\psi : [0,1] \to [0,1]$ with $\psi(r) = 1$ on [0,1/3], $supp(\psi(r)) \subseteq [0,2/3)$, and decreasing, so that $\psi'(r) \leq 0$.

Define $f_{i,j} : N_{i,j} \to N_{i,j}$ by

$$f(x,y,e^{2\pi i\theta}) = (x,y,e^{2\pi i\left[\theta+\psi\left(\sqrt{x^2+y^2}\right)\right]}).$$

Then $f_{i,j}$ is a diffeomorphism of $N_{i,j}$.

Define $F_{i,j}$ by :

$$F_{i,j}(p) = \begin{cases} f_{i,j}(p) & \text{if } p \in N_{i,j} \\ p & p \in M_n - N_{i,j}. \end{cases}$$

If $p \in P'_j$ has spherical coordinates (θ, φ, r), define $g_j : P'_j \to P'_j$ by $g_j(p) = (\theta + \psi(r)\pi, \varphi, r)$.

As g_j respects the identification of A_j^+ and A_j^-, it induces a diffeomorphism on P which we still denote by $g_j : P \to P$. Define G_j by :

$$G_j(\theta,\varphi,r) = \begin{cases} g_j(p) & \text{if } p \in P_j \\ p & p \in M_n - P_j. \end{cases}$$

To see that $F_{i,j}$ actually realizes $R_{i,j}$, consider what $F_{i,j}$ does to the $a'_k s$, as depicted in figure 3 for the case $n=3, i=1$, and $j=2$.

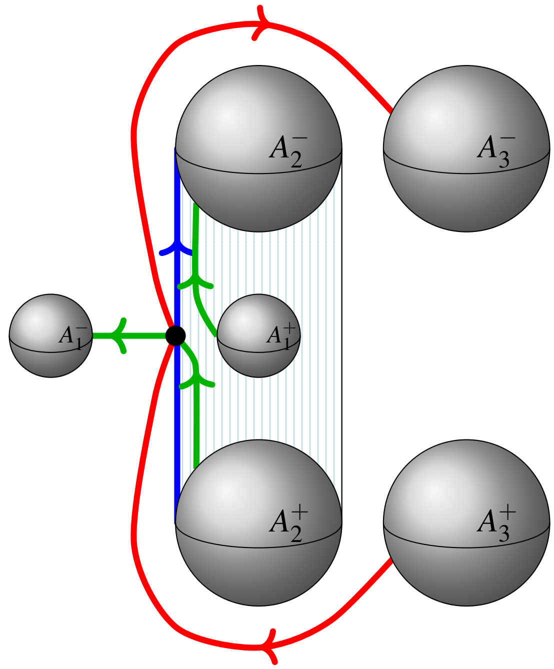

FIGURE 3. The image of a_1 under $F_{1,2}$ is depicted in green, and it is homotopic to $a_1 a_2$.

Thus, $[F_{1,2}(a_1)] = [a_1 a_2]$, and since $F_{1,2}$ fixes the homotopy classes of a_2 and a_3, then F realizes $R_{1,2}$. To see that G_j actually realizes I_j, consider what G_j does to a_j, as depicted in figure 4 for the case $j = 1$. Notice that G_j fixes the subpath of a_j that is in $P_j \cap \{(x,y,z) | 1/9 \leq x^2 + y^2 + z^2 \leq 1\}$.

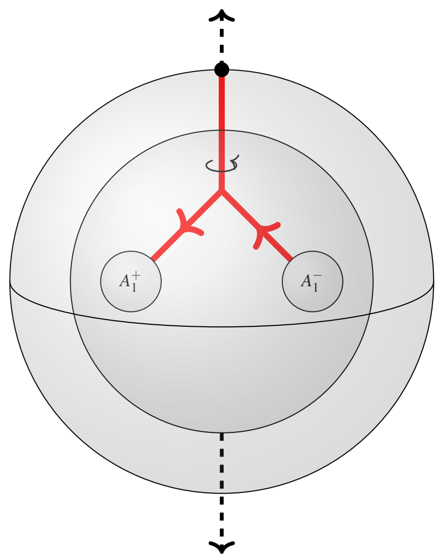

FIGURE 4. The image of a_1 under G_1 is depicted in red, and it is homotopic to a_1^{-1}
.

Hence, s defined on the Nielsen generators by $s([R_{i,j}]) = [F_{i,j}]$, and $s([I_j]) = G_j$ will be a section of ρ, provided that $\mathfrak{T}([F_{i,j}]) = 0$, and $\mathfrak{T}([G_j]) = 0$.

