Orthogonality-preserving lowering differential operator for Laguerre and Jacobi polynomials

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Abstract

In this paper, we describe all Λ_c-classical orthogonal polynomials, where Λ_c, $c \in \mathbb{C}$ is a fourth-order lowering differential operator. The solutions are the Laguerre $\{L_n^{(3)}\}_{n\geq 0}$ and the Jacobi $\{P_n^{(\alpha-4,3)}\}_{n\geq 0}$. As an illustration, some connection formulas between the polynomial solutions are deduced.

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1 Introduction

An orthogonal polynomial sequence $\{P_n(x)\}_{n\geq 0}$ is called classical if $\{DP_n(x)\}_{n\geq 0}$, where $D := \frac{d}{dx}$ is the standard derivative, is also orthogonal (Hermite, Laguerre, Bessel or Jacobi). This is Sonine-Hahn property [18, 24]. In [25], Hahn gave similar characterization theorems for orthogonal polynomials P_n such that the polynomials ΔP_n or $D_q P_n$, $(n \geq 1)$ are again orthogonal. Here ΔP_n is the difference operator and $D_q P_n$ is the q-difference operator.

In a more general setting, let $\mathscr{O}$ be a linear operator acting on the space $\mathscr{P}$ of polynomials in one variable which sends polynomials of degree n to polynomials of degree $n + n_0$ (n_0 is a fixed integer). We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathscr{O}$-classical if there exist a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials such that $\mathscr{O}P_n = Q_{n+n_0}$ (where $n \geq 0$ if $n_0 \geq 0$ and $n \geq n_0$ if $n_0 < 0$). The concept of $\mathscr{O}$-classical orthogonal polynomials has been studied by many authors in the literature, we can see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 9, 15, 26]

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In this paper, we describe all Λ_c-classical orthogonal polynomial sequences where $\Lambda_c,\ c \in \mathbb{C}$ is the following differential operator generalizing the Laguerre derivative

$$\begin{array}{rcl} \Lambda_c : \mathscr{P} & \rightarrow & \mathscr{P} \\ f & \mapsto & \left(\hat{\Omega}_3(x-c)\hat{\Omega}_3 + 2\hat{\Omega}_3\right) f(x), \end{array}$$

where $\hat{\Omega}_m$ is introduced by Dattoli et Ricci (see[21])

$$\hat{\Omega}_m := \frac{d}{dx} x \frac{d}{dx} + m \frac{d}{dx},\ m \neq -n,\ n \in \mathbb{N}.$$

The basic idea has been deduced by starting from the Laguerre derivative given in [21]. On the other hand, by using [28], we can easily prove

$$\left(\hat{\Omega}_3 x \hat{\Omega}_3 + 2\hat{\Omega}_3\right) L^{(3)}_{n+1}(x) = \Theta_n L^{(1)}_n(x),\ n \geq 1,$$

where $\Theta_n := (n+1)(n+2)(n+3)(n+4)$, is the normalisation factor and $\{L^{(\alpha)}_n(x)\}_{n\geq 0},\ \alpha \neq -n,\ n \geq 1$, are the monic orthogonal Laguerre polynomial sequences.

This means that the above family of standard orthogonal polynomials is an Λ-classical polynomial sequence, since it satisfies the Hahn's property for the lowering operator $\Lambda := \hat{\Omega}_3 x \hat{\Omega}_3 + 2\hat{\Omega}_3$.

From a more general point of view, for a given $c \in \mathbb{C}$, let $\Lambda_c : \mathscr{P} \to \mathscr{P}$ be the linear operator defined by

$$\Lambda_c := \hat{\Omega}_3(x-c)\hat{\Omega}_3 + 2\hat{\Omega}_3, \quad (\Lambda_0 = \Lambda).$$

The purpose of this paper is to introduce the concept of the Λ_c-classical polynomial sequence and to provide a full description of this family of orthogonal polynomials. Especially, we prove that the Λ_c-classical polynomial sequences form a subfamily of the D-classical polynomial sequences.

The rest of this paper is organized as follows. In Section 2, we develop the terminology and basic definitions that will be used later on. In Section 3, we exhaustively describe the Λ_c-classical sequences.

2 Preliminaries and notations

Let $\mathscr{P}$ be the linear space of polynomials in one variable with complex coefficients. The algebraic dual space of $\mathscr{P}$ will be represented by $\mathscr{P}'$. We denote by $\langle u, p\rangle$ the action of $u \in \mathscr{P}'$ on $p \in \mathscr{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the sequence of moments of u with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$.

Let us define the following operations in $\mathscr{P}'$. For linear functionals u, any polynomial q, and any $(a,b,c) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}^2$, let $Du = u',\ qu,\ (x-c)^{-1}u,\ \tau_{-b}u$ and $h_a u$ be the linear functionals defined by duality, [14, 17, 18, 29]

$$\begin{aligned}
&\langle qu, p\rangle := \langle u, qp\rangle, \qquad \langle u', p\rangle := -\langle u, p'\rangle,\\
&\langle (x-c)^{-1}u,\ p\rangle := \langle u,\ \theta_c p\rangle, \quad \text{where} \quad \theta_c p(x) = \frac{p(x)-p(c)}{x-c},\\
&\langle \tau_{-b}u,\ p\rangle := \langle u,\ \tau_b p\rangle, \quad \text{where} \quad \tau_b p(x) = p(x-b),\\
&\langle h_a u,\ p\rangle := \langle u,\ h_a p\rangle, \quad \text{where} \quad h_a p(x) = p(ax), \quad \text{for every } p \in \mathscr{P}.
\end{aligned}$$

A linear functional u is called *normalized* if it satisfies $(u)_0 = 1$.
Let $\{P_n\}_{n\geq 0}$ be a infinite sequence of monic polynomials (SMP) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathbb{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical functional associated with the SMP $\{P_n\}_{n\geq 0}$. Recall that any $u \in \mathscr{P}'$ can be represented as $u = \sum_{n=0}^{+\infty}\langle u, P_n\rangle u_n$. So, if $\{u_n^{[1]}\}_{n\geq 0}$ denotes the dual sequence of the SMP $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x)$, $n \geq 0$, then $Du_n^{[1]} = -(n+1)u_{n+1}$, $n \geq 0$ [30]. Likewise, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of the shifted SMP $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a,b) \in \mathbb{C}\backslash\{0\}\times\mathbb{C}$, is given by $\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n, n \geq 0$ [30].

