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EXISTENCE OF INFINITELY MANY HIGH ENERGY SOLUTIONS FOR A FOURTH-ORDER KIRCHHOFF TYPE EQUATION

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ABSTRACT. In this paper, we study the following fourth-order elliptic equations of Kirchhoff type:

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)\Delta u + V(x)u = f(x,u) + \lambda h(x,u) \quad x \in \mathbb{R}^N,$$

where $a > 0$, $b \geq 0$ are constants, we have the potential $V(x) : \mathbb{R}^N \to \mathbb{R}$, $V \in C(\mathbb{R}^N, \mathbb{R})$. The nonlinearity $\lambda h(x,u) + f(x,u)$ may involve a combination of concave and convex terms. Under some suitable conditions on $h, f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $\lambda \in \mathbb{R}$, we show the existence of nontrivial solutions by combining the mountain pass theorem and variational methods. Moreover, we also prove the existence of infinitely many high-energy solutions using the Fountain theorem.

1. Introduction

In this article, we are interested in the existence of solution for the following Kirchhoff-type problem:

$$(1.1) \qquad \begin{cases} \Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)\Delta u + V(x)u = f(x,u) + \lambda h(x,u), & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where a, b are positive constants, $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator. Problem 1.1 arises in the study of travelling waves in suspension bridge and the study of the static deflection of an elastic plate in a fluid, see [1] Problem 1.1 is a nonlocal problem because of the so-called nonlocal term $b\int_{\mathbb{R}^N} |\nabla u|^2 \, dx$ involved in equation (1.1). There are some mathematical difficulties since the presence of a nonlocal term in the equation indicates that (1.1) is not a pointwise identity. Indeed, in general, we do not know $\int_{\mathbb{R}^N} |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |\nabla u|^2$ from $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$. Compared with previous results where the study was based on the case of bounded domain, the case of unbounded domain seems to be more complicated. In this case, the principal difficulty is the lack of compactness of the embedding. In order to recover the compactness, some classical assumptions on $V(x)$ are introduced, such as the condition denoted as (V) below.

If we set $V(x) = 0, \lambda = 0$, replace $\mathbb{R}^N$ by a bounded smooth domain $\Omega \in \mathbb{R}^N$ and set $u = \Delta u = 0$ on $\partial\Omega$, then problem 1.1 is reduced to the following equation

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$$(1.2) \qquad \begin{cases} \Delta^2 u - \left(a + b \int\limits_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad \nabla u = 0 \quad \text{on } \Omega, \end{cases}$$

which is related to the stationary analogue of the Kirchhoff equation

$$(1.3) \qquad \Delta^2 u + u_{tt} - (a + b \int\limits_{\Omega} |\nabla u|^2 \, dx) \Delta u = f(x,u), \qquad x \in \Omega.$$

Equation (1.3) was proposed by Burgreen [2] as a model for the transverse deflection $u(x,t)$ of an extensible beam of natural length I (Ω) whose ends are held a fixed distance apart. The nonlinear term represents the change in the tension of the beam due to its extensibility. The model has also been discussed by Eisley [3], while Woinowsky-Krieger and Ball had given related experimental results [4, 5].

In recent years, many authors have paid attention to Kirchhoff-type problems. For instance, see [6, 7, 9, 10, 13, 16, 17, 18, 19, 20] and the references therein. Meanwhile, little has been done for the existence of infinitely many solutions for fourth-order Kirchhoff-type problems in $\mathbb{R}^N$. It is the first purpose of our paper to investigate the existence of infinitely many solutions for fourth-order Kirchhoff-type problems in $\mathbb{R}^N$.

In [8], Xu and Chen considered the following nonhomogeneous fourth-order Kirchhoff-type

$$(1.4) \qquad \Delta^2 u - \left(a + b \int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + V(x) u = f(x,u) + h(x), \ \ x \in \mathbb{R}^N,$$

using the Mountain Pass Theorem and Ekeland's variational principle, they obtained a multiplicity result to the above problem provided $|h|_2$ is small enough. Later, Zuo et al. [15] studied the existence of nontrivial solution to problem 1.4 using the Mountain Pass Theorem. In addition, they obtained infinitely many high-energy solutions for the homogeneous problem by two kinds of methods: Symmetry Mountain Pass Theorem and Fountain Theorem, when the nonlinearity f satisfies the following condition:

- (V) $\inf\limits_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$ and for any $M > 0$, meas$\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant, "meas" denotes the Lebesgue measure in $\mathbb{R}^N$.
- (f_1) $\lim\limits_{|t| \to 0} \frac{f(x,t)}{|t|} = 0$ uniformly for any $x \in \mathbb{R}^N$.
- (F_2) There are constants $2 < p < 2^{**}$, and $C > 0$ such that
$$|f(x,t)| \leq C\left(|t|^{p-1} + 1\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$
where if $N \leq 2$, let $2^{**} = +\infty$; if $N \geq 2$, let $2^{**} = \frac{2N}{N-2}$.
- (f_3) $\lim\limits_{|t| \to +\infty} \frac{F(x,t)}{t^4} \to +\infty$ uniformly in $x \in \mathbb{R}^N$.

(F_4) There exists $r > 0$ such that

$$tf(x,t) \geq 4F(x,t), \quad \forall x \in \mathbb{R}^N, |t| \geq r.$$

Observe that condition (F_4) plays an important role for proving that any Palais–Smale sequence is bounded in the work.

Motivated by the above works, the purpose of this paper is to study the existence of nontrivial solution of problem 1.1 by combining the Mountain pass theorem and variational methods and the existence of infinitely many high-energy solutions using Fountain theorem. To the best of our knowledge, there are no papers about the existence of infinitely many high-energy solutions for problem 1.1. In what follows, we make the following assumption:

(h_1) $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, there exist constants $1 < \delta_1 < \delta_2 < \cdots < \delta_m < 2$ and functions $\xi_i \in L^{\frac{2}{2-\delta_i}}\left(\mathbb{R}^N, \mathbb{R}^+\right)$ $(i = 1, \ldots, m)$ such that

$$|h(x,t)| \leq \sum_{i=1}^{m} \xi_i(x)|t|^{\delta_i - 1}, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

(f_2) There are constants $4 < p < 2^*$, and $c_1 > 0$ such that

$$|f(x,t)| \leq c_1\left(1 + |t|^{p-1}\right),$$

where if $1 < N \leq 4$, let $2^* = +\infty$; if $4 < N < 8$, let $2^* = \frac{2N}{N-4}$.

