SOME RESULTS ON (m,n,C)-ISOSYMMETRIC COMMUTING d- TUPLES OF OPERATORS ON A HILBERT SPACE

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Abstract. The study of the d-tuples of commuting operators is currently booming topics. In the current paper, closely related to the problem of generalizing the class of m-isometric c.t.o. and n-symmetric c.t.o., we introduce a new class of d-tuples of commuting operators. Specially, we introduce the class of (m,n,C)-isosymmetric c.t.o. and we show a variety of results which improve and extend some works related to (m,C)-isometric and n-complex symmmetric c.t.o.

1. Introduction and preliminaries

We set below the notations used throughout this paper. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. We use the notations $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}_+$ the set of nonnegative integers, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. Recall from [18] that a conjugation on $\mathcal{H}$ is a map $C:\mathcal{H}\longrightarrow\mathcal{H}$ which is antilinear, involutive ($C^2=I_{\mathcal{H}}$). Moreover C satisfies the following properties

$$\begin{cases} \langle Cx \mid Cy\rangle = \langle y \mid x\rangle \quad \text{for } all \ x,y\in\mathcal{H}, \\ CTC\in\mathcal{B}(\mathcal{H}) \ \text{ for every } T\in\mathcal{B}(\mathcal{H}), \\ \big(CTC\big)^r = CT^rC \quad \text{ for all } r\in\mathbb{N}, \\ \big(CTC\big)^* = CT^*C. \end{cases}.$$

See [2] for more properties of conjugation operators.

Recall that an operator $T\in\mathcal{B}(\mathcal{H})$ is said to be:

(i) n-complex symmetric for some conjugation C [13, 14] if

$$\sum_{0\le k\le n}(-1)^k\binom{n}{k}T^{*n-k}CT^kC=0,$$

(ii) (m,C)-isometric for some conjugation C [12] if

$$\sum_{0\le k\le m}(-1)^k\binom{m}{k}T^{*m-k}CT^{m-k}C=0,$$

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(iii) (m,n,C)-isosymmetic for some conjugation C [11] if $\gamma_{m,n}(T,C)=0$ where

$$\begin{aligned}\gamma_{m,n}(T,C) &= \sum_{0\leq k\leq n}(-1)^k\binom{n}{k}T^{*n-k}\Big(\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}CT^{m-k}C\Big)CT^kC\\ &= \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}\Big(\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}T^{*n-k}CT^kC\Big)CT^{m-k}C.\end{aligned}$$

It should be noted that the class of (m,n,C)-isosymmetric operators contains (m,C)-isometric and n-complex symmetric operators.

For $d\in\mathbb{N}$, let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d:=\underbrace{\mathcal{B}(\mathcal{H})\times...\times\mathcal{B}(\mathcal{H})}_{\text{d-times}}$ with $T_j:\mathcal{H}\longrightarrow\mathcal{H}$ be a tuple of commuting bounded linear operators that is $\big[T_i,T_j\big]:=T_iT_j-T_jT_i=0$. Let $\gamma=(\gamma_1,\cdots,\gamma_d)\in\mathbb{Z}_+^d$ and set $|\gamma|:=\sum_{1\leq j\leq d}\gamma_j$ and $\gamma!:=\prod_{1\leq k\leq d}\gamma_k!$. Further, define $\mathbf{T}^\gamma:=T_1^{\gamma_1}\cdots T_d^{\gamma_d}$ where $T^{\gamma_j}=\underbrace{T_j\cdots T_j}_{\gamma_j-\text{times}}$ $(1\leq j\leq d)$ and $\mathbf{T}^*=(T_1^*,\cdots,T_d^*)$.

Several variables operator theory is a relevant part of functional analysis. Due to the importance of this field, the interest in studying tuples of operators has grown considerably in the recent few years, see for instance [1, 3, 4, 5, 6, 9, 10, 15, 16, 19, 20, 21, 22, 24, 28] and the references therein.

Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting d-tuples of operators (abbreviated c.t.o.), we set

$$\Phi_n(\mathbf{T})=\sum_{0\leq k\leq n}(-1)^{n-j}\binom{m}{k}\big(T_1^*+\cdots+T_d^*\big)^k\big(T_1+\cdots+T_d\big)^{n-k}$$

$$\Psi_m(\mathbf{T})=\sum_{0.\leq k\leq m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\mathbf{T}^\gamma\Big)$$

$$\alpha_n(\mathbf{T},C)=\sum_{0\leq k\leq n}(-1)^{n-j}\binom{n}{k}\big(T_1^*+\cdots+T_d^*\big)^kC\big(T_1+\cdots+T_d\big)^{n-k}C,$$

$$\beta_m(\mathbf{T},C)=\sum_{0.\leq k\leq m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}C\mathbf{T}^\gamma C\Big)$$

and

$$\begin{aligned}\Lambda_{m,\,n}(\mathbf{T}) &= \sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}\big(T_1^*+\cdots+T_d^*\big)^k\Psi_m\big(\mathbf{T}\big)\big(T_1+\cdots+T_d\big)^{n-k}\\ &= \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\Phi_n\big(\mathbf{T}\big)\mathbf{T}^\gamma\Big).\end{aligned}$$

A d-tuples of commuting operators $\mathbf{T}=(T_1,\cdots,T_d)$ is said to be

(i) n-symmetric c.t.o. if $\Phi_n(\mathbf{T})=0$ ([9, 10, 19]),

(ii) m-isometric c.t.o. if $\Psi_m(\mathbf{T})=0$ ([22, 24, 20]),

(iii) n-complex symmetric c.t.o. if $\alpha_n(\mathbf{T},C)=0$ ([10]),

(iv) (m,C)-isometric c.t.o. if $\beta_m(\mathbf{T},C)=0$ ([27])

(v) (m,n)-isosymmetric c.t.o. if $\Lambda_{m,\,n}(\mathbf{T})=0$ ([16, 17, 28]).

It should be noted that the class of (m,n)-isosymmetric c.t.o. contains m-isometric c.t.o and n-symmetric c.t.o.

Over the past few years, various aspects of the problem of generalizing the class of m-isometric c.t.o. and n-symmetric c.t.o. have appeared in the literature. For example, (m,C)-isometric c.t.o. [27] and n-complex symmmetric c.t.o. [10] and (m,n)-isosymmetric c.t.o. [28] have been studied in Hilbert spaces. In the current paper, closely related to this problem of generalization, we introduce a new class of operators, and we investigate numerous properties of this class. Specifically, we introduce the class of (m,n,C)-isosymmetric c.t.o. and extend some classical theorems on (m,C)-isometric and n-complex symmmetric c.t.o. to the class of (m,n,C)-isosymmetric c.t.o. Our results provide a natural extension of many known ones in the literature and, in particular, of those obtained in the works [1, 6, 9, 10, 16, 19, 27, 28].