Let us recall that a form u (linear functional) is said to be quasi-definite (regular) if there exists a unique sequence of monic polynomials $\{P_n\}_{n\geq 0}$, such that [14, 17, 18, 29, 30]

$$\langle u, P_nP_m\rangle = r_n\delta_{n,m}, \quad n,\ m \geq 0, \quad r_n \neq 0, \quad n \geq 0. \tag{1}$$

The sequence $\{P_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to u. Note that $u = (u)_0 u_0$, with $(u)_0 \neq 0$. For any quasi-definite linear functional u and any polynomial φ such that $\varphi u = 0$, it is then straightforward to prove that $\varphi = 0$, [30].

Lemma 2.1 *[14, 17, 18, 29]. The SMP $\{P_n\}_{n\geq 0}$, with dual sequence $\{u_n\}_{n\geq 0}$, is orthogonal with respect to u_0 if and only if one of the following statements hold:*

(i) $u_n = \langle u_0, P_n^2\rangle^{-1}P_n u_0$, $n \geq 0$.

(ii) $\{P_n\}_{n\geq 0}$ *satisfies a Three-Term Recurrence Relation*

$$\text{(TTRR)} \quad \begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geq 0, \end{cases} \tag{2}$$

where $\beta_n = \langle u_0, xP_n^2\rangle/\langle u_0, P_n^2\rangle \in \mathbb{C}$ and $\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle/\langle u_0, P_n^2\rangle \in \mathbb{C}\backslash\{0\}$.

A linear functional u is said to be positive-definite if it is quasi-definite i.e. it satisfies (1) and $r_n > 0$ for every nonnegative integer n (see [29]). Note that a linear functional u is quasi-definite but it is not necessarily positive-definite.

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The orthogonality is preserved by a shifting in the variable. Indeed, for the shifted sequence $\{\tilde{P}_n\}_{n\geq 0}$ where $\tilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a,b)\in\mathbb{C}\backslash\{0\}\times\mathbb{C}$, the following Three-Term Recurrence Relation holds (see [20, 29])

$$\text{(TTRR)}\quad \begin{cases} \tilde{P}_0(x)=1, \quad \tilde{P}_1(x)=x-\tilde{\beta}_0, \\ \tilde{P}_{n+2}(x)=(x-\tilde{\beta}_{n+1})\tilde{P}_{n+1}(x)-\tilde{\gamma}_{n+1}\tilde{P}_n(x),\ n\geq 0, \end{cases}$$

where $\tilde{\beta}_n=a^{-1}(\beta_n-b)$ and $\tilde{\gamma}_{n+1}=a^{-2}\gamma_{n+1}$.

Notice that $\{\tilde{P}_n\}_{n\geq 0}$ is orthogonal with respect to the linear functional $\tilde{u}_0=(h_{a^{-1}}\circ\tau_{-b})u_0$.

A linear functional u is said to be D-classical when it is quasi-definite and there exist two polynomials Φ and Ψ, Φ monic, $\deg\Phi=t\leq 2$, and $\deg\Psi=1$, such that u satisfies a Pearson's equation (see [14, 17, 18, 22, 29, 30, 31])

$$\text{(PE)}\quad (\Phi u)'+\Psi u=0. \tag{3}$$

In such a case, the corresponding SMOP $\{P_n\}_{n\geq 0}$ is said to be D-classical.

Any shift leaves invariant the D-classical character. Indeed, the shifted linear functional $\tilde{u}=(h_{a^{-1}}\circ\tau_{-b})u$ fulfils [14, 17, 18, 30]

$$(\tilde{\Phi}\tilde{u})'+\tilde{\Psi}\tilde{u}=0,$$

where $\tilde{\Phi}(x)=a^{-t}\Phi(ax+b)$ and $\tilde{\Psi}(x)=a^{1-t}\Psi(ax+b)$.

It is well-known that any D-classical polynomial sequence $\{P_n\}_{n\geq 0}$ can be characterized by taking into account its orthogonality and a First Structure Relation (FSR) or a Second Structure Relation (SSR) as follows, [13, 17, 20, 22, 23, 30]:

$$\text{(FSR)}\qquad \Phi(x)P'_{n+1}(x)=r(x;n)P_{n+1}(x)+s_nP_n(x),\ n\geq 0, \tag{4}$$

$$\text{(SSR)}\qquad P_n(x)=P_n^{[1]}(x)+a_nP_{n-1}^{[1]}(x)+b_nP_{n-2}^{[1]}(x),\ n\geq 0, \tag{5}$$

The D-classical orthogonal polynomials are essentially the only polynomial (not just orthogonal polynomial) systems that satisfy a Second-Order Differential Equation (SODE, in short), Bochner [16], (see also [12, 31]), of the form

$$\Phi(x)P''_{n+1}(x)-\Psi(x)P'_{n+1}(x)=\omega_nP_{n+1}(x),\ n\geq 0, \tag{6}$$

with $\deg\phi\leq 2$, $\deg\psi=1$ and where $(n+1)\big(\frac{1}{2}\phi''(0)n+\psi'(0)\big)=\omega_n\neq 0,\ n\geq 0$. For the four canonical situations (three positive-definite cases, namely *Hermite*, *Laguerre*, and *Jacobi*, and one quasi-definite case, namely *Bessel*), in the next table we summarize the parameters involved in (2)−(6), (for more details, see [12, 22, 29, 30, 31]).

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Some basic characteristics of classical orthogonal polynomials.

(C_1) Hermite: $P_n(x)=H_n(x),\ n\geq 0,$
$\beta_n=0,\ n\geq 0,\ \gamma_{n+1}=\frac{n+1}{2},\ n\geq 0,$
$\Phi(x)=1,\quad \Psi(x)=2x,$
$r(x;n)=0,\ s_n=n+1,\ n\geq 0,$
$a_n=b_n=0,\ n\geq 0,$
$\omega_n=-2(n+1),\ n\geq 0.$

(C_2) Laguerre: $P_n(x)=L_n^{(\alpha)}(x),\ n\geq 0,\quad (\alpha\neq -n,\ n\geq 1),$
$\beta_n=2n+\alpha+1,\ n\geq 0,\ \gamma_{n+1}=(n+1)(n+\alpha+1),\ n\geq 0,$
$\Phi(x)=x,\quad \Psi(x)=x-\alpha-1,$
$r(x;n)=n+1,\ s_n=\gamma_{n+1},\ n\geq 0,$
$a_n=n,\ b_n=0,\ n\geq 0,$
$\omega_n=-(n+1),\ n\geq 0.$