3. Calculation of $\mathfrak{T}([F_{i,j}])$

Denote $N_{i,j}$ by N, $F_{i,j}$ by F, and f by $f_{i,j}$. Consider the universal cover $\widetilde{N}$ of N, which is given by:

$$\{(x,y,z) | x^2 + y^2 \leq 1\} - \bigcup_{n \in \mathbb{Z}} C_n \subseteq \mathbb{R}^3,$$

where $C_n = \{(x,y,z) | x^2 + y^2 + (z-n)^2 < 1/9\}$.

Denote by π the projection map $\pi : \widetilde{N} \to N$. π is given by $\pi(x,y,z) = (x,y,e^{2\pi i z})$, and is a local diffeomorphism. f lifts to a diffeomorphism $\widetilde{f}$ given by $\widetilde{f}(x,y,z) = (x,y,z+\psi\left(\sqrt{x^2+y^2}\right))$.

We have that $\pi_1(\mathrm{GL}^+(3,\mathbb{R}), id) \cong \pi_1(SO(3), id) \cong \mathbb{Z}/2$ is generated by a loop $l : [0,1] \to SO(3)$ which can be chosen to be

$$l(t) = \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) & 0 \\ \sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for $t \in [0,1]$.

For M a closed oriented 3-manifold, let TM be the tangent bundle of M and define $Fr(TM)$ to be the principal $\mathrm{GL}_3^+(\mathbb{R})$-bundle of oriented frames of TM. That means, $Fr(TM)_x$ is the space of linear isomorphisms $T : \mathbb{R}^3 \to T_x M$. Fix a section σ_0 of $Fr(TM)$. We think of σ_0 as describing a preferred basis $\{\sigma_0(p)(e_1), \sigma_0(p)(e_2), \sigma_0(p)(e_3)\}$ of the tangent space at each point p. Denote by $C(M, \mathrm{GL}^+(3,\mathbb{R}))$ the space of continuous functions from M to $\mathrm{GL}^+(3,\mathbb{R})$.

The derivative crossed homomorphism

$$\mathscr{D} : \mathrm{Diff}^+(M) \to C(M, \mathrm{GL}^+(3, \mathbb{R}))$$

will be defined now. Given a diffeomorphism F of M, the derivative crossed homomorphism evaluated at $[F]$, $\mathscr{D}([F]) : M \to \mathrm{GL}^+(3, \mathbb{R})$, gives for each p a linear transformation $\mathscr{D}([F])(p)$ in $\mathrm{GL}^+(3, \mathbb{R})$, defined as follows. It is the unique linear transformation that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\sigma_0(p)} & T_pN \\ {\scriptstyle \mathscr{D}([F])(p)}\downarrow & & \uparrow{\scriptstyle [DF^{-1}]_{F(p)}} \\ \mathbb{R}^3 & \xrightarrow{\sigma_0(F(p))} & T_{F(p)}N \end{array}$$

Thus, $\mathscr{D}([F])(p)$ is the inverse of the linear transformation that represents the change of basis transformation from the basis
$\{\sigma_0(p)(e_1), \sigma_0(p)(e_2), \sigma_0(p)(e_3)\}$ of T_pN to the basis
$\{DF^{-1}(\sigma_0(F(p))(e_1)), DF^{-1}(\sigma_0(F(p))(e_2)), DF^{-1}(\sigma_0(F(p))(e_3))\}$ of T_pN, as depicted in figure 5.

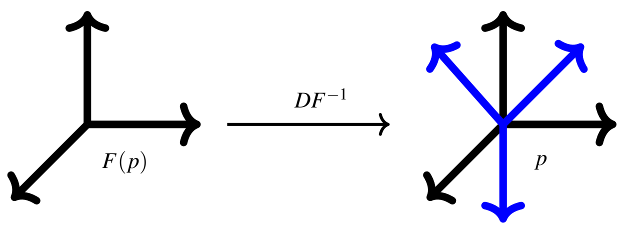

FIGURE 5. The basis at $F(p)$ and p that are determined by σ_0 are depicted black. The basis at $F(p)$ is sent to the basis in blue at p by DF^{-1}. $\mathscr{D}([F])^{-1}(p)$ is the change of basis between the blue and black basis at p

.