2. $(m,\ n,C)$-ISOSYMMETRIC COMMUTING TUPLES OF OPERATORS

This section deals with the study of the class of (m,n,C)-isosymmetric c.t.o.

Definition 2.1. A commuting tuple $\mathbf{T}=(T_1,\cdots,T_d)$ is said to be an $(m,n;\ C)$-isosymmetric c.t.o. if there exists a conjugation C such that $\mathcal{Q}_{m\,n}(\mathbf{T},C)=0$, where

$$\begin{aligned}\mathcal{Q}_{m,\,n}(\mathbf{T},C) &= \sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}\big(T_1^*+\cdots+T_d^*\big)^k\beta_m\big(\mathbf{T},C\big)C\big(T_1+\cdots+T_d\big)^{n-k}C\\ &= \sum_{0.\leq k\leq m}(-1)^{m-k}\binom{m}{k}\bigg(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\bigg).\end{aligned}\tag{2.1}$$

Here are a few straightforward yet important observations.

Remark 2.2. (1) When $d=1$, Definition 2.1 goes back to [11, Definition 3.1].

(2) $\mathcal{Q}_{m,0}\big(\mathbf{T},C\big)=\beta_m\big(\mathbf{T},C\big)$ and $\mathcal{Q}_{0,n}\big(\mathbf{T},C\big)=\alpha_n\big(\mathbf{T},C\big)$.

Example 2.3. (i) If $\mathbf{T}=(T_1,\cdots,T_d)$ is an (m,C)-isometric c.t.o., then $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. for any $n\in\mathbb{N}$.

(ii) If $\mathbf{T}=(T_1,\cdots,T_d)$ is an n-complex symmetric c.t.o., then $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. for any $m\in\mathbb{N}$.

Remark 2.4. We point out the following special cases:

$$\mathcal{Q}_{1,\,0}\big(\mathbf{T},C\big)=\sum_{1\leq k\leq d}T_k^*CT_kC-I,\tag{2.2}$$

$$\mathcal{Q}_{0,\,1}\big(\mathbf{T},C\big)=\sum_{1\leq k\leq d}\big(CT_kC-T_k^*\big),\tag{2.3}$$

$$(2.4)\ \mathcal{Q}_{1,\,1}\big(\mathbf{T},C\big)=\Big(\sum_{k=1}^{d}T_k^*\Big)\Big(\sum_{1\leq j\leq d}T_j^*CT_jC-I\Big)-\Big(\sum_{1\leq j\leq d}T_j^*CT_jC-I\Big)\Big(\sum_{k=1}^{d}CT_kC\Big)$$

or

$$\mathcal{Q}_{1,1}\big(\mathbf{T},C\big)=\sum_{1\leq k\leq d}\Big(T_k^*\sum_{1\leq j\leq d}\big(T_j^*-CT_jC\big)\big(CT_kC\Big)-\sum_{1\leq j\leq d}\big(T_j^*-CT_jC\big). \tag{2.5}$$

In the following example we give a tuple of c.t.o. which is (m,n,C)-isosymmetric but neither (m,C)-isometric nor n-complex symmetric c.t.o.

Example 2.5. (1) Let C be a conjugation on $\mathbb{C}^2$ defined by $C(x_1,x_2)=(\overline{x_2},\overline{x_1})$. Consider $\mathbf{T}=(T_1,T_2)$ such that

$$T_1=T_2=\begin{pmatrix}\frac{1}{\sqrt{2}} & 1\\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix}.$$

Direct computation shows that $\mathbf{T}$ is a $(1,1,C)$-isosymmetric but is not $(1,C)$-isometric tuples and not 1-complex symmetric

From Example 2.5 we observe that the class of (m,n,C)-isosymmetric c.t.o is significantly large than the class of (m,C)-isometric and n-complex symmetric c.t.o.

Remark 2.6. Let $(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d$ and let C be a conjugation on $\mathcal{H}$.

(1) Since the operators $T_1,,\cdots,T_d$ are commuting,then every permutation of an (m,n,C)-isosymmetric c.t.o. is also an (m,n,C)-isosymmetric c.t.o.

(2) $\mathbf{T}=(T_1,\cdots,T_d)$ is an (m,n,C)-isosymmetric c.t.o. if and only if $C\mathbf{T}C:=\big(CT_1C,\cdots,CT_dC\big)$ is an (m,n,C)-isosymmetric c.t.o.

(3) Under some conditions, the classes of (m,n)-isosymmetric c.t.o and (m,n,C)-isosymmetric c.t.o. coincide. In fact, if $T_jC=CT_j$ for all $j=1,2,\cdots,d$, then $\mathbf{T}$ is an (m,n,C)- isosymmetric c.t.o. if and only if $\mathbf{T}$ is an (m,n)-isosymmetric c.t.o.

In the following lemmas, we state some immediate consequences of Definition 2.1.

Lemma 2.7. *Let $\mathbf{T}=(T_1,\cdots,T_d)$ be a c.t.o., The following characterizations hold.*

(i) $\mathbf{T}$ is a $(1,n,C)$-isosymmetric c.t.o. if and only if

$$\sum_{1\leq j\leq d}T_j^*\alpha_n\big(\mathbf{T},C\big)CT_jC-\alpha_n\big(\mathbf{T},C\big)=0. \tag{2.6}$$

(ii) $\mathbf{T}$ is a $(2,n,C)$-isosmmetric c.t.o. if and only if

$$\alpha_n\big(\mathbf{T},C\big)-2\sum_{1\leq j\leq d}T_j^*\alpha_n\big(,\mathbf{T},C\big)CT_jC\ +\sum_{1\leq j\leq d}T_j^{*2}\alpha_n\big(\mathbf{T},C\big)CT_j^2C$$
$$+2\sum_{1\leq j<k\leq d}T_j^*T_k^*\alpha_n\big(\mathbf{T},C\big)CT_jT_kC=0. \tag{2.7}$$

Lemma 2.8. *Let $\mathbf{T}=(T_1,\cdots,T_d)$ be c.t.o., the following hold.*

(i) $\mathbf{T}$ is an $(m,1,C)$-isosymmetric c.t.o. if and only if

$$\Big(\sum_{1\leq k\leq d}T_k^*\Big)\beta_m\big(\mathbf{T},C\big)-\beta_m\big(\mathbf{T},C\big)\Big(\sum_{1\leq k\leq d}CT_kC\Big)=0. \tag{2.8}$$

(ii) $\mathbf{T}$ is an $(m,2,C)$-isosymmetric c.t.o. if and only if

$$\begin{aligned}\beta_m\big(\mathbf{T},C\big)\Big(\sum_{1\leq k\leq d}CT_kC\Big)^2 \;-\; 2\Big(\sum_{1\leq k\leq d}T_k^*\Big)\beta_m\big(\mathbf{T},C\big)\Big(\sum_{1\leq k\leq d}CT_kC\Big)&\\ +\Big(\sum_{1\leq k\leq d}T_k^*\Big)^2\beta_m\big(\mathbf{T},C\big)=0.&\end{aligned} \tag{2.9}$$

Example 2.9. Let $T\in\mathcal{B}(\mathcal{H})$ be an (m,n,C)-isosymmetry single operator, $d\in\mathbb{N}$ and $\lambda=(\lambda_1,\cdots,\lambda_d)\in(\mathbb{R}^d,\ \|.\|_2)$ with $\|\lambda\|_2^2=\sum_{1\leq j\leq d}\lambda_j^2=1$. Then $\mathbf{T}=(T_1,\cdots,T_d)$ where $T_j=\lambda_jT$ for $j=1,\cdots,d$ is an (m,n,C)-isosymmetric c.t.o.