(C_3) Bessel: $P_n(x)=B_n^{(\alpha)}(x),\ n\geq 0,\quad (\alpha\neq -\frac{n}{2},\ n\geq 0),$
$\beta_0=-\frac{1}{\alpha},\ \beta_n=\frac{1-\alpha}{(n+\alpha-1)(n+\alpha)},\ n\geq 0,\ \gamma_n=-\frac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)},\ n\geq 1,$
$\Phi(x)=x^2,\quad \Psi(x)=-2(\alpha x+1),$
$r(x;n)=(n+1)(x-\frac{1}{n+\alpha}),\ s_n=-(2n+2\alpha+1)\gamma_{n+1},\ n\geq 0,$
$a_n=\frac{n}{(n+\alpha-1)(n+\alpha)},\ n\geq 1,\ a_0=0,\quad b_n=\frac{(n-1)n}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)},\ n\geq 2,\ b_0=b_1=0,$
$\omega_n=(n+1)(n+2\alpha),\ n\geq 0.$

(C_4) Jacobi: $P_n(x)=P_n^{(\alpha,\beta)}(x),\ n\geq 0,\quad (\alpha,\ \beta\neq -n,\ \alpha+\beta\neq -n-1,\ n\geq 1),$
$\beta_0=\frac{\alpha-\beta}{\alpha+\beta+2},\quad \beta_n=\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},\ \gamma_n=\frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\ n\geq 1,$
$\Phi(x)=x^2-1,\quad \Psi(x)=-(\alpha+\beta+2)x+\alpha-\beta,$
$r(x;n)=(n+1)(x-\frac{\alpha-\beta}{2n+\alpha+\beta+2}),\ s_n=-(2n+\alpha+\beta+3)\gamma_{n+1},\ n\geq 0,$
$a_n=-\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},\ n\geq 1,\ a_0=0,\ b_n=-\frac{4(n-1)n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\ n\geq 2,\ b_0=b_1=0,$
$\omega_n=(n+1)(n+\alpha+\beta+2),\ n\geq 0.$

Notice that a linear functional D-classical is not necessarily positive-definite.

Now, recall that the latter operator is given by

$$\begin{array}{rcl}\Lambda_c:\mathscr{P} & \longrightarrow & \mathscr{P}\\ f & \longmapsto & \Lambda_c(f)=(x-c)^3f^{(4)}+12(x-c)^2f^{(3)}+36(x-c)f''+24f',\end{array}$$

It is clear that Λ_c lowers the degree by one of any polynomial. Indeed, we have

$$\begin{aligned}\Lambda_c(x-c)^{n+1} &= \Big(\hat{\Omega}_3(x-c)\hat{\Omega}_3+2\hat{\Omega}_3\Big)(x-c)^{n+1}\\ &= (n+1)(n+2)(n+3)(n+4)(x-c)^n,\ n\geq 0,\end{aligned}\tag{7}$$

since $\hat{\Omega}_3(x-c)^{n+1}=(n+1)(n+4)(x-c)^n,\ n\geq 0$, (see [21]).

By transposition of the operator Λ_c, we get

$${}^t\Lambda_c=(x-c)f^{(4)}+(6(x-c)^3-108(x-c)^2)f^{(2)}-(144(x-c)^3-36(x-c)-48)f'-72(x-c)^2f.\tag{8}$$

For any SMP $\{P_n\}_{n\geq 0}$, we define

$$Q_n(x):=\frac{\Lambda_c\ P_{n+1}(x)}{(n+1)(n+2)(n+3)(n+4)},\quad n\geq 0.\tag{9}$$

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Clearly, $\{Q_n\}_{n\geq 0}$ is a SMP and $\deg Q_n = n$. If $\{v_n\}_{n\geq 0}$ denotes the dual sequence of $\{Q_n\}_{n\geq 0}$, then we have

$$ {}^t\Lambda_c v_n = (n+1)(n+2)(n+3)(n+4)u_{n+1}, \quad n \geq 0. \tag{10} $$

Note that for $c = 0$ and using the representation of Laguerre polynomials in terms of hypergeometric series [27]:

$$ L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \frac{\Gamma(n+\alpha+1)}{\Gamma(\nu+\alpha+1)}\, x^\nu, \tag{11} $$

with $\alpha = 3$, we obtain

$$ \Big((x-c)^3D^4+12(x-c)^2D^3+36(x-c)D^2+24D\Big)L_n^{(3)}(x) = (n+1)(n+2)(n+3)(n+4)L_n^{(1)}(x), \quad n \geq 0. $$

Equivalently

$$ L_n^{(1)}(x) = \frac{\Lambda_{3,0}\ L_n^{(3)}(x)}{(n+1)(n+2)(n+3)(n+4)},\ n \geq 0. \tag{12} $$

This implies that $Q_n(x) = L_n^{(1)}(x),\ n \geq 0$, which is an example of solution of our problem. The goal of this manuscript is to describe all of the sequences of Λ_c-classical orthogonal polynomials in the Hahn's sense, where $\Lambda_c,\ c \in \mathbb{C}$ is the above operator.

3 The Λ_c-classical orthogonal polynomials

Definition 3.1 *Let u_0 be a quasi-definite linear functional and let $\{P_n\}_{n\geq 0}$ be the corresponding SMOP. We call $\{P_n\}_{n\geq 0}$ is Λ_c-classical if $\{\Lambda_c P_{n+1}\}_{n\geq 0}$ is also orthogonal. In this case, u_0 is also said to be an Λ_c-classical linear functional.*

Our next goal is to describe all of the the Λ_c-classical polynomial sequences. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are SMOP satisfying

$$ \begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ \gamma_{n+1} \neq 0,\ n \geq 0, \end{cases} \tag{13} $$

$$ \begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \xi_0, \\ Q_{n+2}(x) = (x-\xi_{n+1})Q_{n+1}(x) - \lambda_{n+1}Q_n(x),\ \lambda_{n+1} \neq 0,\ n \geq 0. \end{cases} \tag{14} $$

The dual sequences of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ will be denoted by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, respectively. By Lemma 2.1 (i), we get

$$ u_n = \frac{P_n}{\langle u_0, P_n^2\rangle}u_0,\ n \geq 0 \quad ; \quad v_n = \frac{Q_n}{\langle v_0, Q_n^2\rangle}v_0,\ n \geq 0. \tag{15} $$

Based on the characterizations of D-classical polynomials (see for example [17, 19]), we prove, in the following result, that the sequence $\{Q_n\}_{n\geq 0}$ is D-classical.