(f_4) There exist $L > 0$ and $\rho \in \left[0, \frac{V_0}{2}\right]$ such that

$$4F(x,t) - f(x,t)t \leq \rho|t|^2, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall t \geq L.$$

Theorem 1.1. *Assume that (V), (h_1) and $(f_1) - (f_4)$ hold. Then there exists $\bar{\lambda} > 0$ such that for $\lambda \in [-\bar{\lambda}, \bar{\lambda}]$, problem 1.1 has at least one nontrivial solution.*

Theorem 1.2. *Assume that (V), (h_1) and $(f_1) - (f_4)$ hold and*

(f_5) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(h_2) $h(x,-t) = -h(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Then there exists $\bar{\lambda} > 0$ such that for $\lambda \in [-\bar{\lambda}, \bar{\lambda}]$, problem 1.1 has a sequence of solutions (u_n) with

$$\frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx\right)^2 - \lambda \int_{\mathbb{R}^N} h(x,u_n)u_n \, dx - \int_{\mathbb{R}^N} f(x,u_n)u_n \, dx \to \infty, \quad \text{as } n \to \infty.$$

Corollary 1.3. *The conclusion of Theorem 1.2 holds if we replace (f_3) and (f_4) by the following condition:*

(f_3') There exist $r > 0$ and $\psi_0 > 0$ such that

$$l = \inf_{x \in \mathbb{R}^N, |t| = r} F(x,t) > \psi_0.$$

(f_4') There exist $\mu > 4$, and $\psi \in C(\mathbb{R}^N, \mathbb{R}_+^*)$ such that $\sup_{x \in \mathbb{R}^N} \psi(x) \leq \psi_0$, and

$$\mu F(x,t) - f(x,t)t \leq d|t|^2 + \mu\psi(x), \quad \text{for a.e } x \in \mathbb{R}^N \text{ and } \forall |t| \geq r,$$

where $d \in \left[0, \dfrac{(l-\psi_0)(\mu-2)}{r^2}\right)$.

Corollary 1.4. *The conclusion of Theorem 1.2 holds if we replace* (f_3) *and* (f_4) *by the following conditions:*

(f_3'') *There exists* $r_1 > 0$ *such that*

$$l' = \inf_{x\in\mathbb{R}^N, |t|=r_1} F(x,t) > 0.$$

(f_4'') *There exists* $\mu' > 4$ *such that*

$$\mu' F(x,t) - f(x,t)t \le d'|t|^2, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall |t| \ge r_1,$$

where $d' \in \left[0, \dfrac{l'(\mu-2)}{r_1^2}\right)$.

Remark 1.5. *Since problem 1.1 is defined on the whole space* $\mathbb{R}^N$*, it is well known that the main difficulty is the lack of compactness of the Sobolev embedding. To overcome this difficulty, we always assume that the potential* $V(x)$ *satisfies the condition* (V)*, which was introduced by Bartsch et al.* [11].

Remark 1.6. *Obviously, condition* (f_4) *is much weaker than condition* (F_4)*. It is worth pointing out that from* (F_2)*, one sees that* $2 < p < 2^{**} = \frac{2N}{N-2}$ *implies that* $2^{**} \searrow 2$ *as* $N \to \infty$*. On the other hand, the combination of* (f_1)*,* (f_3) *and* (F_4) *implies that*

$$\frac{f(x,t)}{t^3} \ge \frac{4F(x,t)}{t^4} \to \infty, \quad \text{as } |t| \to \infty.$$

In particular, $f(x,t) \ge O\left(t^3\right)$*. This is consistent with* (F_2) *only when* $N \le 6$*. We were able to improve upon this restriction by considering* $4 < N < 8$ *in* (f_2)*.*

The rest of this article is organized as follows. In Section 2, we establish thevariational framework associated with problem 1.1. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

2. Preliminaries

Hereafter, we shall use c_i, C_i, $i = 1, 2, \cdots$ to denote various positive constants which may change from line to line, and by $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence. We denote $L^p(\mathbb{R}^N)$ as a Lebesgue space with the norm $|u|_p := (\int_{\mathbb{R}^N} |u(x)|^p \, \mathrm{d}x)^{\frac{1}{p}}$, $1 \le p < \infty$. Denote $H^2(\mathbb{R}^N)$ as the usual Sobolev space equipped with the inner product and norm,

$$\langle u, v\rangle_{H^2} = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + uv) \, \mathrm{d}x, \quad \|u\|_{H^2} = \langle u, u\rangle_{H^2}^{\frac{1}{2}}.$$

Define our working space

$$E = \{u \in H^2 : \int_{\mathbb{R}^N} \left(\Delta u^2 + |\nabla u|^2 + V(x)u^2\right) \, \mathrm{d}x < \infty\}.$$

with the inner product and norm

$$\langle u,v\rangle = \int_{\mathbb{R}^N} (\Delta u\Delta v + a\nabla u\cdot\nabla v + V(x)uv)\ \mathrm{d}x, \quad \|u\| = \langle u,u\rangle^{\frac{1}{2}}.$$

Since $V(x)$ satisfies (V), it is easy to see that $\|\cdot\|_{H^2}$ is equivalent to $\|\cdot\|$. Then, E is a Hilbert space. Furthermore, E is continuously embedded in $L^s(\mathbb{R}^N)$ for $2\le s\le 2^*$ under the condition (V), that is, there exists $\eta_s>0$ such that

$$\|u\|_s \le \eta_s\|u\| \quad \forall u\in E. \tag{2.1}$$

Lemma 2.1 ([12], Lemma 3.1)**.** *Under the assumption (V), the embedding $E\hookrightarrow L^s$ is compact for any $s\in[2,2^*)$.*

Lemma 2.2. *We say that $u\in E$ is a weak solution of problem 1.1 if*

$$\langle u,\varphi\rangle + b\int_{\mathbb{R}^N}|\nabla u|^2\ \mathrm{d}x\int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi\ \mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\varphi\ \mathrm{d}x + \lambda\int_{\mathbb{R}^N} h(x,u)\varphi\ \mathrm{d}x, \quad \forall\varphi\in E, \tag{2.2}$$

the energy associated with problem 1.1, is functional $J_\lambda : E\to\mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\ \mathrm{d}x\right)^2 - \int_{\mathbb{R}^N}F(x,u)\ \mathrm{d}x - \lambda\int_{\mathbb{R}^N}H(x,u)\ \mathrm{d}x. \tag{2.3}$$

Consequently, seeking a weak solution of problem 1.1 is equivalent to finding a critical point of the functional J_λ. Moreover, $J_\lambda\in C^1(E,\mathbb{R})$ with

$$\begin{aligned}\langle J'_\lambda(u),v\rangle &= \int_{\mathbb{R}^N}(\Delta u\Delta v + a\nabla u\cdot\nabla v + V(x)uv)\ \mathrm{d}x\\ &\quad + b\int_{\mathbb{R}^N}|\nabla u|^2\ \mathrm{d}x\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\ \mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N}f(x,u)v\ \mathrm{d}x - \lambda\int_{\mathbb{R}^N}h(x,u)v\ \mathrm{d}x, \quad \forall u,v\in E.\end{aligned} \tag{2.4}$$