In fact, it is obvious that $T_jT_k=T_kT_j$ for all $1\leq j,\ k\leq d$. From the multinomial expansion, we get for any natural number q

$$\begin{aligned}1=\Big(\lambda_1^2+\cdots+\lambda_d^2\Big)^q &= \sum_{\gamma_1+\gamma_2+\cdots+\gamma_d=q}\binom{q}{\gamma_1,\cdots,\gamma_p}\prod_{1\leq l\leq d}\lambda_l^{2\gamma_i}\\ &= \sum_{|\gamma|=q}\frac{q!}{\gamma!}|\lambda\gamma|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\beta_m(\mathbf{T},C) &= \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{T}^{*\gamma}C\mathbf{T}^{\gamma}C\Big)\\ &= \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\gamma|=j}\frac{j!}{\gamma!}\prod_{1\leq j\leq d}\lambda_j^{2\gamma_j}R^{*|\gamma|}CR^{|\gamma|}C\Big)\\ &= \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^jC.\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_{m,\,n}(\mathbf{T},C) &= \sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}\big(T_1^*+\cdots+T_d^*\big)^k\beta_m\big(\mathbf{T},C\big)C\big(T_1+\cdots+T_d\big)^{n-k}C\\ &= \Big(\sum_{1\leq j\leq d}\lambda_j\Big)^n\Big(\sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}T^{*k}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}T^{*(m-j)}CT^{m-j}C\Big)CT^{n-k}C\Big)\\ &= 0.\end{aligned}$$

Hence, $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o.

The following theorem will be required later.

Theorem 2.10. *Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators and C be a conjugation on $\mathcal{H}$. The following statements hold:*

$$\mathcal{Q}_{m+1,\,n}\big(\mathbf{T},\ C)\big) = \sum_{1\leq j\leq d} T_j^*\mathcal{Q}_{m,\,n}\big(\mathbf{T},\ C\big)\big(CT_jC\big) - \mathcal{Q}_{m,\,n}\big(\mathbf{T},\ C\big). \tag{2.10}$$

$$\mathcal{Q}_{m,\,n+1}\big(\mathbf{T},\ C)\big) = \sum_{1\leq j\leq d} T_j^*\mathcal{Q}_{m,\,n}\big(\mathbf{T},\ C\big) - \sum_{1\leq j\leq d} \mathcal{Q}_{m,\,n}\big(\mathbf{T},\ C\big)(CT_jC). \tag{2.11}$$

Proof. According to [27, Proposition 1] we have

$$\beta_{m+1}\big(\mathbf{T}, C\big) = \sum_{1\leq i\leq d} T^{*i}\beta_m\big(\mathbf{T}, C\big)\,(CT_iC) - \beta_m\big(\mathbf{T}, C\big).$$

From this we have

$$\begin{aligned}
&\mathcal{Q}_{m+1,\,n}\big(\mathbf{T},\ C)\big) \\
= &\sum_{0\leq k\leq n} (-1)^{n-k}\binom{n}{k}\big(T_1^* + \cdots + T_d^*\big)^k\beta_{m+1}\big(\mathbf{T}, C\big)C\big(T_1 + \cdots + T_d\big)^{n-k}C \\
= &\sum_{0\leq k\leq n} (-1)^{n-k}\binom{n}{k}\big(T_1^* + \cdots + T_d^*\big)^k\left[\sum_{i=1}^{d} T^{*i}\beta_m\big(\mathbf{T}, C\big)\,(CT_iC) - \beta_m\big(\mathbf{T}, C\big)\right] \\
&C\big(T_1 + \cdots + T_d\big)^{m-k}C \\
= &\sum_{i=1}^{d} T^{*i}\left[\sum_{0\leq k\leq n} (-1)^{n-k}\binom{n}{k}\big(T_1^* + \cdots + T_d^*\big)^k\beta_m\big(\mathbf{T}, C\big)C\big(T_1 + \cdots + T_d\big)^{m-k}C\right](CT_iC) \\
&+ \sum_{0\leq k\leq n} (-1)^{n-k}\binom{n}{k}\big(T_1^* + \cdots + T_d^*\big)^k\beta_m\big(\mathbf{T}, C\big)C\big(T_1 + \cdots + T_d\big)^{m-k}C \\
= &\sum_{1\leq i\leq d} T_i^*\mathcal{Q}_{m,\,n}\big(\mathbf{T},\ C\big)CT_iC - \mathcal{Q}_{m,\,n}\big(\mathbf{T},\ C\big).
\end{aligned}$$

□

Corollary 2.11. *If $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is an (m, n, C)-isosymmetric c.t.o., then $\mathbf{T}$ is an $(m'; n',\ C)$-isosymmetric c.t.o. for all $n' \geq n$ and $m' \geq m$.*

Proposition 2.12. *Let $\mathbf{T} = (T_1, \cdots, T_d)$ be a c. t.o. which is a $(2, n, C)$-isosymmetric c.t.o. for some conjugation C. Then the following hold:*

$$\sum_{|\gamma|=k} \frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T}, C\big)C\mathbf{T}^{\gamma}C = (1-k)\alpha_n\big(\mathbf{T}, C\big) + k\bigg(\sum_{1\leq j\leq d} T_j^*\alpha_n\big(\mathbf{T}, C\big)CT_jC\bigg),\ \ \forall\, k \in \mathbb{N}. \tag{2.12}$$

$$\lim_{k\to\infty} \frac{1}{k}\sum_{|\gamma|=k} \frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T}, C\big)C\mathbf{T}^{\gamma}C = \mathcal{Q}_{1,n}\big(\mathbf{T}, C\big). \tag{2.13}$$

