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THE ADJOINT OF THE HIGHER ORDER HEAT OPERATORS ON JACOBI FORMS

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ABSTRACT. We compute the adjoint of higher order heat operators with respect to the Petersson scalar product on the space of Jacobi cusp forms.

1. Introduction

Constructing new modular forms by means of derivatives of modular forms is well known in the theory of modular forms. Recently, Kumar [15] constructed certain cusp forms by computing the adjoint of the Ramanujan-Serre derivative of a cusp form with respect to the Petersson scalar product. The work of Kumar [15] has been extended by Charan [2], where the author constructed cusp form by computing the adjoint of the higher order Ramanujan-Serre derivative $\vartheta_k^{[r]}$. There is a natural generalization of the differential operator ϑ_k in the case of Jacobi forms called the modified heat operator denoted by $\mathscr{L}_{k,m}$, which is defined as

$$\mathscr{L}_{k,m} := L_m - \frac{m}{3}\left(k - \frac{1}{2}\right)E_2,$$

where $L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right)$, $\tau \in \mathscr{H}, z \in \mathbb{C}$. The operator $\mathscr{L}_{k,m}$ maps a Jacobi form φ of weight k and index m to a Jacobi form of weight $k+2$ and index m, and is a linear map.

The main aim of this paper is to construct Jacobi cusp forms using higher order heat operator. To do this, we first define the higher order modified heat operator (defined in Section 3) and compute its adjoint with respect to the Petersson scalar product on the space of Jacobi forms. To prove our result, we consider certain generalized Jacobi Poincaré series. Such Poincaré series was first studied by Williams [19] in the case of modular forms. Jha and Pandey studied similar Poincaré series for Jacobi forms in [10]. For more details on the problems of construction of automorphics forms, we refer to [8, 9, 11, 12, 13, 14, 15, 18].

We now give the outline of the paper. In the next section, we recall the basic definition and properties of Jacobi forms. In Section 3, we state our main result (Theorem 3.1). In Section 4, we provide some results which will be used to prove Theorem 3.1. A proof of Theorem 3.1 is presented in Section 5. Finally, we give some applications of Theorem 3.1 in Section 6.

2. Preliminaries

Let $\mathscr{H}$ and $\mathbb{C}$ denote the complex upper half plane and the field of complex numbers, respectively. The Jacobi group Γ^J is defined by

$$\Gamma^J := SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 = \{(M,(\lambda,\mu)) : M \in SL_2(\mathbb{Z}), \lambda, \mu \in \mathbb{Z}\}$$

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with the group law $(M,X)(M',X') = (MM', XM'+X')$. The Jacobi group Γ^J acts on $\mathscr{H}\times\mathbb{C}$ via

$$\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, (\lambda,\mu)\right)\cdot(\tau,z) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

Let k and m be fixed positive integers. For a complex-valued function φ defined on $\mathscr{H}\times\mathbb{C}$ and $\gamma = \left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right),(\lambda,\mu)\right)\in\Gamma^J$, the slash operator of weight k and index m is defined as

$$(1)\qquad (\varphi|_{k,m}\gamma)(\tau,z) := (c\tau+d)^{-k} e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z\right)}\varphi(\gamma.(\tau,z)),\ (\tau,z)\in\mathscr{H}\times\mathbb{C}.$$

We define $\varphi|_{k,m}M := \varphi|_{k,m}(M,(0,0))$ and $\varphi|_m X := \varphi|_{k,m}\left(\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right), X\right)$ for $M\in SL_2(\mathbb{Z})$ and $X\in\mathbb{Z}^2$. Then it is easy to check that

$$(2)\qquad \varphi|_{k,m}(M,X) = (\varphi|_{k,m}M)|_m X.$$

Definition 2.1. *A Jacobi form of weight k and index m for the Jacobi group Γ^J is a holomorphic function $\varphi:\mathscr{H}\times\mathbb{C}\to\mathbb{C}$ which satisfies $\varphi|_{k,m}\gamma = \varphi$ for all $\gamma\in\Gamma^J$, and has a Fourier series expression of the form*

$$\varphi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4mn\geq r^2}} c_\varphi(n,r) q^n\zeta^r,\ (q=e^{2\pi i\tau}, \zeta = e^{2\pi i z}).$$

Furthermore φ is called Jacobi cusp form if $c_\varphi(n,r) = 0$ for $4nm - r^2 = 0$.