Proof. It follows from (h_1) that

$$|H(x,u)| \le \sum_{i=1}^m \frac{1}{\delta_i}\xi_i(x)|u|^{\delta_i}. \tag{2.5}$$

By (V),(2.5) and the Hölder inequality, for any $u \in E$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |H(x,u)| \, \mathrm{d}x &\leq \int_{\mathbb{R}^N} \sum_{i=1}^{m} \frac{1}{\delta_i} \xi_i(x)|u|^{\delta_i} \, \mathrm{d}x \\
&\leq \sum_{i=1}^{m} \int_{\mathbb{R}^N} \left(\frac{V(x)}{V_0}\right)^{\frac{\delta_i}{2}} \frac{1}{\delta_i} \xi_i(x)|u|^{\delta_i} \, \mathrm{d}x \\
&\leq \sum_{i=1}^{m} \frac{V_0^{\frac{-\delta_i}{2}}}{\delta_i} |\xi_i(x)|_{\frac{2}{2-\delta_i}} \left(\int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x\right)^{\frac{\delta_i}{2}} \\
&\leq C_2 \sum_{i=1}^{m} \|u\|^{\delta_i}.
\end{aligned} \tag{2.6}
$$

By (2.3) and (2.6), J_λ is well defined on E. Now, we show that (2.4) holds. By (h_1), for any $u, v \in E, t \in (0,1)$, $\theta(x) : \mathbb{R}^N \to (0,1)$ and the Hölder inequality we can obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \max_{t \in (0,1)} |h(x, u(x) + t\theta(x)v(x))v(x)| \, \mathrm{d}x \\
=& \int_{\mathbb{R}^N} \max_{t \in (0,1)} |h(x, u(x) + t\theta(x)v(x))||v(x)| \, \mathrm{d}x \\
\leq& \sum_{i=1}^{m} \int_{\mathbb{R}^N} \xi_i(x)(|u(x)|^{\delta_i - 1} + |\theta(x)v(x)|^{\delta_i - 1})|v(x)| \, \mathrm{d}x \\
\leq& V_0^{\frac{-\delta_i}{2}} \sum_{i=1}^{m} \left(\int_{\mathbb{R}^N} |\xi_i|^{\frac{2}{2-\delta_i}} \, \mathrm{d}x\right)^{\frac{2-\delta_i}{2}} \left(\int_{\mathbb{R}^N} V(x)|u(x)|^2 \, \mathrm{d}x\right)^{\frac{\delta_i - 1}{2}} \left(\int_{\mathbb{R}^N} V(x)|v(x)|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&+ V_0^{\frac{-\delta_i}{2}} \sum_{i=1}^{m} \left(\int_{\mathbb{R}^N} |\xi_i|^{\frac{2}{2-\delta_i}} \, \mathrm{d}x\right)^{\frac{2-\delta_i}{2}} \left(\int_{\mathbb{R}^N} V(x)|v(x)|^2 \, \mathrm{d}x\right)^{\frac{\delta_i}{2}} \\
\leq& V_0^{\frac{-\delta_i}{2}} \sum_{i=1}^{m} |\xi_i|_{\frac{2}{2-\delta_i}} \left(\|u\|^{\delta_i - 1} + \|v\|^{\delta_i - 1}\right) \|v\| < +\infty.
\end{aligned} \tag{2.7}
$$

Then by (2.3), (2.7) and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}
\langle J'_\lambda(u),v\rangle &= \lim_{t\to 0^+}\frac{J_\lambda(u+tv)-J_\lambda(u)}{t}\\
&=\lim_{t\to 0^+}\Bigg\{\langle u,v\rangle+\frac{t}{2}\|v\|^2+\frac{b}{4}\Bigg[t^3\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x+4\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,\mathrm{d}x\\
&\quad+2t\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x+4t^2\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,\mathrm{d}x+4t\left(\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,\mathrm{d}x\right)^2\Bigg]\\
&\quad-\frac{1}{t}\int_{\mathbb{R}^N}[F(x,u(x)+tv(x))-F(x,u(x))]\,\mathrm{d}x-\frac{\lambda}{t}\int_{\mathbb{R}^N}[H(x,u(x)+tv(x))-H(x,u(x))]\,\mathrm{d}x\Bigg\}\\
&=\langle u,v\rangle+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)v\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}h(x,u)v\,\mathrm{d}x.
\end{aligned}$$

which implies that (2.4) holds. Moreover, by a standard argument, it is easy to show that $J_\lambda\in C^1(E,\mathbb{R})$. □

Definition 2.3. *We say that J_λ satisfies the Palais-Smale condition at level c $(PS)_c$, i.e., any sequence $\{u_n\}$ has a convergent subsequence in E whenever*

$$J_\lambda(u_n)\to c \quad \text{and} \quad J'_\lambda(u_n)\to 0, \text{ as } n\to\infty. \tag{2.8}$$

Let X be a Banach space with the norm $\|\cdot\|$ and $X=\overline{\bigoplus_{i=1}^{\infty}X_i}$ with $\dim X_i<+\infty$ for each $i\in\mathbb{N}$. Further, we set

$$Y_k=\bigoplus_{i=1}^{k}X_i,\quad Z_k=\overline{\bigoplus_{i=k}^{\infty}X_i}.$$

To prove Theorem 1.1 we state the following mountain pass theorem (see [[14] Theorem 1.17]).

Theorem 2.4 (Mountain Pass Theorem)**.** *Let X be a Banach space, $I\in C^1(X,\mathbb{R})$, $I(0)=0$, and assume that*

(S_1) there exist two positive real numbers α and ρ such that $I(u)\geq\alpha$ for all $\|u\|=\rho$,
(S_2) there exists $e\in X$ with $\|e\|>\rho$ such that $I(e)\leq 0$,

If I satisfies the $(PS)_c$-condition for

$$c=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}I(\gamma(t)),$$

with

$$\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0,\gamma(1)=e\},$$

then c is critical value of I and $c\geq\alpha$.

In order to deduce our results, the following Fountain theorem is a very useful tool.

Theorem 2.5 (Fountain theorem, Bartsch [14]). *Let $I \in C^1(X,\mathbb{R})$ satisfy $I(-u) = I(u)$. Assume that, for every $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that*

(A_1) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} I(u) \leq 0$,

(A_2) $b_k := \inf_{u\in Z_k, \|u\|=\gamma_k} I(u) \to +\infty$ *as* $k \to +\infty$.