Lemma 3.1 (i) *The sequence* $\{Q_n\}_{n\geq 0}$ *satisfies the following relation*

$$\text{(SSR)} \quad Q_n(x) = Q_n^{[1]}(x) + c_n Q_{n-1}^{[1]}(x) + d_n Q_{n-2}^{[1]}(x),\ n \geq 0,$$

where

$$c_n = \frac{n}{4}\big(\beta_{n+1} - \xi_n\big),\ n \geq 0, \quad d_n = \frac{n-1}{4}\big(\frac{n}{n+4}\gamma_{n+1} - \lambda_n\big),\ n \geq 1,\ d_0 = 0.$$

(ii) *The form* v_0 *satisfies*

$$\text{(PE)} \quad (\Phi v_0)' + \Psi v_0 = 0,$$

where

$$\begin{aligned} \kappa\Phi(x) &= d_2(\lambda_1\lambda_2)^{-1}Q_2(x) + c_1\lambda_1^{-1}Q_1(x) + 1, \\ \Psi(x) &= (\kappa\lambda_1)^{-1}Q_1(x).\ (\kappa \textit{ is a normalization factor}) \end{aligned}$$

Proof. Let us introduce the sequence of monic polynomials $\{Z_n\}_{n\geq 0}$ given by

$$(n+1)(n+2)(n+3)Z_n(x) = \big[(x-c)^3 P_n(x)\big]^{(3)},\ n \geq 0. \tag{16}$$

By taking derivatives in both hand sides of (16) and taking into account (9), we get

$$Z_n^{[1]}(x) = Q_n(x),\ n \geq 0. \tag{17}$$

Notice that $Z_n(x)$ is a monic primitive of $Q_n(x)$.
From (13) and (16), we obtain

$$\begin{aligned} (n+3)(n+4)(n+5)Z_{n+2}(x) &= 3\big[(x-c)^3 P_{n+1}(x)\big]^{(2)} + (n+2)(n+3)(n+4)(x-\beta_{n+1})Z_{n+1}(x) \\ &\quad -(n+1)(n+2)(n+3)\gamma_{n+1}Z_n(x),\ n \geq 0. \end{aligned}$$

By differentiating in both sides of the previous identities and inserting (17), we get

$$\begin{aligned} 4(n+2)(n+3)(n+4)Z_{n+1}(x) &= 4(n+2)(n+3)(n+4)Q_{n+1}(x) \\ &\quad -(n+1)(n+2)(n+3)(n+4)(\xi_n - \beta_{n+1})Q_n(x) \\ &\quad -(n+1)(n+2)(n+3)[(n+4)\lambda_n - n\gamma_{n+1}]Q_{n-1}(x),\ n \geq 0. \end{aligned}$$

Then, it follows that

$$Z_{n+1}(x) = Q_{n+1}(x) + e_{n+1}Q_n(x) + f_n Q_{n-1}(x),\ n \geq 0. \tag{18}$$

where

$$e_n = \frac{n+1}{4}(\beta_{n+1} - \xi_n) \text{ and } f_n = \frac{n+1}{4}\big(\frac{n}{n+4}\gamma_{n+1} - \lambda_n\big).$$

By differentiating both hand sides of (18) and using (17), (i) holds.

Let $\{v_n^{[1]}\}_{n\geq 0}$ be the dual sequence of $\{Q_n^{[1]}\}_{n\geq 0}$. From (i), we have $\langle v_0^{[1]}, Q_n\rangle = 0,\ n \geq 3$, $\langle v_0^{[1]}, Q_2\rangle = d_2$, $\langle v_0^{[1]}, Q_1\rangle = c_1$, and $\langle v_0^{[1]}, Q_0\rangle = 1$. So, $v_0^{[1]} = d_2 v_2 + c_1 v_1 + v_0$, and by (15), we get $v_0^{[1]} = \kappa\Phi(x)v_0$, where $\kappa\Phi(x) = d_2\lambda_1^{-1}\lambda_2^{-1}Q_2(x) + c_1\lambda_1^{-1}Q_1(x) + 1$ and κ is a normalization factor. Because $(v_0^{[1]})' = -v_1 = -\lambda_1^{-1}Q_1 v_0$, then $(\Phi v_0)' + \Psi v_0 = 0$, where $\Psi(x) = (\kappa\lambda_1)^{-1}Q_1(x)$. Hence, (ii) holds. ■

Lemma 3.2 *There are four polynomials E, F, G and H, such that*

(i) $(x-c)^3 v_0 = E(x) u_0$.

(ii) $E(x)Q_n^{(4)}(x) + F(x)Q_n^{(3)}(x) + G(x)Q_n''(x) + H(x)Q_n' + \rho_0 P_1 Q_n(x) = \rho_n P_{n+1}(x),\ n \geq 0,$ *where*

$$\begin{aligned}
H(x) &= \rho_1 P_2(x) - \rho_0 P_1 Q_1(x), \\
G(x) &= \frac{1}{2}\big[\rho_2 P_3(x) - \rho_0 P_1 Q_2 - H(x)Q_2'\big], \\
F(x) &= \frac{1}{6}\big[\rho_3 P_4(x) - \rho_0 P_1 Q_3 - H(x)Q_3' - G(x)Q_3''\big], \\
E(x) &= \frac{1}{24}\big[\rho_4 P_5(x) - \rho_0 P_1 Q_4 - H(x)Q_4' - G(x)Q_4'' - F(x)Q_4^{(3)}\big], \\
\rho_n &= (n+1)(n+2)(n+3)(n+4)\frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+1}^2\rangle}.
\end{aligned}$$

(iii) *The following relations hold*

(a)

$$\begin{aligned}
E(x)C(x) &= \rho_0 \Phi^4(x) P_1(x), \\
4E(x)B(x) &= \Phi^3(x) H(x), \\
6E(x)A(x) &= \Phi^2(x) G(x), \\
4E(x)\tilde{A}(x) &= \Phi(x) F(x),
\end{aligned}$$

(b) $\gamma_1 \Phi^4(x) = \big[\{(\rho_0^{-1} - \rho_1^{-1})x + \rho_1^{-1}\xi_0 - \rho_0^{-1}\beta_1\}C(x) - 4\rho_1^{-1}\phi B(x)\big]E(x),$

where, $\tilde{A} = -(\Phi' + \Psi),\ A = -(2\Phi' + \Psi)\tilde{A} + \Phi\tilde{A}',\ B = -(3\Phi' + \Psi)A + \Phi A',\ C = -(4\Phi' + \Psi)B + \Phi B'.$

Proof. From (10) and (15), we obtain

$$(x-c)^3\big[Q_n(x)v_0^{(4)} + 4Q_n'(x)v_0^{(3)} + 6Q_n''(x)v_0'' + 4Q_n^{(3)}v_0' + Q_n^{(4)}v_0\big] = \rho_n P_{n+1}(x)u_0,\ n \geq 0, \quad (19)$$

where

$$\rho_n = (n+1)(n+2)(n+3)(n+4)\frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+1}^2\rangle}.$$