Let $J_{k,m}$ (resp. $J_{k,m}^{cusp}$) denote the vector space of all Jacobi forms (resp. Jacobi cusp forms) of weight k and index m. The space $J_{k,m}^{cusp}$ is a finite dimensional Hilbert space with respect to the Petersson scalar product defined as

$$\langle\varphi,\psi\rangle := \int_{\Gamma^J\backslash\mathscr{H}\times\mathbb{C}} \varphi(\tau,z)\overline{\psi(\tau,z)} v^k e^{-4\pi m y^2/v} dV,$$

where $dV := v^{-3}dx\,dy\,du\,dv, (\tau = u+iv, z = x+iy)$ and $\varphi,\ \psi$ are Jacobi cusp form of weight k and index m. For details on the theory of Jacobi forms we refer to [6].

2.1. *Poincaŕe series.*

We now define the Jacobi Poincaŕe series of exponential type.

Definition 2.2. *Let m,n and r be fixed integers with $r^2 < 4mn$. Then the (n,r)-th Poincaŕe series of weight k and index m is defined by*

$$(3)\qquad P_{k,m}^{n,r}(\tau,z) := \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} e^{2\pi i(n\tau+rz)}|_{k,m}\gamma,$$

where $\Gamma_\infty^J := \left\{\left(\left(\begin{smallmatrix} 1 & \lambda\\ 0 & 1\end{smallmatrix}\right),(0,\mu)\right) | \lambda,\mu\in\mathbb{Z}\right\}$ is the stabilizer of $q^n\zeta^r$ in Γ^J. It is well-known that $P_{k,m}^{n,r}\in J_{k,m}^{cusp}$ for $k>2$ (see [7]).

The next lemma shows that the Petersson scalar product of a Jacobi cusp form $\varphi\in J_{k,m}^{cusp}$ and $P_{k,m}^{n,r}$ yields the (n,r)-th Fourier coefficient of φ up to some constant.

Lemma 2.3 (eqn. 1, p. 519, [7])**.** *Let $\varphi\in J_{k,m}^{cusp}$. Then we have*

$$(4)\qquad \langle\varphi, P_{k,m}^{n,r}\rangle = \alpha_{k,m}(4mn-r^2)^{-k+\frac{3}{2}} c_\varphi(n,r),$$

where $c_\varphi(n,r)$ is the (n,r)-th Fourier coefficient of φ and $\alpha_{k,m} = \dfrac{m^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-\frac{3}{2}}}$.

For more details about the Poincaré series $P_{k,m}^{n,r}$ for Jacobi forms, we refer to [7].

Next we define the generalized Jacobi Poincaré series which will be used to prove Theorem 3.1.

Definition 2.4 (Generalized Jacobi Poincaré series)**.** *Let $f(\tau,z)$ be a holomorphic function on $\mathscr{H}\times\mathbb{C}$ with Fourier expansion*

$$f(\tau,z)=\sum_{\substack{n.r\in\mathbb{Z},\\ 4nm>r^2}} a_f(n,r)q^n\zeta^r.$$

where the coefficients $a_f(n,r)$ satisfy the bound $a_f(n,r)=O((4nm-r^2)^{\frac{k}{2}-6-\varepsilon})$. We define the generalized Poincaré series $\mathbb{P}_{k,m}(f)(\tau,z)$ associated to the seed function f as

$$\mathbb{P}_{k,m}(f)(\tau,z)=\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} f|_{k,m}\gamma. \tag{5}$$

Remark 2.1. The bound satisfied by the coefficient in the above definition is needed for the convergence of the series. More details can be found in [10].

2.2. *Higher order heat operator for Jacobi forms.* Recall that the classical heat operator is defined as

$$L_m=\frac{1}{(2\pi i)^2}\left(8\pi im\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial z^2}\right).$$

We now state a lemma without proof which gives the relation between higher order classical heat operator and slash operator defined in (1).