If I satisfies the $(PS)_c$ condition for every $c > 0$, then I has an unbounded sequence of critical values.

3. Proof of Theorems 1.1

We begin verifying the follow compactness lemma which shows that the functional J_λ satisfies (PS)-condition

Lemma 3.1. *Let assumptions (V), (h_1) and $(f_1)-(f_4)$ hold. Then for every $\lambda \in \mathbb{R}$, any Palais-Smale sequence of J_λ is bounded.*

Proof. Let $\{u_n\} \subset E$ be any Palais-Smale sequence of J_λ. Then, up to a subsequence, there exists $c \in \mathbb{R}$ such that

$$J_\lambda(u_n) \to c, \text{ and } J'_\lambda(u_n) \to 0.$$

For n large enough, by (f_3) we have

$$\begin{aligned}
c+1+\|u\| \geq & J_\lambda(u_n)-\frac{1}{4}\left\langle J_\lambda'(u_n),u_n\right\rangle \\
= & \frac{1}{4}\int_{\mathbb{R}^N}\left(|\Delta u_n|^2+a|\nabla u_n|^2\right)\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x+\int_{\mathbb{R}^N}\widetilde{F}(x,u_n)\,\mathrm{d}x \\
& -\lambda\int_{\mathbb{R}^N}\widetilde{H}(x,u_n)\,\mathrm{d}x \\
\geq & \frac{1}{4}\int_{\mathbb{R}^N}\left(|\Delta u_n|^2+a|\nabla u_n|^2\right)\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x-\frac{\rho}{4}\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x \\
& +\int_{\mathbb{A}_n}\widetilde{F}(x,u_n)\,\mathrm{d}x-|\lambda|\int_{\mathbb{R}^N}\widetilde{H}(x,u_n)\,\mathrm{d}x \\
\geq & \frac{1}{4}\int_{\mathbb{R}^N}\left(|\Delta u_n|^2+a|\nabla u_n|^2\right)\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x-\frac{1}{8}\int_{\mathbb{R}^N}V_0u_n^2\,\mathrm{d}x \\
& +\int_{\mathbb{A}_n}\widetilde{F}(x,u_n)\,\mathrm{d}x-|\lambda|\int_{\mathbb{R}^N}\widetilde{H}(x,u_n)\,\mathrm{d}x \\
\geq & \frac{1}{4}\int_{\mathbb{R}^N}\left(|\Delta u_n|^2+a|\nabla u_n|^2\right)\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x-\frac{1}{8}\int_{\mathbb{R}^N}V_0u_n^2\,\mathrm{d}x \\
& +\int_{\mathbb{A}_n}\widetilde{F}(x,u_n)\,\mathrm{d}x-|\lambda|\int_{\mathbb{R}^N}\widetilde{H}(x,u_n)\,\mathrm{d}x \\
\geq & \frac{1}{16}\|u_n\|^2+\frac{1}{16}\int_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x+\int_{\mathbb{A}_n}\widetilde{F}(x,u_n)\,\mathrm{d}x-|\lambda|\int_{\mathbb{R}^N}\widetilde{H}(x,u_n)\,\mathrm{d}x,
\end{aligned}$$

where $\widetilde{F}(x,u_n)=\frac{1}{4}f(x,u_n)u_n-F(x,u_n)$, $\widetilde{H}(x,u_n)=H(x,u_n)-\frac{1}{4}h(x,u_n)u_n$ and $\mathbb{A}_n=\{x\in\mathbb{R}^N:|u_n|\leq L\}$.

By (h_1), (2.1) and Hölder's inequality we can obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}\widetilde{H}(x,u_n)\right| = & \left|\int_{\mathbb{R}^N}\left(H(x,u_n)-\frac{1}{4}h(x,u_n)u_n\right)\right| \\
\leq & \int_{\mathbb{R}^N}|H(x,u_n)|+|\frac{1}{4}h(x,u_n)u_n| \\
\leq & \sum_{i=1}^{m}\left(\frac{1}{\delta_i}+\frac{1}{4}\right)|\xi_i(x)|_{\frac{2}{2-\delta_i}}\eta_2^{\delta_i}\|u_n\|^{\delta_i},
\end{aligned}\tag{3.1}$$

Hence

(3.2)

$$c+1+\|u_n\|+|\lambda|\sum_{i=1}^{m}\left(\frac{1}{\delta_i}+\frac{1}{4}\right)|\xi_i(x)|_{\frac{2}{2-\delta_i}}\eta_2^{\delta_i}\|u_n\|^{\delta_i}\geq\frac{1}{16}\|u_n\|^2+\frac{1}{16}\int\limits_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x+\int\limits_{\mathbb{A}^N}\widetilde{F}(x,u_n)\,\mathrm{d}x.$$

For any $\varepsilon>0$, by (f_1), (f_2), there exists $C(\varepsilon)>0$ such that

$$\begin{aligned}&|f(x,t)|\leq 2\varepsilon|t|+pC(\varepsilon)|t|^{p-1}, && \forall(x,t)\in\mathbb{R}^N\times\mathbb{R},\\ &|F(x,t)|\leq\int_0^1|f(x,st)t|ds\leq\varepsilon|t|^2+C(\varepsilon)|t|^p, && \forall(x,t)\in\mathbb{R}^N\times\mathbb{R}.\end{aligned}\tag{3.3}$$

For $x\in\mathbb{R}^N$ and $|u_n|\leq L$, by (3.3), we have

$$\begin{aligned}\left|\widetilde{F}(x,u_n)\right|\leq&\frac{1}{4}|f(x,u_n)||u_n|+|F(x,u_n)|\\ \leq&\frac{5}{4}\varepsilon|u_n|^2+\frac{5}{4}C(\varepsilon)|u_n|^p\\ =&\frac{5}{4}\left[\varepsilon+C(\varepsilon)|u_n|^{p-2}\right]|u_n|^2\\ \leq&\frac{5}{4}\left[\varepsilon+C(\varepsilon)L^{p-2}\right]|u_n|^2\\ \leq&C_3|u_n|^2,\end{aligned}$$

Take $M>\max\{16C_3,V_0\}$, then

$$\widetilde{F}(x,u_n)\geq-\frac{M}{16}|u_n|^2,\quad\forall x\in\mathbb{R}^N,|u_n|\leq L.\tag{3.4}$$