From (19) with $n = 0$, we obtain

$$(x-c)^3 v_0^{(4)} = \rho_0 P_1 u_0. \quad (20)$$

Using (19) and (20), it follows that

$$(x-c)^3\big[4Q_n'(x)v_0^{(3)} + 6Q_n''(x)v_0'' + 4Q_n^{(3)}v_0' + Q_n^{(4)}v_0\big] = \big[\rho_n P_{n+1}(x) - \rho_0 P_1 Q_n\big]u_0. \quad (21)$$

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For $n=1$, (21) becomes

$$4(x-c)^3 v_0(3) = H(x)u_0, \tag{22}$$

where

$$H(x) = \rho_1 P_2(x) - \rho_0 P_1 Q_1(x).$$

By inserting (22) in (21), we obtain

$$(x-c)^3\big[6Q_n''(x)v_0'' + 4Q_n^{(3)}v_0' + Q_n^{(4)}v_0\big] = \big[\rho_n P_{n+1}(x) - \rho_0 P_1 Q_n - H(x)Q_n'\big]u_0. \tag{23}$$

Taking $n=2$ in (23), we obtain

$$6(x-c)^3 v_0'' = G(x)u_0, \tag{24}$$

where

$$G(x) = \frac{1}{2}\big[\rho_2 P_3(x) - \rho_0 P_1 Q_2 - H(x)Q_2'\big].$$

By inserting (24) in (23), we obtain

$$(x-c)^3\big[4Q_n^{(3)}v_0' + Q_n^{(4)}v_0\big] = \big[\rho_n P_{n+1}(x) - \rho_0 P_1 Q_n - H(x)Q_n' - G(x)Q_n''\big]u_0. \tag{25}$$

Taking $n=3$ in (25), we obtain

$$4(x-c)^3 v_0' = F(x)u_0, \tag{26}$$

where

$$F(x) = \frac{1}{6}\big[\rho_3 P_4(x) - \rho_0 P_1 Q_3 - H(x)Q_3' - G(x)Q_3''\big].$$

By inserting (26) in (25), we obtain

$$(x-c)^3 Q_n^{(4)}v_0 = \big[\rho_n P_{n+1}(x) - \rho_0 P_1 Q_n - H(x)Q_n' - G(x)Q_n'' - F(x)Q_n^{(3)}\big]u_0. \tag{27}$$

Hence, taking $n=4$ in (27), (i) holds.

Meanwhile, by substituting $(x-c)^3 v_0 = Eu_0$ in (27) and taking into account the quasi-definiteness of u_0, we deduce (ii).

By using Lemma 3.1 (ii), we can write

$$\Phi v_0' = \tilde{A}v_0, \quad \Phi^2 v_0'' = Av_0, \quad \Phi^3 v_0^{(3)} = Bv_0, \quad \Phi^4 v_0^{(4)} = Cv_0, \tag{28}$$

where $\tilde{A} = -(\Phi' + \Psi)$, $A = -(2\Phi' + \Psi)\tilde{A} + \Phi\tilde{A}'$, $B = -(3\Phi' + \Psi)A + \Phi A'$, $C = -(4\Phi' + \Psi)B + \Phi B'$.

In contrast, if we multiply (20), (22), (24) and (26) by Φ^4, Φ^3, Φ^2, Φ successively and we take into account (28), (i) and also the quasi-definiteness of u_0, we get (iii) (a).

Using (20), (22), (24), (26) and we take into account (i) and also the quasi-definiteness of u_0, we get

$$\big[Q_n^{(4)}\phi^{(4)} + 4\tilde{A}Q_n^{(3)}\phi^{(3)} + 6AQ_n''\phi^2 + 4Q_n^{'}B\phi + Q_nC\big]E(x) = \rho_n P_{n+1}(x)\phi^4(x),\ n \geq 0, \tag{29}$$

Hence, taking $n=1$ in (29), and using (13), (14), we deduce (iii) (b). ∎

Now, we will describe all of the Λ_c-classical polynomial sequences.

Theorem 3.1 *The Λ_c-classical polynomial sequences are, up to a suitable affine transformation in the variable, one of the following D-classical polynomial sequences*

(i) $P_n(x) = L_n^{(3)}(x)$ *and* $Q_n(x) = L_n^{(1)}(x)$, $n \geq 0$, *with* $c = 0$.

(ii) $P_n(x) = P_n^{(\alpha-4,3)}(x)$ *and* $Q_n(x) = P_n^{(\alpha,1)}(x)$, $n \geq 0$, *with* $c = 1$ *and where* $\alpha \neq -n+4$, $n \geq 1$.

Proof. From Lemma 3.1, $\{Q_n\}_{n\geq 0}$ is D-classical. We will analyze the following situations:

(S$_1$). $\{Q_n\}_{n\geq 0}$ is the Hermite SMOP. From Table 2 **(C$_1$)**, $C(x) = 16x^4 - 48x^2 + 12$. Then, from Lemma 3.2 (iii) (a), we get $E(x)C(x) = \rho_0 P_1(x)$. This yields a contradiction.

(S$_2$). $\{Q_n\}_{n\geq 0}$ is the Laguerre SMOP. From Table 2 **(C$_2$)**, $C(x) = x^4 - 4\alpha x^3 + 6\alpha(\alpha-1)x^2 - 4\alpha(\alpha-1)(\alpha-2)x + \alpha(\alpha-1)(\alpha-2)(\alpha-3)$. Therefore, from Lemma 3.2, (iii), we have

$$E(x)C(x) = \rho_0 x^4 P_1(x), \tag{30}$$

From (30), we can deduce, $\deg E = 1$, so there exists $(a,b) \in \mathbb{C}\backslash\{0\}\times\mathbb{C}$ such as $E(x) = ax+b$. According to (13), equation(30) becomes

$$(ax+b)\big[x^4 - 4\alpha x^3 + 6\alpha(\alpha-1)x^2 - 4\alpha(\alpha-1)(\alpha-2)x + \alpha(\alpha-1)(\alpha-2)(\alpha-3)\big] = \rho_0 x^4(x-\beta_0), \tag{31}$$

which gives, after comparing the degrees

$$\begin{aligned}
a &= \rho_0, & (32)\\
b &= -\rho_0(\beta_0 - 4\alpha), & (33)\\
\alpha\big(3(\alpha-1) + 2(\beta_0 - 4\alpha)\big) &= 0, & (34)\\
\alpha(\alpha-1)\big(2(\alpha-2) + 3(\beta_0 - 4\alpha)\big) &= 0, & (35)\\
\alpha(\alpha-1)(\alpha-2)\big(\alpha - 3 + 4(\beta_0 - 4\alpha)\big) &= 0, & (36)\\
\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\beta_0 - 4\alpha) &= 0. & (37)
\end{aligned}$$

Note that, we need $b = 0$ to ensure that $\{P_n\}_{n\geq 0}$ is an orthogonal sequence. Indeed, if we suppose that $b \neq 0$. In view of (34), we can remark two possibilities: either (i) $\alpha \neq 0$ or (ii) $\alpha = 0$.