Proof. We agree (2.12) by induction on k. For $k=0,1$ it is obvious. Assume that (2.12) is true for k. Direct calculation gives

$$\begin{aligned}
&\sum_{|\gamma|=k+1}\frac{(k+1)!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\\
&= \sum_{\gamma_1+\cdots+\gamma_d=k+1}\frac{(k+1)k!}{\gamma_1!\cdots\gamma_d!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\\
&= \sum_{\gamma_1+\cdots+\gamma_d=k+1}\frac{(\gamma_1+\cdots+\gamma_d)k!}{\gamma_1!\cdots\gamma_d!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\\
&= \sum_{1\le r\le d}\Bigg(\sum_{\gamma_1+\cdots+\gamma_r-1+\cdots+\gamma_d=k}\frac{k!}{\gamma_1!\cdots(\gamma_r-1)!\cdots\gamma_d!}T_r^*T_1^{*\gamma_1}\cdots T_r^{*\gamma_r-1}\cdots T_d^{*\gamma_d}\\
&\qquad\qquad \cdot\alpha_n\big(\mathbf{T},C\big)CT_1^{\gamma_1}\cdots T_r^{\gamma_r-1}\cdots T_d^{\gamma_d}T_rC\Bigg)\\
&= \sum_{1\le j\le d}T_j^*\Bigg(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\Bigg)CT_jC \quad (\text{since } C^2=I_{\mathcal{H}}).
\end{aligned}$$

By taking into account that $\mathbf{T}$ is a $(2,n,C)$-isosymmetric c.t.o., the induction hypothesis and (2.7) we may write

$$\begin{aligned}
&\sum_{|\gamma|=k+1}\frac{(k+1)!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\\
&= \sum_{1\le r\le d}T_r^*\Bigg(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\Bigg)CT_rC\\
&= \sum_{1\le j\le d}T_j^*\Bigg((1-k)\alpha_n\big(\mathbf{T},C\big)+k\sum_{1\le j\le d}T_j^*\alpha_n\big(\mathbf{T},C\big)CT_jC\Bigg)CT_jC\\
&= (1-k)\sum_{1\le j\le d}T_j^*\alpha_n\big(\mathbf{T},C\big)CT_jC+k\underbrace{\sum_{1\le j,r\le d}T_r^*T_j^*\alpha_n\big(\mathbf{T},C\big)CT_rT_jC}_{K}\\
&= (1-k)\sum_{1\le r\le d}T_r^*\alpha_n\big(\mathbf{T},C\big)CT_rC+k\underbrace{\Bigg(\sum_{1\le r\le d}T_r^{*2}\alpha_n\big(\mathbf{T},C\big)CT_r^2C+2\Bigg(\sum_{1\le j<r\le d}T_j^*T_r^*\alpha_n\big(\mathbf{T},C\big)CT_jT_rC\Bigg)\Bigg)}_{J}\\
&= (1-k)\sum_{1\le r\le d}T_r^*\alpha_n\big(\mathbf{T},C\big)CT_rC+k\Bigg(-\alpha_n\big(\mathbf{T},C\big)I+2\sum_{1\le j\le d}T_j^*\alpha_n\big(\mathbf{T},C\big)CT_jC\Bigg)\\
&= -k\alpha_n\big(\mathbf{T},C\big)+(k+1)\Bigg(\sum_{1\le r\le d}T_r^*\alpha_n\big(\mathbf{T},C\big)CT_rC\Bigg).
\end{aligned}$$

Therefore, (2.12) holds for $k+1$. From identity (2.12) we get

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C &= -\alpha_n\big(\mathbf{T},C\big)+\Big(\sum_{1\le r\le d}T_r^*\alpha_n\big(\mathbf{T},C\big)CT_rC\Big).\\ &= \mathcal{Q}_{1,n}\big(\mathbf{T},C\big).\end{aligned}$$

□

Theorem 2.13. *Let $\mathbf{T}=(T_1,\cdots,T_d)$ be a c.t.o. and C be a conjugation on $\mathcal{H}$. Then the following hold:*

$$\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C=\sum_{0\le j\le k}\binom{k}{j}\mathcal{Q}_{j,n}\big(\mathbf{T},C\big), \tag{2.14}$$

for every integer $k\ge 1$.

(i) $\mathbf{T}$ *is an (m,n,C)-isosymmetric c.t.o. if and only if*

$$\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C=\sum_{0\le j\le m-1}\binom{k}{j}\mathcal{Q}_{j,n}\big(\mathbf{T},C\big);\quad \forall\, k\in\mathbb{N}. \tag{2.15}$$

(ii) *If $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o., then*

$$\mathcal{Q}_{m-1,n}\big(\mathbf{T},C\big)=\lim_{k\to\infty}\frac{1}{\binom{k}{m-1}}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C. \tag{2.16}$$

Proof. We argue (2.14) by induction. For $k=1$ it is obvious that (2.14) is true. Now assume that (2.14) is true for k. We shall deduce it at step $n+1$. By taking into account (2.1) and (2.14), we get

$$\begin{aligned}
&\sum_{|\gamma|=k+1}\frac{k!}{\beta!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\\
= &\ \mathcal{Q}_{k+1,n}\big(\mathbf{T},C\big)-\sum_{0\le j\le k}(-1)^{k+1-j}\binom{k+1}{j}\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\\
= &\ \mathcal{Q}_{k+1,n}\big(\mathbf{T},C\big)-\sum_{0\le j\le k}(-1)^{k+1-j}\binom{k+1}{j}\sum_{0\le r\le j}\binom{j}{r}\mathcal{Q}_{r,n}\big(\mathbf{T},C\big))\\
= &\ \mathcal{Q}_{k+1,n}\big(\mathbf{T},C\big)-\sum_{0\le r\le k}\mathcal{Q}_{r,n}\big(\mathbf{T},C\big)\sum_{r\le j\le k}(-1)^{k+1-j}\binom{k+1}{j}\binom{j}{r}\\
= &\ \mathcal{Q}_{k+1,n}\big(\mathbf{T},C\big)-\sum_{0\le r\le j}\mathcal{Q}_{r,n}\big(\mathbf{T},C\big)\Big(\sum_{r\le j\le k}(-1)^{k+1-j}\binom{k+1}{r}\binom{k+1-r}{j-r}\Big)\\
= &\ \mathcal{Q}_{k+1,n}\big(\mathbf{T},C\big)-\sum_{0\le r\le k}\binom{k+1}{r}\mathcal{Q}_{r,n}\big(\mathbf{T},C\big)\Big(\sum_{r\le j\le k}(-1)^{k+1-j}\binom{k+1-r}{j-r}\Big)\\
= &\ \mathcal{Q}_{k+1,n}\big(\mathbf{T},C\big)-\sum_{0\le r\le k}\binom{k+1}{r}\mathcal{Q}_{r,n}\big(\mathbf{T},C\big)\Big(-1+\underbrace{\sum_{0\le r\le k+1-j}(-1)^{k+1-j-r}\binom{k+1-j}{r}}_{=0}\Big)\\
= &\sum_{0\le j\le k+1}\binom{k+1}{j}\mathcal{Q}_{j,n}\big(\mathbf{T},C\big).
\end{aligned}$$