Let $\widetilde{\mathbb{A}}=\left\{x\in\mathbb{R}^N:V(x)\leq M\right\}$. By (V) we know that $\mathrm{meas}(\widetilde{\mathbb{A}})<+\infty$. On the other hand, it follows from (3.4) that

$$\begin{aligned}\frac{1}{16}\int\limits_{\mathbb{R}^N}V(x)u_n^2\,\mathrm{d}x+\int\limits_{\mathbb{A}_n}\widetilde{F}(x,u_n)\,\mathrm{d}x\geq&\frac{1}{16}\int\limits_{|u_n|\leq L}(V(x)-M)|u_n|^2\,\mathrm{d}x\\ \geq&\frac{1}{16}\int\limits_{\widetilde{\mathbb{A}}\cap\mathbb{A}_n}(V(x)-M)L^2\,\mathrm{d}x\\ \geq&\frac{1}{16}(V_0-M)L^2\,\mathrm{meas}(\widetilde{\mathbb{A}}\cap\mathbb{A}_n)\\ \geq&\frac{1}{16}(V_0-M)L^2\,\mathrm{meas}(\widetilde{\mathbb{A}}).\end{aligned}\tag{3.5}$$

Combining (3.2) and (3.5), we get that

$$c+1+\|u_n\|+|\lambda|\sum_{i=1}^{m}\left(\frac{1}{\delta_i}+\frac{1}{4}\right)|\xi_i(x)|_{\frac{2}{2-\delta_i}}\eta_2^{\delta_i}\|u_n\|^{\delta_i}\geq\frac{1}{16}\|u_n\|^2+\frac{1}{16}(V_0-M)L^2\,\mathrm{meas}(\widetilde{\mathbb{A}}),$$

which implies that $\{u_n\}$ is bounded in E. Thus, this completes the proof. □

Lemma 3.2. *Let assumptions (V), (h_1) and $(f_1)-(f_4)$ hold and $\{u_n\}$ is a bounded Palais-Smale sequence of J_λ, then for every $\lambda \in \mathbb{R}$, $\{u_n\}$ has a strongly convergent subsequence in E.*

Proof. Since $\{u_n\}$ is bounded in E, then there exists a constant $M > 0$ such that

$$\|u_n\| \leq M, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Going if necessary to a subsequence, we may assume that there is a $u \in E$ such that

$$\begin{aligned} &u_n \rightharpoonup \quad \text{in } E, \\ &u_n \to u \quad \text{in } L^s\left(\mathbb{R}^N\right)(2 \leq s < 2^*), \\ &u_n \to u \quad \text{a.e. on } \mathbb{R}^N. \end{aligned} \tag{3.7}$$

By (2.4), we have

$$\begin{aligned}
&\left\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u \right\rangle \\
&= \int_{\mathbb{R}^N} (\Delta u_n - \Delta u)^2 + V(x)|u_n - u|^2 \,\mathrm{d}x + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x\right) \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla (u_n - u) \,\mathrm{d}x \\
&- \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x\right) \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u_n - u) \,\mathrm{d}x - \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u) \,\mathrm{d}x \\
&- \lambda \int_{\mathbb{R}^N} (h(x,u_n) - h(x,u))(u_n - u) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} (\Delta u_n - \Delta u)^2 + V(x)|u_n - u|^2 \,\mathrm{d}x + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x\right) \int_{\mathbb{R}^N} |\nabla (u_n - u)^2 \,\mathrm{d}x \\
&- \left(\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x\right) \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u_n - u) \,\mathrm{d}x - \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u) \,\mathrm{d}x \\
&- \lambda \int_{\mathbb{R}^N} (h(x,u_n) - h(x,u))(u_n - u) \,\mathrm{d}x \\
&\geq \|u_n - u\|^2 - b\left(\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x\right) - \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u_n - u) \,\mathrm{d}x \\
&- \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u) \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} (h(x,u_n) - h(x,u))(u_n - u) \,\mathrm{d}x.
\end{aligned}$$

Therefore, one has

$$\|u_n-u\|^2 \le \langle J'_\lambda(u_n)-J'_\lambda(u),u_n-u\rangle + b\left(\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x-\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x\right)\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)\,\mathrm{d}x+\lambda\int_{\mathbb{R}^N}(h(x,u_n)-h(x,u))(u_n-u)\,\mathrm{d}x.$$

Then, it follows from (3.3), (3.6), and the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)||u_n-u|\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^N}(|f(x,u_n)|+|f(x,u)|)|u_n-u|\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^N}2\varepsilon(|u_n|+|u|)|u_n-u|\,\mathrm{d}x+pC(\varepsilon)\int_{\mathbb{R}^N}\left(|u_n|^{p-1}+|u|^{p-1}\right)|u_n-u|\,\mathrm{d}x\\
&\le 2\varepsilon\left[\left(\int_{\mathbb{R}^N}|u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\right]\left(\int_{\mathbb{R}^N}|u_n-u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&+pC(\varepsilon)\left[\left(\int_{\mathbb{R}^N}|u_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}+\left(\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}\right]\left(\int_{\mathbb{R}^N}|u_n-u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le 2\varepsilon(\eta_2M+|u|_2)\|u_n-u\|_2+pC(\varepsilon)(\eta_p^{p-1}M^{p-1}+|u|_p^{p-1})\|u_n-u\|_p\to 0, n\to+\infty.
\end{aligned}\tag{3.8}$$

Let $\phi_u : E \to \mathbb{R}$ such that $\phi_u(v)=\int_{\mathbb{R}^3}\nabla u\cdot\nabla v\,\mathrm{d}x$. Since $\phi_u(v)\le\|u\|\|v\|$, we can deduce that ϕ_u is continuous (linear and bounded) on E, using (3.7), then we have

$$\int_{\mathbb{R}^N}\nabla u\cdot\nabla u_n\,\mathrm{d}x=\phi_u(u_n)\to\phi_u(u)=\int_{\mathbb{R}^N}\nabla u\cdot\nabla u\,\mathrm{d}x,\quad \text{as } n\to\infty. \tag{3.9}$$

Thus, we get from $u_n \rightharpoonup u$ in H, (3.9), and the boundedness of $\{u_n\}$ that

$$b\left(\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x-\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x\right)\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)\,\mathrm{d}x\to 0,\ \text{as } n\to\infty. \tag{3.10}$$

By (h_1), (2.1), and (3.6), using the Hölder inequality, we can conclude

$$\begin{aligned}
&\int_{\mathbb{R}^N} |h(x,u_n)-h(x,u)||u_n-u|\mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} |h(x,u_n)||u_n-u|\mathrm{d}x + \int_{\mathbb{R}^N} |h(x,u)||u_n-u|\mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \sum_{i=1}^m \xi_i(x)|u_n|^{\delta_i-1}|u_n-u|\mathrm{d}x + \int_{\mathbb{R}^N} \sum_{i=1}^m \xi_i(x)|u|^{\delta_i-1}|u_n-u|\mathrm{d}x \\
&\leq \sum_{i=1}^m |\xi_i|_{\frac{2}{2-\delta_i}}(|u_n|_2^{\delta_i-1}+|u|_2^{\delta_i-1})|u_n-u|_2 \\
&\leq \sum_{i=1}^m |\xi_i|_{\frac{2}{2-\delta_i}}(\eta_2^{\delta_i-1}M^{\delta_i-1}+|u|_2^{\delta_i-1})|u_n-u|_2.
\end{aligned}$$