Thus, we get:

$$\mathscr{D}([F])^{-1}(p) = \sigma_0^{-1}(p)[DF^{-1}]_{F(p)}\sigma_0(F(p)).$$

In the particular case of $M = M_n$, we study the derivative crossed homomorphism of F using a lift of it on the universal cover of N. This simplifies the computation of the derivative crossed homomorphism of F. Let q be in the interior of $\widetilde{N}$. There is an isomorphism of vector spaces $b_q : \mathbb{R}^3 \to T_q\widetilde{N} \cong T_q\mathbb{R}^3$, defined by $b_q(e_1) = \left.\frac{\partial}{\partial x}\right|_q, b_q(e_2) = \left.\frac{\partial}{\partial y}\right|_q$, and $, b_q(e_3) = \left.\frac{\partial}{\partial z}\right|_q$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Then, define $\sigma(q) \in Fr(T\widetilde{N})$ by $\sigma(q) = b_q$. Since π is a local diffeomorphism, it induces an isomorphism of vector spaces $D\pi_q : T_q\widetilde{N} \to T_{\pi(q)}N$. Let p be in the interior of N. Select any q in the interior of $\widetilde{N}$ such that $\pi(q) = p$. Then $\sigma_0 : \mathbb{R}^3 \to T_pN$ is defined by $\sigma_0(p) := D\pi_q \circ \sigma(q)$. We want to show that σ_0 doesn't depend on the lift q of p. Let q' be another lift of p. Consider the Deck transformation Γ of $\widetilde{N}$ that sends q' to q, and is given by $\Gamma(x, y, z) = (x, y, z + k)$, for some $k \in \mathbb{Z}$.

Then, $D\Gamma_{q'}\left(\frac{\partial}{\partial x_i}\Big|_{q'}\right)=\frac{\partial}{\partial x_i}\Big|_q$. Since Γ is a Deck transformation of $\widetilde{N}$, it satisfies $\pi\circ\Gamma=\pi$. Then, $D\pi_q\circ D\Gamma_{q'}=D\pi_{q'}$. Hence:

$$\left[D\pi_{q'}\circ\sigma(q')\right](e_i)=\left[D\pi_{q'}\right](\sigma(q')(e_i))=\left[D\pi_q\circ D\Gamma_{q'}\right]\left(\frac{\partial}{\partial x_i}\Big|_{q'}\right)=D\pi_q\left(\frac{\partial}{\partial x_i}\Big|_q\right)=$$

$$\left[D\pi_q\circ\sigma(q)\right](e_i).$$

Thus, $D\pi_{q'}\circ\sigma(q')=D\pi_q\circ\sigma(q)$, so $\sigma_0(p)$ doesn't depend on the lift q of p. Thus, σ_0 is in fact a smooth section $\sigma_0 : Int(N)\to Fr(T(Int(N)))$ of the frame bundle of $Int(N)$.

Lemma 3.1. *Let $p\in Int(N)$, and $q\in\widetilde{N}$ with $\pi(q)=p$. Then, $\mathscr{D}([F])^{-1}_{ki}(p)=\frac{\partial\widetilde{f_k}^{-1}}{\partial x_i}\Big|_{\widetilde{f}(q)}$.*

Proof. Since $\pi\circ\widetilde{f}=f\circ\pi$, then by the chain rule we get:

$$D\pi_{\widetilde{f}(q)}\circ D\widetilde{f}_q=Df_{\pi(q)}\circ D\pi_q.$$

Since $D\pi_q : T_q\widetilde{N}\to T_pN$ is a linear isomorphism for each q, then:

$$D\pi_{\widetilde{f}(q)}\circ D\widetilde{f}_q\circ[D\pi_q]^{-1}=Df_p,$$

and thus:

$$D\pi_q\circ D\widetilde{f}^{-1}_{\widetilde{f}(q)}\circ\left[D\pi_{\widetilde{f}(q)}\right]^{-1}=Df^{-1}_{f(p)}.$$

Thus:

$$\sigma_0^{-1}(p)Df^{-1}_{f(p)}\sigma_0(f(p))=\sigma_0^{-1}(p)\left[D\pi_q\circ D\widetilde{f}^{-1}_{\widetilde{f}(q)}\circ\left[D\pi_{\widetilde{f}(q)}\right]^{-1}\right]\sigma_0(f(p))=\sigma^{-1}(q)D\widetilde{f}^{-1}_{\widetilde{f}(q)}\sigma(\widetilde{f}(q)).$$

Therefore:

$$\mathscr{D}([F])^{-1}(p)=\sigma^{-1}(q)D\widetilde{f}^{-1}_{\widetilde{f}(q)}\sigma(\widetilde{f}(q)).$$