(i) If $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o., then $\mathcal{Q}_{k,n}\big(\mathbf{T},C\big)=0$ for all $k\ge m$ (by Corollary 2.11). Hence (2.15) *follows from* (2.14)*. However, if* (2.15) *holds for all $k\ge 1$. Then $\mathcal{Q}_{k,n}\big(\mathbf{T},C\big)=0$ for $k\ge m$ by* (2.14)*, therefore $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o.*

(ii) If $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o., we have by (2.15),

$$\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C=\sum_{0\le j\le m-2}\binom{k}{j}\mathcal{Q}_{j,n}\big(\mathbf{T},C\big)+\binom{k}{m-1}\mathcal{Q}_{m-1,n}\big(\mathbf{T},C\big).$$

If we put this equation in the form

$$\frac{1}{\binom{k}{m-1}}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C=\sum_{0\le j\le m-2}\frac{1}{\binom{k}{m-1}}\binom{k}{j}\mathcal{Q}_{j,n}\big(\mathbf{T},C\big)+\mathcal{Q}_{m-1,n}\big(\mathbf{T},C\big),$$

then, we find that

$$\mathcal{Q}_{m-1,n}\big(\mathbf{T},C\big)=\lim_{k\to\infty}\frac{1}{\binom{k}{m-1}}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C.$$

□

To establish our next result, we require the following lemma. which is quoted from [8].

Lemma 2.14. ([8]) *Let* $(e_r)_{r\geq 0}$ *and* $(h_j)_{j\geq 0}$ *be sequences of real numbers and let* $(c_{r,j})_{r,j\geq 0}$ *be a double sequence of real numbers. Then*

$$\sum_{0\leq r\leq k} e_r\Big(\sum_{0\leq j\leq r} c_{r,j}h_j\Big) = \sum_{0\leq j\leq r} h_j\Big(\sum_{j\leq r\leq k} c_{r,j}e_r\Big). \tag{2.17}$$

The following Corollary gives a description of an (m,n,C)-isosymmetric c.t.o.

Corollary 2.15. *Let* $\mathbf{T}=(T_1,\cdots,T_d)$ *be c.t.o. and* C *be a conjugation on* $\mathcal{H}$*. Then* $\mathbf{T}$ *is an* (m,n,C)*-isosymmetric c.t.o. if and only if*

$$\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C = \sum_{0\leq j\leq m-1}\Big(\sum_{j\leq r\leq m-1}(-1)^{r-j}\binom{k}{r}\binom{r}{j}\Big)\Big(\sum_{|\gamma|=r}\frac{r!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\Big). \tag{2.18}$$

Proof. Suppose that $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. According to (2.15), (2.1) and (2.17), we my write

$$\begin{aligned}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}(\mathbf{T},C)C\mathbf{T}^{\beta}C &= \sum_{0\leq j\leq m-1}\binom{k}{j}\mathcal{Q}_{j,n}(\mathbf{T},C)\\ &= \sum_{0\leq j\leq m-1}\binom{k}{j}\Big(\sum_{0\leq r\leq j}(-1)^{j-r}\binom{j}{r}\sum_{|\gamma|=r}\frac{r!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\Big)\\ &= \sum_{0\leq r\leq m-1}\Big(\sum_{r\leq j\leq m-1}(-1)^{j-r}\binom{k}{j}\binom{j}{r}\Big)\sum_{|\gamma|=r}\frac{r!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C.\end{aligned}$$

So, (2.18) is now verified.

Now assume that (2.18) holds. According to [22, Lemma 2.3] and Corollary 2.15 it follows that $\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C$ is a polynomial in j of degree $\leq m-1$, that is,

$$\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C = \psi_0(\mathbf{T},C,n)+\psi_1(\mathbf{T},C,n)j+\cdots+\psi_{m-1}(\mathbf{T},C,n)j^{m-1}.$$

As an application of the identities

$$\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}j^q=0 \qquad \text{for } q=0,1,\cdots,m$$

(see [23, Lemma 3.3]), it is not hard to see that

$$\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\Big)=0.$$

This yields that $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. □

We give the following theorem, which is one of the most important results of this section.

Theorem 2.16. *Let $\mathbf{T}=(T_1,\cdots,T_d)$ be an (m,n,C)-isosymmetric c.t.o. and satisfies the following condition*

$$\sup_k \Big\| \sum_{|\gamma|=k} \frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\Big\| < \infty. \tag{2.19}$$

then

$$\sum_{1\leq j\leq m} T_j^*\alpha_n\big(\mathbf{T},C\big)(CT_jC)-\alpha_n\big(\mathbf{T},C\big)=0 \tag{2.20}$$

i.e., $\mathbf{T}$ is a $(1,n,C)$-isosymmetric c.t.o.

Proof. by referring to the assumption that we have $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. and (2.15) we can see that

$$\sum_{|\gamma|=k} \frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C = \sum_{0\leq j\leq m-1}\binom{k}{j}\mathcal{Q}_{j,n}\big(\mathbf{T},C\big)$$

for every $k\in\mathbb{N}$. To do this, there exist operators $\psi_j\big(\mathbf{T},C,n\big)$ for $j=0,1,\cdots,m-1$ which may be written

$$\sum_{1\leq i\leq n} T_i^{*k}\alpha_n\big(\mathbf{T},C\big)(CT_i^kC)=\sum_{0\leq j\leq m-1}\psi_j\big(\mathbf{T},C,n\big)k^j. \tag{2.21}$$

In that case, (2.19) tells us that

$$M=\sup_k \Big\| \sum_{|\gamma|=k} \frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n\big(\mathbf{T},C\big)C\mathbf{T}^{\gamma}C\Big\| < \infty.$$

Then we have

$$0\leq \sup\Big\{\|\sum_{0\leq j\leq m-1}\psi_j\big(\mathbf{T},C,n\big)k^j\| \;:\; k=1,2,...\Big\}\leq M.$$

Since k is arbitrary, we have $\psi_j\big(\mathbf{T},C,n\big)=0$ for $j=1,\cdots,m-1$. Hence

$$\sum_{1\leq j\leq m} T_j^{*k}\alpha_n\big(\mathbf{T},C\big)(CT_j^kC)-\alpha_n\big(\mathbf{T},C\big)=0$$

Since k is arbitrary, letting $k=1$ we have a desired equality. □

It has been proven in [27] that the class of (m,C)-isometric c.t.o. is norm closed in $\mathcal{B}(\mathcal{H})^d$. So here we would like to prove that this property remains valid for the class of (m,n,C)-isometric c.t.o.