Therefore, it follows from (3.7) that

$$\int_{\mathbb{R}^N} h(x,u_n)-h(x,u)(u_n-u)\mathrm{d}x \to 0, \quad \text{as } n\to\infty. \tag{3.11}$$

Clearly,

$$\left\langle J_\lambda'(u_n)-J_\lambda'(u), u_n-u\right\rangle \to 0, \text{ as } n\to\infty. \tag{3.12}$$

It follows from (3.8), (3.10), (3.11) and (3.12) that $\|u_n-u\|\to 0$. □

Lemma 3.3. *If (V), (h_1), $(f_1)-(f_3)$ hold, then there exist α, ρ and $\bar{\lambda}>0$ such that $J_\lambda(u)\geq\alpha$ whenever $\|u\|=\rho$ and $\lambda\in[-\bar{\lambda},\bar{\lambda}]$.*

Proof. (h_1) together with (2.1) and Hölder's inequality imply that

$$\begin{aligned}
\int_{\mathbb{R}^N} |H(x,u)| &\leq \sum_{i=1}^m \int_{\mathbb{R}^N} \sum_{i=1}^m \frac{1}{\delta_i}|\xi_i(x)||u|^{\delta_i} \\
&= \sum_{i=1}^m \frac{V_0^{\frac{-\delta_i}{2}}}{\delta_i}|\xi_i(x)|_{\frac{2}{2-\delta_i}}\|u\|^{\delta_i} \\
&\leq C_4\|u\|^{\delta_m},
\end{aligned} \tag{3.13}$$

for $\|u\|$ large enough. It follows from (3.3) and (3.13) that

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} F(x,u) - \lambda \int_{\mathbb{R}^N} H(x,u) \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u) - |\lambda| \int_{\mathbb{R}^N} H(x,u) \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left(\varepsilon|u|^2 + C(\varepsilon)|u|^p\right) - |\lambda| C_4 \|u\|^{\delta_m} \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon\eta_2^2\|u\|^2 - C(\varepsilon)\eta_p^p\|u\|^p - |\lambda| C_4\|u\|^{\delta_m} \\ &= \|u\|^2\left(\frac{1}{2} - \varepsilon\eta_2^2 - C(\varepsilon)\eta_p^p\|u\|^{p-2} - |\lambda|C_4\|u\|^{\delta_m - 2}\right). \end{aligned}$$

For $\varepsilon \leq \frac{1}{4\eta_2^2}$, we have

$$J_\lambda(u) \geq \|u\|^2\left(\frac{1}{4} - C(\varepsilon)\eta_p^p\|u\|^{p-2} - |\lambda|C_4\|u\|^{\delta_m-2}\right).$$

Let

$$g(t) = C_5 t^{p-2} + |\lambda| C_4 t^{\delta_m - 2}, \quad t \geq 0, \tag{3.14}$$

and then we get $\lim_{t\to+\infty} g(t) = \lim_{t\to 0^+} g(t) = +\infty$, which implies that $g(t)$ is bounded below, thus $g(t)$ admits a minimizer t_0 :

$$t_0 = \left(\frac{|\lambda|C_4(\delta_m - 2)}{C_5(2-p)}\right)^{\frac{1}{p-\delta_m}}.$$

It follows from (3.14) that

$$\begin{aligned} \inf_{t\in[0,+\infty)} g(t) &= g(t_0) \\ &= C_5\left(\frac{|\lambda|C_4(\delta_m-2)}{C_5(2-p)}\right)^{\frac{p-2}{p-\delta_m}} + |\lambda|C_4\left(\frac{|\lambda|C_4(\delta_m-2)}{C_5(2-p)}\right)^{\frac{\delta_m-2}{p-\delta_m}} \\ &= |\lambda|^{\frac{p-2}{p-\delta_m}} G, \end{aligned}$$

with $G = C_5\left(\frac{C_4(\delta_m-2)}{C_5(2-p)}\right)^{\frac{p-2}{p-\delta_m}} + C_4\left(\frac{C_4(\delta_m-2)}{C_5(2-p)}\right)^{\frac{\delta_m-2}{p-\delta_m}}$.

Then for $|\lambda| \leq \left(\frac{1}{4G}\right)^{\frac{p-q_m}{p-2}} := \bar{\lambda}$, we have

$$g(t_0) < \frac{1}{4},$$

thus, whenever $\lambda \in [-\bar{\lambda}, \bar{\lambda}]$, there exists $\rho = t_0 > 0$ such that $\|u\| = \rho$ and $J_\lambda(u) \geq \alpha > 0 = J_\lambda(0)$. □

Lemma 3.4. *Assume that (h_1), $(f_1)-(f_3)$ hold, then there exists $e \in E$, such that $J_\lambda(e) < 0$ with $\|e\| > \rho$.*

Proof. For every $M > 0$, by $(f_1)-(f_3)$, there exists $C(M) > 0$ such that

$$F(x,t) \geq M|t|^4 - C(M)|t|^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.15}$$

Choose $\phi \in E$ with $|\phi|_4 = 1$, then for $t > 0$, we have

$$\begin{aligned} J_\lambda(t\phi) =& \frac{t^2}{2}\|\phi\|^2 + \frac{t^4 b}{4}\left(\int_{\mathbb{R}^N} |\nabla\phi|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,t\phi) - \lambda \int_{\mathbb{R}^N} H(x,t\phi) \\ \leq& \frac{t^2}{2}\|\phi\|^2 + \frac{t^4 b}{4}\|\phi\|^4 - Mt^4|\phi|_4^4 + t^2 C(M)|\phi|_2^2 + t^{\delta_m}|\lambda| C_4 \|\phi\|^{\delta_m}. \end{aligned}$$

which implies $J_\lambda(t\phi) \to -\infty$ as $t \to +\infty$ by taking $M > \frac{b}{4}\|\phi\|^4$. So, there exists $e = t_0\phi$ such that $\|e\| > \rho$ and $J_\lambda(e) < 0 = J_\lambda(0)$. □

Proof of Theorems 1.1. We have $J_\lambda \in C^1(E,\mathbb{R})$ and $J_\lambda(0) = 0$. On the other hand, condition (S_1) is satisfied whenever $|\lambda| \leq \bar{\lambda}$ due to Lemma 3.3, and the functional J_λ satisfies the conditions (S_2) due to Lemma 3.4. Moreover, by Lemmas 3.1, 3.2 J_λ satisfies the $(PS)_c$ condition. Therefore, the functional J_λ has at least one nontrivial solution $u \in E$, whenever $|\lambda| \leq \bar{\lambda}$. □

4. Proof of Theorems 1.2

In this section, we prove the existence of sequence of solutions with high energy to problem 1.1.