Lemma 2.5 (Lemma 3.1, p. 98, [3])**.** *Let φ be a Jacobi form of weight k and index m. Then for a non-negative integer ν, $M=\begin{pmatrix}* & *\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$, and $X\in\mathbb{Z}^2$, we have*

$$L^\nu_m(\varphi|_{k,m}M)=\sum_{l=0}^{\nu}(-1)^{\nu-l}\binom{\nu}{l}\left(\frac{2mc}{\pi i}\right)^{\nu-l}\frac{(k+\nu-\frac{3}{2})!}{(k+l-\frac{3}{2})!}\frac{L^l_m(\varphi)|_{k+2l,m}M}{(c\tau+d)^{\nu-l}}$$

and

$$L^\nu_m(\varphi|_mX)=(L^\nu_m\varphi)|_mX.$$

An easy consequence of the above lemma is the following (take $\nu=1$):

$$(L_m\varphi)|_{k+2,m}M=L_m\varphi+\frac{m(2k-1)c}{\pi i(c\tau+d)}\varphi. \tag{6}$$

Therefore, in view of (2), $L_m\varphi$ is not a Jacobi form. However, we have the following:

Proposition 2.6. *The modified heat operator $\mathscr{L}_{k,m}$ defined as*

$$\mathscr{L}_{k,m}:=L_m-\frac{m}{3}(k-1/2)E_2, \tag{7}$$

maps Jacobi forms (resp. Jacobi cusp forms) of weight k and index m to Jacobi forms (resp. Jacobi cusp forms) of weight $k+2$ and index m, where $E_2=1-24\sum_{n\geqslant1}\sigma(n)q^n\left(\sigma(n)=\sum_{d|n}d\right)$ is the Eisenstein series of weight 2.

Proof. The proof is just straightforward calculations, hence omitted. □

Remark 2.2. From now on, we drop the index notation for $\mathscr{L}_{k,m}$ and write only $\mathscr{L}$; as the index is fixed throughout the paper and the weight will be clear from the context.

3. Statement of results

For $n \geqslant 1$, we define the modified heat operator of order n, $\mathscr{L}^n : J_{k,m} \to J_{k+2n,m}$ recursively as follows:

$$\mathscr{L}^0\varphi = \varphi, \quad \mathscr{L}^1\varphi = L_m\varphi - \frac{m}{3}(k-1/2)E_2\varphi, \tag{8}$$

and

$$\mathscr{L}^{n+1}\varphi = \mathscr{L}(\mathscr{L}^n\varphi) - \left(\frac{m}{3}\right)^2 n(k+n-3/2)E_4\mathscr{L}^{n-1}\varphi. \tag{9}$$

By Proposition 2.6, $\mathscr{L}^n : J^{cusp}_{k,m} \to J^{cusp}_{k+2n,m}$ is a $\mathbb{C}$-linear map. Therefore, it has an adjoint map $\mathscr{L}^{n^*} : J^{cusp}_{k+2n,m} \to J^{cusp}_{k,m}$ such that $\langle \mathscr{L}^{n^*}\varphi, \psi\rangle = \langle \varphi, \mathscr{L}^n\psi\rangle$ for any $\varphi \in J^{cusp}_{k+2n,m}$ and $\psi \in J^{cusp}_{k,m}$. The aim of this paper is to compute the adjoint map of $\mathscr{L}^n$ with respect to the Petersson scalar product. This is done by explicitly computing the Fourier coefficients of the image of φ under $\mathscr{L}^{n^*}$ for any $\varphi \in J^{cusp}_{k+2n,m}$. We now state the main result of this paper.

Theorem 3.1. *Let $k,m,n \in \mathbb{N}$ with $k \geqslant 4n+4$. For any $\varphi \in J^{cusp}_{k+2n,m}$, the (N,R)-th Fourier coefficient of $\mathscr{L}^{n^*}\varphi \in J^{cusp}_{k,m}$ is given by*

$$b(N,R) = \left(\frac{m}{\pi}\right)^{2n} \frac{\Gamma(k+2n-3/2)}{\Gamma(k-3/2)}(4Nm-R^2)^{k-3/2} \sum_{t\geqslant 0} \frac{a(t+N,R)\,\Omega^{N,R}_{k,m,n}(t)}{(4(t+N)m-R^2)^{k+2n-3/2}}, \tag{10}$$

where $a(s,t)$ is the (s,t)-th Fourier coefficient of φ, the constants $\Omega^{N,R}_{k,m,n}(t)$ is given by