Theorem 2.17. *Let $\big(\mathbf{T}_q=(T_{1q},\cdots,T_{dq})\big)_q$ be a sequence of an (m,n,C)-isosymmetric c.t.o. for some conjugation C such that $\mathbf{T}_q\longrightarrow\mathbf{T}:=(T_1,\cdots,T_d)$ as $q\to\infty$ in the strong topology of $\mathcal{B}(\mathcal{H})^d$. Then $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. where $\|\mathbf{T}\|^2=\sum_{1\leq j\leq d}\|T_j\|^2$.*

Proof. We consider $\big(\mathbf{T}_q=(T_{1q},\cdots,T_{dq})\big)_q$ be a sequence of an (m,n,C)-isosymmetric c.t.o. for which

$$\|\mathbf{T}_q-\mathbf{T}\|^2=\sum_{1\leq j\leq d}\|T_{qj}-T_j\|^2\longrightarrow 0 \text{ as } q\longrightarrow\infty.$$

We note in particular that

$$\|T_{qj}-T_j\|\longrightarrow 0 \quad (q\longrightarrow\infty) \quad \text{for all} \quad j=1,2,\cdots,d,$$

We will furthermore see that

$$\|T_{qj}^{\gamma_j}-T_j^{\gamma_j}\|\longrightarrow 0 \quad (q\longrightarrow\infty) \text{ for all } j=1,2,\cdots,d$$

which ensures that

$$\|\mathbf{T}_q^{\gamma}-\mathbf{T}^{\gamma}\|\longrightarrow 0 \quad (q\longrightarrow\infty).$$

However

$$\lim_{q\longrightarrow\infty}\Big\|\sum_{1\leq j\leq d}\big(T_{jq}-T_j\big)\Big\|=\lim_{q\longrightarrow\infty}\Big\|\sum_{1\leq j\leq d}\big(T_{jq}^*-T_j^*\big)\Big\|=0$$

and

$$\lim_{q\longrightarrow\infty}\Big\|\sum_{1\leq j\leq d}\big(CT_{jq}C-CT_jC\big)\Big\|=0.$$

Depending on these properties we want to show that $\mathcal{Q}_{m,n}(\mathbf{T})=0$. Given that $\mathcal{Q}_{m,n}(\mathbf{T}_q,C)=0$ and $\|C\|=1$, it will be with us

$$\begin{aligned}
& \|\mathcal{Q}_{m,n}(\mathbf{T},C)\| \\
= & \|\mathcal{Q}_{m,n}(\mathbf{T}_q,C)-\mathcal{Q}_{m,n}(\mathbf{T},C)\| \\
= & \Big\|\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}_q^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}_q^{\gamma}C-\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\Big\| \\
\leq & \Big\|\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}_q^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}_q^{\beta}C-\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}_q^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\Big\| \\
& +\Big\|\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}_q^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C-\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{T}^{*\gamma}\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\Big\| \\
\leq & \sum_{0\leq k\leq m}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\|\mathbf{T}_q^{*\gamma}\alpha_n(\mathbf{T},C)C\big(\mathbf{T}_q^{\gamma}-\mathbf{T}^{\gamma}\big)C\|+\sum_{0\leq k\leq m}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\|\big(\mathbf{T}_q^{*\gamma}-\mathbf{T}^{*\gamma}\big)\alpha_n(\mathbf{T},C)C\mathbf{T}^{\gamma}C\| \\
\leq & \sum_{0\leq k\leq m}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\|\mathbf{T}_q^{*\gamma}\|\|\alpha_n(\mathbf{T},C)\|\|\big(\mathbf{T}_q^{\gamma}-\mathbf{T}^{\gamma}\big)\|+\sum_{0\leq k\leq m}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\|\big(\mathbf{T}_q^{*\gamma}-\mathbf{T}^{*\gamma}\big)\|\|\alpha_n(\mathbf{T},C)\|\|\mathbf{T}^{\gamma}\|
\end{aligned}$$

By taking $q\to\infty$ we get $\mathcal{Q}_{m,n}(\mathbf{T},C)=0$. . ∎

3. Perturbation by a nilpotent operator

Perturbation theory has long been a very useful tool in operator theory and has been developed by several authors. A considerable amount of research has been done on the perturbation of m-isometric operators and n-symmetric operators in single and multivariable operators on a Hilbert space, principally by T. Bermúdez et al.[7], Mecheri et al.[25], Chō et al.[10], Duggal et al.[16], C. Gu [22], Rabaouiet al. [26] and Sid Ahmed et al.[27]. In the following main theorem, we will combine these results and we devote much effort to extend them for (m,n,C))-isosymmetric c.t.o.

The following proposition will be required later.

Proposition 3.1. *Let $\mathbf{T}=(T_1,\cdots,T_d)$, $\mathbf{N}=(N_1,\cdots,N_d)$ be two c.t.o for which $\big[T_j,N_i\big]=\big[CT_jC,N_i^*\big]=0$ for all $j,i\in\{1,\cdots,d\ \}$ and C be a conjugation on $\mathcal{H}$. Then, for a positive integers m and n, the following identity holds:*

(3.22)

$$\mathcal{Q}_{m,\,n}\big(\mathbf{T}+\mathbf{N},\ C\big)=\sum_{j=0}^{n}\sum_{|\beta|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\,\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}$$

where $\alpha_j\big(\mathbf{N},\mathbf{C}\big)=0$ if $j\geq 2q$ and $\displaystyle\binom{m}{\beta,\gamma,k}=\frac{m!}{\beta!\gamma!k!}$.

Proof. We prove by two-dimensional induction principle on $(m,n)\in\mathbb{N}^2$. We first check that it is true for $(m,n)=(1,1)$. In fact,

$$\begin{aligned}
&\sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=1}\binom{1}{j}\binom{1}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\,\mathbf{N}^{*\gamma}\Lambda_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\\
=\ &\sum_{j=0}^{1}\Bigg\{\sum_{i=1}^{d}(T_i^*+N_i^*)\,\mathcal{Q}_{0,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)N_i\\
&+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{0,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)T_i+\mathcal{Q}_{1,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\Bigg\}\\
=\ &\sum_{i=1}^{d}(T_i^*+N_i^*)\,\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)\alpha_0\big(\mathbf{N},\ C\big)N_i+\sum_{i=1}^{d}(T_i^*+N_i^*)\,\mathcal{Q}_{0,0}\big(\mathbf{T},\ C\big)\alpha_1\big(\mathbf{N},\ C\big)N_i\\
&+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)\alpha_0\big(\mathbf{N},\ C\big)T_i+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{0,0}\big(\mathbf{T},\ C\big)\alpha_1\big(\mathbf{N},\ C\big)T_i\\
&+\mathcal{Q}_{1,1}\big(\mathbf{T},\ C\big)\alpha_0\big(\mathbf{N},\ C\big)+\mathcal{Q}_{1,0}\big(\mathbf{T},\ C\big)\alpha_1\big(\mathbf{N},\ C\big)
\end{aligned}$$