(i) If $\alpha \neq 0$, the equation (34) we gives $2\beta_0 = 5\alpha - 3$ and $\alpha \neq 1$. By substituting this result in (35), we obtain $\alpha = \frac{1}{5}$. Then, equation (37) we gives $\beta_0 - 4\alpha = 0$, which contradicts $b \neq 0$.

(ii) If $\alpha = 0$, equation (33) we gives $b = -\rho_0\beta_0$, then $E(x) = \rho_0 P_1(x)$ and $C(x) = x^4$. Using Lemma 3.2 (iii) (a), we obtain $H(x) = -4\rho_0 P_1(x)$, $G(x) = 6\rho_0 P_1(x)$, $F(x) = -4\rho_0 P_1(x)$. Replaced x by β_0 in Lemma 3.2 (ii), we obtain $P_{n+1}(\beta_0) = 0$, $n \geq 0$. Then, $P_2(\beta_0) = -\gamma_1 = 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.

Then, $b=0$, $E(x)=\rho_0 x$ and the equations (32), (33), (34), (35) and (36) becomes

$$\begin{aligned} a &= \rho_0, && (38)\\ \beta_0 &= 4\alpha, && (39)\\ \alpha(\alpha-1) &= 0, && (40)\\ \alpha(\alpha-1)(\alpha-2) &= 0, && (41)\\ \alpha(\alpha-1)(\alpha-2)(\alpha-3) &= 0. && (42)\\ & && (43)\end{aligned}$$

In view of (40), we can remark two possibilities: either $\alpha=0$ or $\alpha=1$. If $\alpha=0$, equation (33) we gives $\beta_0=0$, then $E(x)=\rho_0 x$ and $C(x)=x^4$. Using Lemma 3.2 (iii) (a), we obtain $H(x)=-4\rho_0 x$, $G(x)=6\rho_0 x$, $F(x)=-4\rho_0 x$. Replaced x by 0 in Lemma 3.2 (ii), we obtain $P_{n+1}(0)=0,\ n\geq 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.

Then $\alpha=1$, $C(x)=x^3(x-4),\ B(x)=-x^2(x-3),\ A(x)=x(x-2),\ \tilde{A}(x)=x-1$ and $Q_n(x)=L_n^{(1)}(x)$, where $\{L_n^{(1)}\}_{n\geq 0}$ is is the sequence polynomials of Laguerre with parameter 1. In this case $\beta_0=4$ and using Lemma 3.2 (iii) (a), we obtain $H(x)=-4\rho_0(x-3)$, $G(x)=6\rho_0(x-2)$, $F(x)=-4\rho_0(x-1)$.

Now, Lemma 3.2 (iii) (b) gives $\gamma_1=\big[\{(\rho_0^{-1}-\rho_1^{-1})x+\rho_1^{-1}\xi_0-\rho_0^{-1}\beta_1\}(x-4)-4\rho_1^{-1}(x-3)\big]\rho_0$. Use the fact that $\xi_0=2$, we obtain

$$\gamma_1\rho_0^{-1}=(\rho_0^{-1}-\rho_1^{-1})x^2+[-4\rho_0^{-1}+10\rho_1^{-1}-\rho_0^{-1}\beta_1]x-20\rho_1^{-1}+4\rho_0^{-1}\beta_1,$$

which gives, after comparing the degrees $\rho_0=\rho_1$, $\beta_1=6$ and $\gamma_1=4$.

According to Lemma 3.2 (ii) and Lemma 3.1 (i), we have $\rho_0=24\gamma_1^{-1}=6$, $\beta_{n+1}=2n+6,\ n\geq 0$ and $\gamma_{n+1}=(n+1)(n+4)$.

Now, using the previous results and Lemma 3.2 (ii), we obtain

$$6xQ_n^{(4)}-24(x-1)Q_n^{(3)}+36(x-2)Q_n''-24(x-3)Q_n'+6P_1Q_n=\rho_nP_{n+1},\ n\geq 0,$$

which gives, after comparing the degrees $\rho_n=6,\ n\geq 0$. therefore

$$xQ_n^{(4)}-4(x-1)Q_n^{(3)}+6(x-2)Q_n''-4(x-3)Q_n'+P_1Q_n=P_{n+1},\ n\geq 0,$$

In contrast, we have by Lemma 3.1 (i), $\beta_n=2n+4,\ n\geq 1$ and $\gamma_{n+1}(n+1)(n+4),\ n\geq 2$. Then, $P_n(x)=L_n^{(3)}(x)$, $n\geq 0$, with $Q_n(x)=L_n^{(1)}(x),\ n\geq 0$. Making $n=1$ in (9), we get $c=0$. Consequently, the following relations hold:

$$L_n^{(1)}(x)=\frac{\Lambda_{3,0}L_n^{(3)}(x)}{(n+1)(n+2)(n+3)(n+4)},\ n\geq 0, \tag{44}$$

$$\begin{aligned} x\{L_n^{(1)}\}^{(4)}(x)-4(x-1)\{L_n^{(1)}\}^{(3)}(x)+6(x-2)\{L_n^{(1)}\}^{(2)}(x)-4(x-3)\{L_n^{(1)}\}^{(1)}(x)\\ +(x-4)L_n^{(1)}(x)=L_{n+1}^{(3)}(x),\ n\geq 0,. \end{aligned} \tag{45}$$

($\mathbf{S}_3$). $\{Q_n\}_{n\geq 0}$ is the Bessel SMOP with parameter $\alpha \neq -n/2,\ n\geq 0$. In this case, we get

$$\begin{aligned}
A(x) &= 2(1-\alpha)(3-2\alpha)x^2+4(2\alpha-3)x+4,\\
B(x) &= -4\big[(1-\alpha)(2-\alpha)(3-2\alpha)x^3+3(2-\alpha)(2\alpha-3)x^2+6(2-\alpha)x-2\big],\\
C(x) &= 4\big[(1-\alpha)(2-\alpha)(3-2\alpha)(5-2\alpha)x^4+4(2-\alpha)(2\alpha-3)(5-2\alpha)x^3\\
&\qquad\qquad\qquad +12(2-\alpha)(5-2\alpha)x^2-8(5-2\alpha)x+4\big].
\end{aligned}$$

Using Lemma 3.2 (iii) (a), we obtain $E(x)C(x)=\rho_0x^8P_1(x)$. This requires that $\deg C=4$ and $\deg E=5$ because $\deg C\leq 4$, $\deg E\leq 5$, and $\deg C+\deg E=9$. However, from the previous equation, we must have $C(0)=0$, that contradicts the fact that $C(0)=16$.