$$\Omega^{N,R}_{k,m,n}(t) = \begin{cases} \displaystyle\sum_{r=0}^{n} \binom{n}{r} \frac{(k+n-3/2)!}{(k+n-r-3/2)!} \left(-\frac{m}{3}\right)^r (4Nm-R^2)^{n-r} & \text{if } t=0 \\ \displaystyle\sum_{r=1}^{n} \binom{n}{r} \frac{(k+n-3/2)!}{(k+n-r-3/2)!} \left(-\frac{m}{3}\right)^r (4Nm-R^2)^{n-r}\mathsf{e}_r(t) & \text{if } t>0, \end{cases} \tag{11}$$

and $E_2(\tau)^r = \sum_{t\geqslant 0} \mathsf{e}_r(t)q^t$.

4. Preliminary results

We first state few results which will play a crucial role in the proof of Theorem 3.1.

Proposition 4.1. *Let n be a positive integer and φ a Jacobi form of weight k and index m. Then we have*

$$\mathscr{L}^n\varphi = \sum_{r=0}^{n} A^{n,k,m}_r E_2^r L_m^{n-r}\varphi,$$

where $A^{n,k,m}_r = \binom{n}{r} \frac{(k+n-3/2)!}{(k+n-r-3/2)!} \left(-\frac{m}{3}\right)^r$.

Proof. This can easily be verified by simple induction arguments and we omit the proof. □

Next, we describe the image of Jacobi Poincaré series under $\mathscr{L}^n$ in terms of the generalized Poincaré series.

Proposition 4.2. *Let n be a positive integer and $k \geqslant 4n+4$. Then we have*

$$\mathscr{L}^n P_{k,m}^{N,R} = \mathbb{P}_{k+2n,m}(\varphi),$$

where the seed function φ is given by $\varphi(\tau,z) = q^N \zeta^R \sum_{r=0}^{n} A_r^{n,k,m} E_2^r (4Nm-R^2)^{n-r}$.

Proof. Since the proof is just a routine calculation, we give a brief sketch of the proof. For any $\varepsilon > 0$, we have $\mathsf{e}_n(t) = O(t^{2n-1+\varepsilon})$, thus the convergence condition for $\mathbb{P}_{k+2n,m}(\varphi)$ becomes $\frac{k}{2} - 2n + \varepsilon > 2$. Thus, for $k \geqslant 4n+4$, $\mathbb{P}_{k+2n,m}(\varphi)$ is given by

$$\sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} \varphi|_{k+2n,m}\gamma = \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} \sum_{r=0}^{n} \frac{A_r^{n,k,m}}{(c\tau+d)^{2r}} E_2\left(\frac{a\tau+b}{c\tau+d}\right)^r \left(L_m^{n-r}(q^N\zeta^R)\right)|_{k+2n-2r,m}\gamma. \tag{12}$$

Since $E_2\left(\frac{a\tau+b}{c\tau+d}\right)^r = \sum_{j=0}^{r} \binom{r}{j} (c\tau+d)^{r+j} \left(\frac{12c}{2\pi i}\right)^{r-j} E_2^j$, after a change of order of summation the right hand side of (12) is reduced to

$$\sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} \sum_{j=0}^{n} A_j^{n,k,m} E_2^j \left[\sum_{r=0}^{n-j} \binom{n-j}{r} \left(-\frac{2mc}{\pi i}\right)^{n-r-j} \frac{(k+n-j-3/2)!}{(k+r-3/2)!} \frac{L_m^r(q^N\zeta^R)|_{k+2r,m}\gamma}{(c\tau+d)^{n-r-j}}\right].$$

Since the expression inside bracket is $L_m^{n-j}(q^N\zeta^R|_{k,m}\gamma)$, by linearity of L_m and slash operator, we can write the above expression as $\sum_{j=0}^{n} A_j^{n,k,m} E_2^j L_m^{n-j}\left(\sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} q^N\zeta^R|_{k,m}\gamma\right)$. Now the statement follows from Proposition 4. □