Since $\mathcal{Q}_{0,0}\big(\mathbf{T},\ C\big)=I$ and $\alpha_0\big(\mathbf{N},\ C\big)=I$, we get

$$\sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=1}\binom{1}{j}\binom{1}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\,\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}=A+B$$

where

$$A=\sum_{i=1}^{d}\left(T_i^*+N_i^*\right)\Lambda_{0,1}\big(\mathbf{T},\ C\big)N_i+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)T_i$$

and

$$B=\mathcal{Q}_{1,1}\big(\mathbf{T},\ C\big)+\mathcal{Q}_{1,0}\big(\mathbf{T},\ C\big)\alpha_1\big(\mathbf{N},\ C\big).$$

We will using Theorem 2.10 to calculate B

$$\begin{aligned}
B &= \mathcal{Q}_{1,1}\big(\mathbf{T},\ C\big)+\mathcal{Q}_{1,0}\big(\mathbf{T},\ C\big)\alpha_1\big(\mathbf{N},\ C\big)\\
&= \left\{\sum_{i=1}^{d}T_i^*\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)CT_iC-\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)\right\}+\left\{\sum_{i=1}^{d}T_i^*CT_iC-I\right\}\alpha_1\big(\mathbf{N},\ C\big)\\
&= \sum_{i=1}^{d}T_i^*\left[\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)+\alpha_1\big(\mathbf{N},\ C\big)\right]CT_iC-\Big(\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)+\alpha_1\big(\mathbf{N},\ C\big)\Big)\\
&= \sum_{i=1}^{d}T_i^*\mathcal{Q}_{0,1}\big(\mathbf{T}+\mathbf{N},\ C\big)CT_iC-\mathcal{Q}_{0,1}\big(\mathbf{T}+\mathbf{N},\ C\big)
\end{aligned}$$

Therefore

$$\begin{aligned}
A+B &= \left(\sum_{i=1}^{d}\left(T_i^*+N_i^*\right)\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)N_i+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{0,1}\big(\mathbf{T},\ C\big)T_i+\sum_{i=1}^{d}T_i^*\mathcal{Q}_{0,1}\big(\mathbf{T}+\mathbf{N},\ C\big)CT_iC\right)\\
&\quad -\mathcal{Q}_{0,1}\big(\mathbf{T}+\mathbf{N},\ C\big)\\
&= \sum_{i=1}^{d}\left(T_i^*+N_i^*\right)\mathcal{Q}_{0,1}\big(\mathbf{T}+\mathbf{N},\ C\big)C\left(T_i+N_i\right)C-\mathcal{Q}_{0,1}\big(\mathbf{T}+\mathbf{N},\ C\big)
\end{aligned}$$

So the identity (3.22) holds for $(m,n)=(1,1)$. Assume that (3.22) holds for $(m,1)$ and prove it for $(m+1,1)$. Thanks to the Theorem 2.10, we get

$$\begin{aligned}
&\mathcal{Q}_{m+1,\,1}\big(\mathbf{T}+\mathbf{N},\ C)\big)\\
&= \sum_{1\le i\le d}\left(T_i^*+N_i^*\right)\mathcal{Q}_{m,\,1}\big(\mathbf{T}+\mathbf{N},\ C\big)C\left(T_i+N_i\right)C-\mathcal{Q}_{m,\,1}\big(\mathbf{T}+\mathbf{N},\ C\big)\\
&= \sum_{i=1}^{d}\left(T_i^*+N_i^*\right)\left\{\sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\beta,\gamma,k}\left(\mathbf{T}+\mathbf{N}\right)^{*\beta}\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\right\}\\
&\quad C\left(T_i+N_i\right)C\\
&\quad -\sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\beta,\gamma,k}\left(\mathbf{T}+\mathbf{N}\right)^{\beta}\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\\
&= \sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\beta,\gamma,k}\left(\mathbf{T}+\mathbf{N}\right)^{*\beta}\mathbf{N}^{*\gamma}A_1\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta},
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sum_{i=1}^{d}(T_i^*+N_i^*)\,\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)C\,(T_i+N_i)\,C-\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\\
&= \sum_{i=1}^{d}T_i^*\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)CT_i^*C-\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\\
&\quad+\sum_{i=1}^{d}(T_i^*+N_i^*)\,\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)CN_iC+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)CN_iC\\
&= \mathcal{Q}_{k+1,\,1-j}\big(\mathbf{T},\ C\big)+\sum_{i=1}^{d}(T_i^*+N_i^*)\,\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)CN_iC+\sum_{i=1}^{d}N_i^*\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)CT_iC
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathcal{Q}_{m+1,\,1}\big(\mathbf{T}+\mathbf{N},\ C)\big)\\
&= \sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}N^{*\gamma}\mathcal{Q}_{k+1,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\\
&\quad+\sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\beta,\gamma,k}\sum_{i=1}^{d}(\mathbf{T}+\mathbf{N})^{*\beta}(T_i^*+N_i^*)\,\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}CN_iC\mathbf{N}^{\beta}\\
&\quad+\sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\beta,\gamma,k}\sum_{i=1}^{d}(\mathbf{T}+\mathbf{N})^{*\beta}\mathbf{N}^{*\gamma}N_i^*\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)CT_iC\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\\
&= \sum_{j=0}^{1}\sum_{|\beta|+|\gamma|+k=m+1}\binom{1}{j}\binom{m+1}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}N^{*\gamma}\mathcal{Q}_{k,\,1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}
\end{aligned}$$

Now assume that (3.22) holds for (m,n) and we prove that holds for $(m,n+1)$. By using the Theorem 2.10, we get

$$\begin{aligned}
&\mathcal{Q}_{m,\,n+1}\big(\mathbf{T}+\mathbf{N},\ C)\big)\\
&= \sum_{1\le i\le d}(T_i^*+N_i^*)\,\mathcal{Q}_{m,\,n}\big(\mathbf{T}+\mathbf{N},\ C\big)-\sum_{1\le i\le d}\mathcal{Q}_{m,\,n}\big(\mathbf{T}+\mathbf{N},\ C\big)C\,(T_i+N_i)\,C\\
&= \sum_{1\le i\le d}(T_i^*+N_i^*)\sum_{j=0}^{n}\sum_{|\beta|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\\
&\quad-\sum_{1\le i\le d}\sum_{j=0}^{n}\sum_{|\beta|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}C\,(T_i+N_i)\,C\\
&= \sum_{|\beta|+|\gamma|+k=m}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\mathbf{N}^{*\gamma}A_2\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}
\end{aligned}$$