Since $E \hookrightarrow L^2$, and L^2 is a separable Hilbert space, E has a countable orthogonal basis $\{e_i\}_{i=1}^{\infty}$. Set

$$E_i = span\{e_i\}, \quad Y_k = \bigoplus_{i=1}^{k} E_i, \quad Z_k = \overline{\bigoplus_{i=k+1}^{\infty} E_i}, \quad k \in \mathbb{N}^*.$$

Then, $E = \overline{\bigoplus_{i=1}^{\infty} E_i}$ and Y_k is finite dimensional.

Lemma 4.1 ([14], Lemma 3.8)**.** *If $2 \leq s < 2^*$ then we have that*

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} |u|_s \to 0, \quad as\ k \to \infty.$$

Proof. It is clear that $0 < \beta_{k+1} \leqslant \beta_k$, so $\beta_k \to \beta \geqslant 0 (k \to \infty)$. For every $k \in \mathbb{N}$ (by the definition of β_k), there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and

$$|u_k|_s > \frac{\beta}{2} > 0. \tag{4.1}$$

For any $v=\sum_{i=1}^{\infty} v_i e_i$, we have, by the Cauchy-Schwartz inequality,

$$\begin{aligned}|\langle u_k, v\rangle| &= \left|\left\langle u_k, \sum_{i=1}^{\infty} v_i e_i\right\rangle\right| = \left|\left\langle u_k, \sum_{i=k+1}^{\infty} v_i e_i\right\rangle\right| \\ &\leqslant \|u_k\| \left\|\sum_{i=k+1}^{\infty} v_i e_i\right\| \\ &= \left(\sum_{i=k+1}^{\infty} v_i^2\right)^{1/2} \to 0,\end{aligned}$$

as $k\to\infty$, which implies that $u_k \rightharpoonup 0$ in E. The compact embedding of $E \hookrightarrow L^s\left(\mathbb{R}^N\right)$ $(2\leqslant s<2^*)$ implies that

$$u_k \to 0 \quad \text{in } L^s\left(\mathbb{R}^N\right)(2\leqslant s<2^*).$$

Hence, letting $k\to\infty$ in (4.1), we get $\beta=0$, which completes the proof. □

Lemma 4.2. *Assume that (V), (h_1), (f_1) and (f_2) hold, then there exist $\bar\lambda>0$ and $\gamma_k>0$ such that*

$$\inf_{u\in Z_k, \|u\|=\gamma_k} J_\lambda(u) \to +\infty \text{ as } k\to\infty,$$

whenever $\lambda\in[-\bar\lambda,\bar\lambda]$.

Proof. Lemma 4.1 implies that

$$|u|_s \le \beta_k \|u\|, \quad 1\le s<2^*. \tag{4.2}$$

Thus, by (3.13), (3.3), Lemma 3.1 and (4.2) we have,

$$\begin{aligned}J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} F(x,u) - \lambda\int_{\mathbb{R}^N} H(x,u) \\ &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u) - \lambda\int_{\mathbb{R}^N} H(x,u) \\ &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left(\varepsilon|u|^2 + C(\varepsilon)|u|^p\right) - |\lambda| C_4 \|u\|^{\delta_m} \\ &\ge \frac{1}{2}\|u\|^2 - \varepsilon\eta_2^2\|u\|^2 - C(\varepsilon)\beta_k^p\|u\|^p - |\lambda| C_4\|u\|^{\delta_m} \\ &= \|u\|^2\left(\frac{1}{2} - \varepsilon\eta_2^2 - C(\varepsilon)\beta_k^p\|u\|^{p-2} - |\lambda| C_4\|u\|^{\delta_m-2}\right).\end{aligned}$$

Taking $\varepsilon\le\frac{1}{4\eta_2^2}$, we have

$$J_\lambda(u) \ge \|u\|^2\left(\frac{1}{4} - C(\varepsilon)\beta_k^p\|u\|^{p-2} - |\lambda| C_4\|u\|^{\delta_m-2}\right).$$

Choose

$$\gamma_k = \left(\frac{|\lambda| C_4 (\delta_m - 2)}{C(\varepsilon) \beta_k^p (2-p)} \right)^{\frac{1}{p-\delta_m}}.$$

such that

$$b_k := \inf_{u \in Z_k, \|u\| = \gamma_k} J_\lambda(u) \geq \frac{1}{8} \gamma_k^2.$$

Since $\beta_k \to 0$, $1 < \delta_m < 2$ and $4 < p < 2^*$ thus

$$b_k \to +\infty, \quad \text{as } k \to +\infty.$$

□

Lemma 4.3. *Assume that (V), (h_1), and $(f_1) - (f_3)$ hold, then for any finite dimensional subspace $Y_k \subset E$, there holds*

$$\max_{u \in Y_k, \|u\| = \rho_k} J_\lambda(u) \leq 0.$$

Proof. Let Y_k be any finite dimensional subspace of E, by (3.13) and (3.15) we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^N} F(x,u) - \lambda \int_{\mathbb{R}^N} H(x,u) \\ &\leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M \int_{\mathbb{R}^N} |u|^4 + C(M) \int_{\mathbb{R}^N} |u|^2 + |\lambda| \int_{\mathbb{R}^N} H(x,u) \\ &\leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M|u|_4^4 + C(M)\eta_2^2 \|u\|^2 + |\lambda| C_4 \|u\|^{\delta_m}. \end{aligned}$$

Since on the finite dimensional space Y_k all norms are equivalent, so we can choose a constant $c_2 > 0$ such that

$$|u|_4 \geq c_2 \|u\|, \quad \forall u \in Y_k.$$

Therefore, one has

$$J_\lambda(u) \leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M c_2^4 \|u\|^4 + C(M)\eta_2^2 \|u\|^2 + |\lambda| C_4 \|u\|^{\delta_m}.$$