5. Proof of Theorem 3.1

We write D and D_t for $(4Nm-R^2)$ and $(4(t+N)m-R^2)$ respectively. By Proposition 4.2, we have

$$\mathscr{L}^n P_{k,m}^{N,R} = \sum_{r=0}^{n} A_r^{n,k,m} D^{n-r} \mathbb{P}_{k+2n,m}\left(\sum_{t \geqslant 0} \mathsf{e}_r(t) q^{t+N}\zeta^R\right) = \sum_{t \geqslant 0} \Omega_{k,m,n}^{N,R}(t) P_{k+2n,m}^{t+N,R},$$

where $\Omega_{k,m,n}^{N,R}(t)$ is given by (11). Using Lemma 2.3 and the definition of the adjoint map, we obtain

$$\alpha_{k,m} D^{-k+3/2} b(N,R) = \left\langle \mathscr{L}^{n^*}\varphi, P_{k,m}^{N,R}\right\rangle = \left\langle \varphi, \sum_{t \geqslant 0} \Omega_{k,m,n}^{N,R}(t) P_{k+2n,m}^{t+N,R}\right\rangle = \alpha_{k+2n,m} \sum_{t=0}^{\infty} \Omega_{k,m,n}^{N,R}(t) \frac{a(t+N,R)}{D_t^{k+2n-3/2}}.$$

Upon simplification, the above equation yields

$$b(N,R) = \left(\frac{m}{\pi}\right)^{2n} \frac{\Gamma(k+2n-3/2)}{\Gamma(k-3/2)} (4Nm-R^2)^{k-3/2} \sum_{t=0}^{\infty} \frac{a(t+N,R)\,\Omega_{k,m,n}^{N,R}(t)}{(4(t+N)m-R^2)^{k+2n-3/2}}.$$

6. Applications

In this section, we apply Theorem 3.1 in some of the special cases and obtain special evaluation of certain L-series. Let E_4 and E_6 be the Eisenstein series of weight 4 and 6, respectively. We use the following

Fourier series expansion of E_4 :

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = \sum_{n=0}^{\infty}\sigma_3(n)q^n, \text{ where } \sigma_3(0) := \frac{1}{240}.$$

Example 1. Consider $\varphi_{10,1} = \frac{1}{144}(E_6E_{4,1} - E_4E_{6,1}) \in J_{10,1}^{cusp}$, where $E_{k,1} \in J_{k,1}$ is the Eisenstein series of weight k and index 1. By taking $\varphi = \varphi_{10,1}$ in Theorem 3.1, we have $\mathscr{L}^*\varphi_{10,1} \in J_{8,1}^{cusp} = \{0\}$, and consequently

$$0 = \sum_{t=0}^{\infty}\frac{c_{10}(t+N,R)\ \Omega_{8,1,1}^{N,R}(t)}{(4(t+N)-R^2)^{8+2-3/2}},$$

where $c_{10}(N,R)$ is the (N,R)-th Fourier coefficient of $\varphi_{10,1}$. Computing the involved constants explicitly, we have $\Omega_{8,1,1}^{N,R}(0) = (4N - R^2 - 5/2)$, and $\Omega_{8,1,1}^{N,R}(t) = -60\sigma(t)$ for $t > 0$. Therefore,

$$60\sum_{t=1}^{\infty}\sigma(t)\frac{c_{10}(t+N,R)}{(4(t+N)-R^2)^{17/2}} = \frac{(4N-R^2-5/2)c_{10}(N,R)}{(4N-R^2)^{17/2}}.$$

Now, if we take $N = 1$ and $R = 1$, we obtain the following special evaluation of the above series.

$$\sum_{t\geqslant 1}\sigma(t)\frac{c_{10}(4t+3)}{(4t+3)^{17/2}} = \frac{1}{3^{19/2}\times 2^2\times 5}.$$

Example 2. Consider the Jacobi cusp form $\varphi_1 = E_4\varphi_{12,1}$ of weight 16 and index 1, where $\varphi_{12,1} = \frac{1}{144}(E_4^2E_{4,1} - E_6E_{6,1}) \in J_{12,1}^{cusp}$. Then by taking $\varphi = \varphi_1$ in Theorem 3.1, we have $\mathscr{L}^{2^*}\varphi_1 \in J_{12,1}^{cusp} = \mathbb{C}\varphi_{12,1}$, and

$$c_{\varphi_1}C_{12}(N,R) = \left(\frac{1}{\pi}\right)^4\frac{\Gamma(16-3/2)}{\Gamma(12-3/2)}(4Nm-R^2)^{12-3/2}\sum_{t\geqslant 0}\frac{c(t+N,R)\ \Omega_{12,1,2}^{N,R}(t)}{(4(t+N)-R^2)^{16-3/2}}$$