($\mathbf{S}_4$). $\{Q_n\}_{n\geq 0}$ is the Jacobi SMOP with parameters α and β satisfying $\alpha,\ \beta\neq -n,\ \alpha+\beta\neq -n-1,\ n\geq 1$. Then, we have

$$\begin{aligned}
A(x) &= (\alpha+\beta-1)\big[(\alpha+\beta)x^2+2(\beta-\alpha)x\big]+(\alpha-\beta)^2-(\alpha+\beta),\\
B(x) &= (\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)x^3+3(\alpha+\beta-1)(\alpha+\beta-2)(\beta-\alpha)x^2\\
&\quad +3(\alpha+\beta-2)\big[(\alpha-\beta)^2-(\alpha+\beta)\big]x+(\beta-\alpha)\big[(\alpha-\beta)^2-3\alpha-3\beta+2\big],\\
C(x) &= (\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)x^4\\
&\qquad +4(\alpha+\beta-3)(\alpha+\beta-1)(\alpha+\beta-2)(\beta-\alpha)x^3\\
&\qquad +6(\alpha+\beta-2)(\alpha+\beta-3)\big[(\alpha-\beta)^2-(\alpha+\beta)\big]x^2\\
&\qquad +4(\beta-\alpha)(\alpha+\beta-3)\big[(\alpha-\beta)^2-3\alpha-3\beta+2\big]x\\
&\quad +(\alpha-\beta)^2\big[(\alpha-\beta)^2-3\alpha-3\beta+2\big]-3(\alpha+\beta-2)\big[(\alpha-\beta)^2-(\alpha+\beta)\big].
\end{aligned}$$

By using Lemma 3.2 (iii) (a), we obtain $E(x)C(x)=\rho_0\phi^4(x)P_1(x)$. The fact that $\deg E+\deg C=9$, $\deg E\leq 5$ and $\deg C\leq 4$, yields $\deg E=5$ and $\deg C=4$. Meanwhile, the previous equation becomes $E(x)$ divides $(x-1)^4(x+1)^4$, hence there are four situations to be considered. Either $E(x)=\mu(x-1)(x+1)^4$, $E(x)=\mu(x-1)^4(x+1)$, $E(x)=\mu(x-1)^2(x+1)^3$, or $E(x)=\mu(x-1)^3(x+1)^2$, where μ is a non-zero real number.

($\mathbf{S}_{4,1}$). $E(x)=\mu(x-1)(x+1)^4, \mu\neq 0$. According to Lemma 3.2 (iii), we easily obtain

$$\mu C(x)=\rho_0\big[x^4-(3+\beta_0)x^3+3(1+\beta_0)x^2-(1+3\beta_0)x+\beta_0\big] \tag{46}$$

From Lemma 3.2 (iii) (a) and (46), we get

$$\begin{cases}
\rho_0\beta_0=\mu\{(\alpha-\beta)^2\big[(\alpha-\beta)^2-3\alpha-3\beta+2\big]-3(\alpha+\beta-2)\big[(\alpha-\beta)^2-(\alpha+\beta)\big]\},\\
-\rho_0(1+3\beta_0)=\mu\{(\alpha-\beta)^2-3\alpha-3\beta+2\}4(\beta-\alpha)(\alpha+\beta-3),\\
3\rho_0(1+\beta_0)=\mu\{(\alpha-\beta)^2-(\alpha+\beta)\}6(\alpha+\beta-2)(\alpha+\beta-3),\\
-\rho_0(3+\beta_0)=\mu\{(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)(\beta-\alpha)\}4,\\
\rho_0=\mu(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3).
\end{cases}$$

The last three equations of the system, we gives

$$\frac{\mu}{\rho_0}=\frac{1}{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)},\quad \beta_0=\frac{\alpha-7\beta}{\alpha+\beta},\quad \beta(\beta-1)=0.$$

We can remark two possibilities: either $\beta = 0$ or $\beta = 1$. If $\beta = 0$, we obtain

$$C(x) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)(x-1)^4,$$

Then, by virtue of Lemma 3.2 (iii) (a), we get

$$P_1(1) = H(1) = G(1) = F(1) = 0.$$

Replaced x by 1 in Lemma 3.2 (ii), we obtain $P_{n+1}(1) = 0,\ n \geq 0$. Then, $P_2(1) = -\gamma_1 = 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.

Then, $\beta = 1$ and by virtue of Lemma 3.2 (ii) and (iii) (a), we obtain successively

$$\begin{aligned} H(x) &= (\rho_1-\rho_0)x^2 - [\rho_1(\beta_0+\beta_1) - \rho_0(\beta_0+\xi_0)]x + \rho_1(\beta_0\beta_1-\gamma_1) - \rho_0\beta_0\xi_0, \\ H(x) &= 4\mu(\alpha-1)\big[\alpha(\alpha+1)x^2 + 6\alpha x - \alpha(\alpha-5)\big], \end{aligned}$$

which gives, since $\rho_1 = 120\lambda_1\gamma_1^{-1}\gamma_2^{-1},\ \rho_0 = 24\gamma_1^{-1}$ and after comparing the degrees $\gamma_2 = \frac{40(\alpha+1)(\alpha-2)}{(\alpha+2)(\alpha+3)^2(\alpha+4)}$, $\beta_1 = \frac{(\alpha-1)(\alpha-7)}{(\alpha+1)(\alpha+3)}$ *and* $\gamma_1 = \frac{16(\alpha-3)}{(\alpha+1)^2(\alpha+2)}$.