Hence, choosing $M > \frac{b}{4c_2^4}$, we conclude that there exists $\rho_k > \gamma_k > 0$ such that

$$\max_{u \in Y_k, \|u\| = \rho_k} J_\lambda(u) \leq 0.$$

□

Proof of Theorems 1.2. Evidently, the functional J_λ defined in (2.3) is an even functional in view of (f_5) and (h_2) with $J_\lambda(0) = 0$. Besides, Lemma 2.2 shows that $J_\lambda \in C^1(E, \mathbb{R})$ and satisfies conditions (A_1) and (A_2) in Theorem 2.5. Thus, by Theorem 2.5, we get a sequence of nontrivial critical points $\{u_k\} \subset E$ of J_λ satisfying $J_\lambda(u_k) \to 0$ as $k \to \infty$ and $u_k \to 0$ in E as $k \to \infty$. whenever $|\lambda| \leq \bar{\lambda}$, that is, problem 1.1 possesses infinitely many solutions. This ends the proof. □

Proof of Corollary 1.3. It is sufficient to show that $(f_3^{'})$, $(f_4^{'})$ imply (f_3), (f_4). Indeed, For any $(x,z) \in \mathbb{R}^N \times \mathbb{R}$, set

$$\tau(t) := F\left(x, t^{-1}z\right) t^{\mu}, \quad \forall t \in \left[1, \frac{|z|}{r}\right].$$

By (f_4'), for $|z| \geq r$, one has

$$\tau'(t) = f\left(x, \frac{z}{t}\right)\left(-\frac{z}{t^2}\right) t^{\mu} + \mu F\left(x, \frac{z}{t}\right) t^{\mu-1},$$
$$t^{\mu-1}\left[\mu F\left(x, \frac{z}{t}\right) - f\left(x, \frac{z}{t}\right)\frac{z}{t}\right] \leq d t^{\mu-3}|z|^2 + \mu|\psi(x)| t^{\mu-1}.$$

Thus,

$$\tau\left(\frac{|z|}{r}\right) - \tau(1) = \int_1^{\frac{|z|}{r}} \tau'(t)dt \leq \frac{d|z|^{\mu}}{(\mu-2)r^{\mu-2}} - \frac{d|z|^2}{\mu-2} + \frac{|z|^{\mu}|\psi(x)|}{r^{\mu}} - |\psi(x)|.$$

Hence, for any $x \in \mathbb{R}^N$ and $|z| \geq r$, by (f_3'), one has

$$\begin{aligned} F(x,z) = \tau(1) &\geq \tau\left(\frac{|z|}{r}\right) + \frac{d|z|^2}{\mu-2} - \frac{d|z|^{\mu}}{(\mu-2)r^{\mu-2}} - \frac{|z|^{\mu}|\psi(x)|}{r^{\mu}} + |\psi(x)| \\ &\geq \inf_{x\in\mathbb{R}^N, |t|=r} F(x,t)\left(\frac{|z|}{r}\right)^{\mu} - \frac{d|z|^{\mu}}{(\mu-2)r^{\mu-2}} - \frac{|z|^{\mu}|\psi(x)|}{r^{\mu}} \\ &\geq C_7|z|^{\mu}, \end{aligned} \tag{4.3}$$

where $C_7 = \frac{l}{r^{\mu}} - \frac{d}{(\mu-2)r^{\mu-2}} - \frac{\psi_0}{r^{\mu}}$, $C_7 > 0$ in view of $d \in \left[0, \frac{(l-\psi_0)(\mu-2)}{r^2}\right)$.
We obtain from (4.3) that

$$\frac{F(x,z)}{z^4} \geq C_7|z|^{\mu-4}, \quad \forall x \in \mathbb{R}^N \text{ and } |z| \geq r. \tag{4.4}$$

Noticing that $\mu > 4$, then (4.4) implies (f_3). Furthermore, it follows from (4.4) and (f_4') that

$$4F(x,z) - f(x,z)z = \mu F(x,z) - f(x,z)z + (4-\mu)F(x,z) \leq d|z|^2 + \mu\psi_0 - (\mu-4)C_4|z|^{\mu}$$

for all $x \in \mathbb{R}^N$ and $|z| \geq r$. This, together with $\mu > 4$, shows there exists $L > 0$ such that

$$4F(x,z) - f(x,z)z < 0 \quad \forall x \in \mathbb{R}^N \text{ and } |z| \geq L,$$

which implies (f_4). □

Proof of Corollary 1.4. The proof of this Corollary is almost the same to the one of Corollary 1.3. So we omit it here. □

Corollary 4.4. *The conclusion of Corollary 1.4 holds if we replace $(f_4^{''})$ by the following condition:*

$(f_4^{'''})$ *$t \to f(x,t)/|t|^{\mu-1}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$.*

Proof. In fact, if $t > 0$, from $(f_4^{'''})$ we have

$$F(x,t) = \int_0^1 f(x,st)t \, \mathrm{d}s = \int_0^1 \frac{f(x,st)}{(st)^{\mu-1}} t^{\mu} s^{\mu-1} \, \mathrm{d}s \leq \int_0^1 \frac{f(x,t)}{t^{\mu-1}} t^{\mu} s^{\mu-1} \, \mathrm{d}s \leq \frac{1}{\mu} f(x,t)t + \frac{d'}{\mu}|t|^2. \tag{4.5}$$

Otherwise, if $t < 0$, then

$$
\begin{aligned}
F(x,t) = \int_0^1 f(x,st)t \, \mathrm{d}s &= -\int_0^1 \frac{f(x,st)}{(-st)^{\mu-1}}(-t)^{\mu} s^{\mu-1} \, \mathrm{d}s \\
&= -\int_0^1 \frac{f(x,st)}{|st|^{\mu-1}}|t|^{\mu} s^{\mu-1} \, \mathrm{d}s \\
&\leq -\int_0^1 \frac{f(x,t)}{|t|^{\mu-1}}|t|^{\mu} s^{\mu-1} \, \mathrm{d}s \\
&\leq \frac{1}{\mu} f(x,t)t + \frac{d'}{\mu}|t|^2.
\end{aligned} \tag{4.6}
$$

Therefore, (4.5) and (4.6) show that (f_4'') holds. □

Corollary 4.5. *If the following condition (f_3''') is used in palace of (f_3'') of Corollary 1.4.*

(f_3''') *There exist $4 < \alpha < 2^*$ such that*

$$\liminf_{|u|\to\infty} \frac{F(x,u)}{|u|^{\alpha}} > 0, \text{ uniformly for } x \in \mathbb{R}^N,$$

then Corollary 1.4 remains true.

Proof. We only need to prove (f_3''). Indeed, by (f_3'''), we can take a $\omega \in \left(0, \liminf_{|u|\to\infty} \frac{F(x,u)}{|u|^{\alpha}}\right)$ small enough such that

$$F(x,u) \geq \omega |u|^{\alpha}, \quad \text{for } |u| \text{ large enough.}$$

then though the above inequality, we know that (f_3''') implies (f_3''). It means that Corollary 1.4 generalizes Corollary 4.5. This proof ends. □

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