for some $c_{\varphi_1} \in \mathbb{C}$, where $C_{12}(\alpha,\beta)$ and $c(\alpha,\beta)$ are the (α,β)-th Fourier coefficients of $\varphi_{12,1}$ and φ_1, respectively. Since

$$\varphi_1 = 240\left(\sum_{t_1\geqslant 0}\sigma_3(t_1)q^{t_1}\right)\left(\sum_{t_2,r\in\mathbb{Z};4t_2-r^2>0}C_{12}(t_2,r)q^{t_2}\zeta^r\right),$$

we have

$$c(t+N,R) = \sum_{t_1+t_2-N=t}\sigma_3(t_1)C_{12}(t_2,R).$$

Consequently, we have

$$\delta_{\varphi_1}C_{12}(N,R) = \sum_{t\geqslant 0}\frac{\Omega_{12,1,2}^{N,R}(t)\sum_{t_1+t_2-N=t}\sigma_3(t_1)C_{12}(t_2,R)}{(4(t+N)-R^2)^{16-3/2}},$$

for some complex constant δ_{φ_1}. Again, by putting particular values of N and R, one can obtain various arithmetic relations between the Fourier coefficients of $\varphi_{12,1}$ and E_4.

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References

[1] J. H. Bruinier, G. Geer, G. Harder, and D. Zagier, *The 1-2-3 of Modular Forms*, Lectures at a Summer School in Nordfjordeid, Norway *Springer Berlin, Heidelberg*, (2008).

[2] M. Charan, On the adjoint of higher order Serre derivatives, *Res. number theory* **9** (2023), Paper No. 30.

[3] Y. Choie, Jacobi forms and the heat operator, *Math. Z.* **1** (1997), 95–101.

[4] Y. Choie and M. H. Lee, Notes on Rankin-cohen brackets, *Ramanujan J.* **25** (2011), 141–147.

[5] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.* **217** (1977), 81–94.

[6] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. in Math. vol. 55, Birkhäuser, Boston, (1985).

[7] B. Gross, W. Kohnen and D. Zagier, Heegner Points and Derivatives of L-series II, *Math. Ann.* **278** (1987), 497–562.

[8] S. D. Herrero, The adjoint of some linear maps constructed with the Rankin–Cohen brackets, *Ramanujan J* **36** (2015), 529–536.

[9] A. K. Jha and A. Kumar, *Construction of cusp forms using Rankin-Cohen brackets*, Analytic number theory, modular forms and q-hypergeometric series, Springer Proc. Math. Stat., 221, Springer, Cham, (2017), 329–341.

[10] A. K. Jha and S. Pandey, Rankin-Cohen bracket of Jacobi forms and Poincaré series, Preprint (2023).

[11] A. K. Jha and B. Sahu, Rankin–Cohen brackets on Jacobi forms and the adjoint of some linear maps, *Ramanujan J.* **39**(3) (2016), 533–544.

[12] A. K. Jha and B. Sahu, Rankin-Cohen brackets on Siegel modular forms and special values of certain Dirichlet series, *Ramanujan J.* **44**(1) (2017), 63–73.

[13] A. K. Jha and B. Sahu, Rankin–Cohen brackets on Jacobi forms of several variables and special values of certain Dirichlet series, *Int. J. Number Theory* **15**(5) (2019), 925–933.

[14] W. Kohnen, Cusp forms and special value of certain Dirichlet series, *Math. Z.* **207** (1991), 657–660.

[15] A. Kumar, The adjoint map of the Serre derivative and special values of shifted Dirichlet series, *J. Number Theory* **177** (2017), 516–527.

[16] R. A. Rankin, The Construction of automorphic forms from the derivatives of a given form, *J. Indian Math. Soc.* **20** (1956), 103–116.

[17] R. A. Rankin, The construction of automorphic forms from the derivatives of given forms, *Michigan Math. J.* **4** (1957), 181–186.

[18] S. Sumukha and S. K. Singh, Rankin–Cohen brackets on Hermitian Jacobi forms and the adjoint of some linear maps, *Funct. Approx. Comment. Math.* **65**(1) (2021), 61–72.

[19] B. Williams, Rankin-Cohen brackets and Serre derivatives as Poincaré series *Res. Number Theory* **4** (2018), Paper No. 37.

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