From Table 2 $\mathbf{(C_4)}$ and by Lemma 3.1 (i), we finally get

$$\begin{aligned} \beta_n &= \frac{(\alpha-1)(\alpha-7)}{(2n+\alpha-1)(2n+\alpha+1)},\ n \geq 0, \\ \gamma_{n+1} &= \frac{4(n+1)(n+4)(n+\alpha)(n+\alpha-3)}{(2n+\alpha)(2n+\alpha+1)^2(2n+\alpha+2)},\ n \geq 0. \end{aligned}$$

Thus, we conclude that $P_n(x) = P_n^{(\alpha-4,3)}(x)$, and $Q_n(x) = P_n^{(\alpha,1)}(x),\ n \geq 0$ with $\alpha \neq -n+4,\ n \geq 1$. Now, for $n = 1$, in (9), we get $c = \frac{1}{3}(5\xi_0 - \beta_0 - \beta_1) = 1$. By Lemma 3.2 (i), $(x-1)^3 v_0 = \frac{3(\alpha+1)(\alpha+2)}{2\alpha(\alpha-1)(\alpha-2)(\alpha-3)}(x-1)(x+1)^4 u_0$. Consequently,

$$P_n^{(\alpha,1)}(x) = \frac{\Lambda_{3,1}P_{n+1}^{(\alpha-4,3)}(x)}{(n+1)(n+2)(n+3)(n+4)}, \quad n \geq 0. \tag{47}$$

$\mathbf{(S_{4,2})}$. $E(x) = \mu(x-1)^4(x+1)$, $\mu \neq 0$. By a similar computation as in $\mathbf{(S_{4,1})}$, we get $P_n(x) = P_n^{(3,\beta-4)}(x)$, $n \geq 0$, where $\beta \neq -n+4,\ n \geq 1$, and also $Q_n(x) = P_n^{(1,\beta)}(x),\ n \geq 0$, and $c = -1$.

$\mathbf{(S_{4,3})}$. $E(x) = \mu(x-1)^2(x+1)^3$, $\mu \neq 0$. According to Lemma 3.2 (iii) (a), we obtain

$$\mu C(x) = \rho_0\big[x^4 - (\beta_0+1)x^3 + (\beta_0-1)x^2 + (\beta_0+1)x - \beta_0\big], \tag{48}$$

From Lemma 3.2 (iii) (a) and (46), we get

$$\begin{cases} -\rho_0\beta_0 = \mu\{(\alpha-\beta)^2\big[(\alpha-\beta)^2 - 3\alpha - 3\beta + 2\big] - 3(\alpha+\beta-2)\big[(\alpha-\beta)^2 - (\alpha+\beta)\big]\}, \\ \rho_0(\beta_0+1) = \mu\{(\alpha-\beta)^2 - 3\alpha - 3\beta + 2\}4(\beta-\alpha)(\alpha+\beta-3), \\ \rho_0(\beta_0-1) = \mu\{(\alpha-\beta)^2 - (\alpha+\beta)\}6(\alpha+\beta-2)(\alpha+\beta-3), \\ -\rho_0(\beta_0+1) = \mu\{(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)(\beta-\alpha)\}4, \\ \rho_0 = \mu(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3). \end{cases}$$

The last three equations of the system, we gives

$$\frac{\mu}{\rho_0} = \frac{1}{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)}, \quad \beta_0 = \frac{3\alpha-5\beta}{\alpha+\beta}, \quad \alpha^2+3\beta^2-2\alpha\beta-\alpha-\beta = 0,$$

and by addition the second and the fourth equations of the last system, we obtain

$$\begin{cases} \alpha^2+3\beta^2-2\alpha\beta-\alpha-\beta = 0, \\ (\beta-\alpha)(\alpha^2+\beta^2-3\alpha-3\beta+2) = 0. \end{cases}$$

We can remark two possibilities: either $\beta = \alpha$ or $\beta \neq \alpha$. Now, if $\alpha = \beta$, then $\beta_0 = -1$ and $\alpha(\alpha-1) = 0$, from the first equation of the last system. This gives a contradiction, since $\rho_0 \neq 0$.

So $\beta \neq \alpha$ and $\alpha^2+\beta^2-3\alpha-3\beta+2 = 0$. Consequently, since $\rho_0 \neq 0$, the unique solutions (α, β) of the last system are $(3,1),$ *and* $(3,2)$.

($\mathbf{S}_{4,31}$). If $(\alpha,\beta) = (3,1)$, then $\beta_0 = 1$. By virtue of Lemma 3.2 (ii) and (iii) (a), we obtain successively

$$\begin{aligned} H(x) &= (\rho_1-\rho_0)x^2 - [\rho_1(\beta_0+\beta_1) - \rho_0(\beta_0+\xi_0)]x + \rho_1(\beta_0\beta_1-\gamma_1) - \rho_0\beta_0\xi_0, \\ H(x) &= 4\mu(2x^2-x-1), \end{aligned}$$

which gives, since $\rho_1 = 120\lambda_1\gamma_1^{-1}\gamma_2^{-1}$, $\rho_0 = 24\gamma_1^{-1}$ and after comparing the degrees $\gamma_2 = \frac{8}{63}$, $\beta_1 = \frac{-1}{3}$ *and* $\gamma_1 = 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.

($\mathbf{S}_{4,32}$). If $(\alpha,\beta) = (3,2)$, then $\beta_0 = -\frac{1}{5}$. By virtue of Lemma 3.2 (ii) and (iii) (a), we obtain successively

$$\begin{aligned} H(x) &= (\rho_1-\rho_0)x^2 - [\rho_1(\beta_0+\beta_1) - \rho_0(\beta_0+\xi_0)]x + \rho_1(\beta_0\beta_1-\gamma_1) - \rho_0\beta_0\xi_0, \\ H(x) &= 48\mu(5x^2+2x-1), \end{aligned}$$

which gives, since $\rho_1 = 120\lambda_1\gamma_1^{-1}\gamma_2^{-1}$, $\rho_0 = 24\gamma_1^{-1}$ and after comparing the degrees $\gamma_2 = \frac{5}{49}$, $\beta_1 = \frac{-3}{35}$ *and* $\gamma_1 = \frac{4}{25}$.
From Table 2 **($\mathbf{C}_4$)** and by Lemma 3.1 (i), we finally get

$$\begin{aligned} \beta_n &= \frac{-3}{(2n+3)(2n+5)}, \; n \geq 0, \\ \gamma_{n+1} &= \frac{(n+1)(n+4)}{(2n+5)^2}, \; n \geq 0. \end{aligned}$$

Thus, we conclude that $P_n(x) = P_n^{(1,2)}(x)$, and $Q_n(x) = P_n^{(3,2)}(x),\ n \geq 0$. Now, for $n = 1$, in (9), we get $c = \frac{1}{3}(5\xi_0 - \beta_0 - \beta_1) = \frac{1}{3}$. By Lemma 3.2 (i),
$\frac{-16}{189} = \langle (x-1)^3 v_0, 1\rangle \neq \langle E(x)u_0, 1\rangle = \frac{-451}{42}$. This yields a contradiction.

($\mathbf{S}_{4,4}$). $E(x) = \mu(x-1)^3(x+1)^2$, $\mu \neq 0$. By a similar computation as in **($\mathbf{S}_{4,3}$)**, we get a contradiction. ■

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