For p in the interior of N, and q with $\pi(q)=p$, evaluation of e_i produces:

$$\sigma^{-1}(q)D\widetilde{f}^{-1}_{\widetilde{f}(q)}\sigma(\widetilde{f}(q))(e_i)=\sigma^{-1}(q)D\widetilde{f}^{-1}_{\widetilde{f}(q)}(\frac{\partial}{\partial x_i}\Big|_{\widetilde{f}(q)})=\sigma^{-1}(q)\left(\frac{\partial\widetilde{f}^{-1}}{\partial x_i}\Big|_{\widetilde{f}(q)}\right)=$$

$$\sigma^{-1}(q)\left(\sum_{k=1}^{3}\left(\frac{\partial\widetilde{f_k}^{-1}}{\partial x_i}\Big|_{\widetilde{f}(q)}\right)\frac{\partial}{\partial x_k}\Big|_q\right)=\sum_{k=1}^{3}\left(\frac{\partial\widetilde{f_k}^{-1}}{\partial x_i}\Big|_{\widetilde{f}(q)}\right)\sigma^{-1}(q)\left(\frac{\partial}{\partial x_k}\Big|_q\right)=$$

$$\sum_{k=1}^{3}\left(\frac{\partial\widetilde{f_k}^{-1}}{\partial x_i}\Big|_{\widetilde{f}(q)}\right)e_k.$$

Thus:

$$\mathscr{D}([F])^{-1}_{ki}(p) = \frac{\partial \widetilde{f}_k^{-1}}{\partial x_i}\Big|_{\widetilde{f}(q)}$$

□

This lemma permits the calculation of $\mathscr{D}([F])$ in terms of $\widetilde{f}$ as we mentioned.

Now, we define the twisting crossed homomorphism

$$\mathfrak{T}: Mod(M) \to Hom(\pi_1(M,x_0), \pi_1(\mathrm{GL}^+(3,\mathbb{R}), id)) = H^1(M;\mathbb{Z}/2).$$

The twisting crossed homomorphism evaluated at $[F]$, $\mathfrak{T}([F]) : \pi_1(M,x_0) \to \pi_1(\mathrm{GL}^+(3,\mathbb{R}), id)$, is the homomorphism that sends $[\gamma] \in \pi_1(M,x_0)$ to the class $\mathfrak{T}(F)[\gamma] = [\mathscr{D}([F])(\gamma)]$, where $\mathscr{D}([F])(\gamma)$ is the loop:

$$[0,1] \xrightarrow{\gamma} M \xrightarrow{\mathscr{D}([F])} \mathrm{GL}^+(3,\mathbb{R}).$$

In other words, $\mathfrak{T}([F])$ is the map induced by $\mathscr{D}([F])$ on fundamental groups.

Because the derivative of F is the identity on a_k for $k \neq i$, we get that $\mathfrak{T}([F])[a_k]$ is trivial in $\pi_1(\mathrm{GL}^+(3,\mathbb{R}), id)$ for every $k \neq i$.

Choose $\gamma \in [a_i]$, and one of its lifts $\widetilde{\gamma}$, such that $\widetilde{\gamma}$ intersects $\widetilde{N}$ as $(\mathfrak{s},0,0)$, for $\mathfrak{s} \in [0,1]$. For $t \in [0,1]$ satisfying that $\gamma(t) \notin Int(N)$, we have that the derivative of F is trivial at $\gamma(t)$, thus $\mathscr{D}([F])(\gamma(t))$ is the trivial matrix for such t. Hence, we are only interested in the case $\gamma(t) \in Int(N)$, and in this case $F = f$.

Notice that $\widetilde{f}^{-1}(x,y,z) = (x,y,z-\psi(\sqrt{x^2+y^2}))$. Then we obtain :

$$\mathscr{D}([F])^{-1}(\gamma(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-x}{\sqrt{x^2+y^2}}\frac{d\psi}{dr}(\sqrt{x^2+y^2}) & \frac{-y}{\sqrt{x^2+y^2}}\frac{d\psi}{dr}(\sqrt{x^2+y^2}) & 1 \end{bmatrix} (\mathfrak{s},0,\psi(\mathfrak{s})) =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{d\psi(\mathfrak{s})}{dr} & 0 & 1 \end{bmatrix}.$$

Hence,

$$\mathscr{D}([F])^{-1}(\gamma(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{d\psi(\mathfrak{s})}{dr} & 0 & 1 \end{bmatrix}.$$