where

$$\begin{aligned}
A_2 &= \sum_{j=0}^{n}\binom{n}{j}\sum_{1\le i\le d}(T_i^*+N_i^*)\,\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\\
&\quad -\sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\sum_{1\le i\le d}C\,(T_i+N_i)\,C\\
&= \sum_{j=0}^{n}\binom{n}{j}\left\{\sum_{1\le i\le d}T_i^*\Lambda_{k,\,n-j}\big(\mathbf{T},\ C\big)-\sum_{1\le i\le d}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)CT_iC\right\}\alpha_j\big(\mathbf{N},\ C\big)\\
&\quad +\sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\left\{\sum_{1\le i\le d}N_i^*\alpha_j\big(\mathbf{N},\ C\big)-\sum_{1\le i\le d}\alpha_j\big(\mathbf{N},\ C\big)CN_iC\right\}\\
&= \sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_{k,\,n+1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)+\sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_{j+1}\big(\mathbf{N},\ C\big)\\
&= \mathcal{Q}_{k,\,n+1}\big(\mathbf{T},\ C\big)\alpha_0\big(\mathbf{N},\ C\big)+\sum_{j=1}^{n}\binom{n}{j}\mathcal{Q}_{k,\,n+1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\\
&\quad +\sum_{j=0}^{n-1}\binom{n}{j}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_{j+1}\big(\mathbf{N},\ C\big)+\mathcal{Q}_{k,\,0}\big(\mathbf{T},\ C\big)\alpha_{n+1}\big(\mathbf{N},\ C\big)\\
&= \mathcal{Q}_{k,\,n+1}\big(\mathbf{T},\ C\big)\alpha_0\big(\mathbf{N},\ C\big)+\sum_{j=1}^{n}\left(\binom{n}{j}+\binom{n}{j-1}\right)\mathcal{Q}_{k,\,n+1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\\
&\quad +\mathcal{Q}_{k,\,0}\big(\mathbf{T},\ C\big)\alpha_{n+1}\big(\mathbf{N},\ C\big)\\
&= \sum_{j=0}^{n+1}\binom{n+1}{j}\mathcal{Q}_{k,\,n+1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)
\end{aligned}$$

As a result that

$$\begin{aligned}
&\mathcal{Q}_{m,\,n+1}\big(\mathbf{T}+\mathbf{N},\ C)\big)\\
&= \sum_{j=0}^{n+1}\sum_{|\beta|+|\gamma|+k=m}\binom{n+1}{j}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\,\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,n+1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}
\end{aligned}$$

Then for all positive integers m and n we have

$$\mathcal{Q}_{m,\,n}\big(\mathbf{T}+\mathbf{N},\ C\big)=\sum_{j=0}^{n}\sum_{|\beta|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\beta,\gamma,k}(\mathbf{T}+\mathbf{N})^{*\beta}\,\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\,n-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}$$

□

Following [22], a tuple $\mathbf{N}=(N_1,\cdots,N_d)\in\mathcal{B}(\mathcal{H})^d$ of c.t.o., is said to be q-nilpotent, $q>0$, if $\mathbf{N}^{\omega}=N_1^{\omega_1}\cdots N_d^{\omega_d}=0$ for all $\omega=(\omega_1,\cdots,\omega_d)\in\mathbb{Z}_+^d$ such that $|\omega|=q$.

Theorem 3.2. *Let $\mathbf{T} = (T_1, \cdots, T_d)$ and $\mathbf{N} = (N_1, \cdots, N_d)$ be two c.t.o. for which $\big[T_j, N_i\big] = \big[CT_jC, N_i^*\big] = 0$ for all $j, i \in \{1, \cdots, d\ \}$ and C be a conjugation on $\mathcal{H}$. If $\mathbf{T}$ is an (m,n,C)-isosymmetric c.t.o. and $\mathbf{N}$ is a nilpotent c.t.o. of order q, then $\mathbf{T}+\mathbf{N}$ is an $(m+2q-2, n+2q-1,\ C)$-isosymmetric c.t.o.*

Proof. Under the hypotheses of the theorem we can using the Proposition 3.1, we get that

$$\begin{aligned}\mathcal{Q}_{m+2q-2,\, n+2q-1}\big(\mathbf{T}+\mathbf{N},\ C\big) &= \sum_{j=0}^{n+2q-1} \sum_{|\beta|+|\gamma|+k=m+2q-2} \binom{n+2q-1}{j}\binom{m+2q-2}{\beta,\gamma,k} \\ &\quad \times (\mathbf{T}+\mathbf{N})^{*\beta}\,\mathbf{N}^{*\gamma}\mathcal{Q}_{k,\, n+2-1-j}\big(\mathbf{T},\ C\big)\alpha_j\big(\mathbf{N},\ C\big)\mathbf{T}^{\gamma}\mathbf{N}^{\beta}\end{aligned}$$

- If $j \geq 2q$ or $q \leq \max(|\beta|, |\gamma|)$, then $\alpha_j\big(\mathbf{N},\ C\big) = 0$ or $\mathbf{N}^{*\delta} = 0$ or $\mathbf{N}^{\beta} = 0$.
- Else if $j \leq 2q$ and $q-1 \geq \max(|\beta|, |\gamma|)$, then $n+2q-1-j \geq n$ and $k = m+2q-2-|\beta|-|\gamma| = m+(q-1-|\beta|)+(q-1-|\gamma|) \geq m$. According to Corollary 2.11, we get that $\mathcal{Q}_{k,\, n+2-1-j}\big(\mathbf{T},\ C\big) = 0$

$\square$

From the previous theorem, we derive the next corollary.

Corollary 3.3. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be an (m,n,C)-isosymmetric c.t.o. and let $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}(\mathcal{H})^d$ be a q-nilpotent c.t.o. Then $\mathbf{T}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{N} := (T_1\otimes I+I\otimes N_1, \cdots, T_d\otimes I+I\otimes N_d) \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})^d$ is an $(m+2q-2, n+2q-1, C\otimes C)$-isosymmetric c.t.o. where C is a conjugation on $\mathcal{H}$.*

Proof. It is evident to see that

$$\big[(T_k\otimes I), (I\otimes N_j)\big] = \big[(C\otimes C)(R_k\otimes I)(C\otimes C), (I\otimes N_j)^*\big] = 0,$$

for all $j, k = 1, \cdots, d$. Moreover, we infer that $\mathbf{T}\otimes\mathbf{I} = (T_1\otimes I,\ \cdots, T_d\otimes I)$ is an $(m,n,C\otimes C)$-isosymmetric c.t.o. and $\mathbf{I}\otimes\mathbf{N} = \big(I\otimes N_1, \cdots, I\otimes N_d\big)$ is a nilpotent c.t.o. of order q. The result now follows from Theorem 3.2. $\square$

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