So,

$$\mathscr{D}([F])(\gamma(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{d\psi(\mathfrak{s})}{dr} & 0 & 1 \end{bmatrix}.$$

For $(\mathscr{s},0,0)$, $0\leq \mathscr{s}\leq 1$.
We have an homotopy from the trivial path to this path,

$$H:[0,1]\times[0,1]\to \mathrm{GL}^+(3,\mathbb{R}),$$

given by:

$$H(\mathscr{s},t)=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ t\dfrac{d\psi(\mathscr{s})}{dr} & 0 & 1\end{bmatrix}.$$

Therefore, $[\mathscr{D}([F])(\gamma)]=1$ for every $t\in Dom(\gamma)$.
Hence, $\mathfrak{T}([F])[a_i]$ is trivial in $\pi_1(\mathrm{GL}^+(3,\mathbb{R}),id)$.
Therefore, the twisting crossed homomorphism $\mathfrak{T}$ evaluated at F, $\mathfrak{T}([F])$, is trivial.

4. Calculation of $\mathfrak{T}([G_j])$

Now, we analyse the case of G_j . Denote G_j by G, and P_j by P. Given $\gamma\in[a_j]$, define $\gamma_1(t):=\gamma(t/3)$, $\gamma_2(t):=\gamma(1/3+t/3)$ and $\gamma_3(t):=\gamma(2/3+t/3)$ for $0\leq t\leq 1$. Then, $\gamma=\gamma_1*\gamma_2*\gamma_3$. Homotope γ such that $\gamma_2\subseteq\{(x,y,z)|x^2+y^2+z^2\leq 1/9\}$, and that $\gamma_1(t)=(0,0,\varepsilon(t))$ and $\gamma_3(t)=(0,0,\varepsilon(1-t))$, for some smooth function $\varepsilon:[0,1]\to[0,1]$. For $(x,y,z)\in P_j$, G is the diffeomorphism that sends (x,y,z) to

$$\begin{bmatrix}\cos(\psi(\sqrt{x^2+y^2+z^2})\pi) & -\sin(\psi(\sqrt{x^2+y^2+z^2})\pi) & 0\\ \sin(\psi(\sqrt{x^2+y^2+z^2})\pi) & \cos(\psi(\sqrt{x^2+y^2+z^2})\pi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Evaluating G on γ_1 we get:

$$G(\gamma_1(t))=\begin{bmatrix}\cos(\psi(\varepsilon(t))\pi) & -\sin(\psi(\varepsilon(t))\pi) & 0\\ \sin(\psi(\varepsilon(t))\pi) & \cos(\psi(\varepsilon(t))\pi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}0\\0\\ \varepsilon(t)\end{pmatrix}=(0,0,\varepsilon(t)),$$

for $0\leq t\leq 1$.
Evaluating G on γ_2 we get:

$$G(\gamma_2(t))=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\gamma_2(t)=\overline{\gamma_2(t)},$$

for $0\leq t\leq 1$.
Evaluating G on γ_3 we get:

$$G(\gamma_3(t))=\begin{bmatrix}\cos(\psi(\varepsilon(1-t))\pi) & -\sin(\psi(\varepsilon(1-t))\pi) & 0\\ \sin(\psi(\varepsilon(1-t))\pi) & \cos(\psi(\varepsilon(1-t))\pi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}0\\0\\ \varepsilon(1-t)\end{pmatrix}=(0,0,\varepsilon(1-t)),$$

for $0\leq t\leq 1$.

Denote $\mathscr{D}([G])^{-1}(\gamma_1(t))$ by $w_1(t)$ and $\mathscr{D}([G])^{-1}(\gamma_3(t))$ by $w_3(t)$, for $0 \leq t \leq 1$. The fact that $G(\gamma_1(t)) = \overline{G(\gamma_3(t))}$, implies that $w_1(t) = \overline{w_3(t)}$ for $0 \leq t \leq 1$.

On the other hand, $\mathscr{D}([G])^{-1}(\gamma_2(t))$ is the constant path:

$$\mathscr{D}([G])^{-1}(\gamma_2(t)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: w_2(t),$$

for $0 \leq t \leq 1$

Therefore, $\mathscr{D}([G])^{-1}(\gamma) = w_1 * w_2 * \overline{w_1}$, which implies that $[\mathscr{D}([G])^{-1}(\gamma)]$ is trivial for $t \in Dom(\gamma)$, because $w_2(t)$ is a constant path. Therefore, the twisting crossed homomorphism $\mathfrak{T}$ evaluated at G, $\mathfrak{T}([G])$, is trivial.

5. References

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