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GEHMAN DENDRITE G_4 AS GENERALIZED INVERSE LIMIT ON [0,1] WITH SINGLE UPPER SEMI-CONTINUOUS BONDING FUNCTION

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ABSTRACT. In this paper we prove that the Gehman dendrite G_4 can be obtained as a generalized inverse limit space with a single upper semi-continuous bonding function on $[0,1]$. This answers a question of Farhan and Mena, [3]. Moreover, we find an uncountable family of inverse sequences on $[0,1]$ whose inverse limit spaces are homeomorphic to the Gehman dendrite G_4.

1. Introduction and Definitions

Generalized inverse limits, where bonding maps are replaced by upper semi-continuous set valued functions, were first introduced by Mahavier [7]. The paper [4] and book [6] by Mahavier and Ingram helped popularize the subject in the continuum theory community and beyond. Since this beginning hundreds of papers on the subject have been published. One aspect about generalized inverse limits that generated a great deal of interest is their ability to produce a wide variety of exotic continua, using simple bonding functions on $[0,1]$, which could not be obtained using traditional inverse limits. One such example, [5, Example 2.22], Ingram showed that the inverse limit space obtained when the graph of the bonding function looks like the letter "H" on it's side is a dendroid having all ramification points being of order 3 and the set of endpoints is a Cantor set. Charatonik and Mena, [2], gave conditions on bonding functions that guaranteed that the inverse limit space is locally connected. This result implies that Ingram's example is a dendrite, in particular, the Gehman dendrite, G_3. In a recent paper Farhan and Mena, [3], showed that there is an uncountably infinite family of inverse sequences that have G_3 as the inverse limit. They also showed examples of inverse limit spaces that are dendrites having ramification points of orders both 3 and 4. They asked if it was possible to obtain the Gehman dendrite G_4, that is a dendrite having all ramification points of order 4 and the set of endpoints being the Cantor set. In this paper, we obtain an uncountable family of inverse limit sequences, each having a single upper semi-continuous bonding function defined on $[0,1]$, which have as their inverse limit space G_4. We generalized this results to not requiring a single bonding function in the inverse sequences.

A continuum is a non-empty, compact, connected, metric space. A subset of a continuum X which itself a continuum is called subcontinuum of X. A continuum X is said to be locally connected continuum if whenever $x \in X$ and each neighborhood N of x, the component of N to which x belongs is neighborhood of x. Let X and Y be continua, a set valued function $f : X \to 2^Y$, where 2^Y is the hyperspace of all closed subsets of Y, is upper semi-continuous at x provided that for any open set V in Y which contains $f(x)$, there exist an open set U in X with $x \in U$ such that if $t \in U$, then $f(t) \subseteq V$.

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If a function $f: X \to 2^Y$ is upper semi-continuous at x for each $x \in X$, we say that f is an upper semi-continuous. Let X and Y be compact metric spaces and $f: X \to 2^Y$ be a set valued function, then f is an upper semi-continuous if and only if the graph of f, $G(f) = \{(x,y) : x \in X, y \in f(x)\}$, is closed in $X \times Y$ [5, p. 3]. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of continua and for each $i \in \mathbb{N}$, let $f_i : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function. The generalized inverse limit of the sequence $\{X_i, f_i\}$ is denoted by $\varprojlim \{X_i, f_i\}$ and defined by $\varprojlim \{X_i, f_i\} = \{(x_i)_{i=1}^{\infty} : x_i \in X_i, x_i \in f_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\}$.

We denote the projection from the inverse limit space onto the n^{th} factor space by π_n. All inverse limits considered in this paper will be generalized inverse limits. In this paper, we will have for all $i \in \mathbb{N}$, $X_i = I = [0,1]$ and we denote the inverse limit space by $\varprojlim\{I, f_i\}$. The topology is the subspace topology of the Hilbert cube or equivalently the metric topology given by the metric $d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$, where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. Additionally, if $f_i = f$ for all i we denote the inverse limit space by $\varprojlim\{I, f\}$ and say f is the single bonding function of the inverse limit. If p is a point and S is a set, the distance from p to S is defined as $\inf\{d(x,y) \colon y \in S\}$. The open ball of radius r centered at p is $B_r(p) = \{x \colon d(p,x) < r\}$. More information about generalized inverse limits of continua with upper semi-continuous bonding functions can be found in [6] and [5].

A *dendrite* is a locally connected continuum that contains no simple closed curve. Let p be a point in a dendrite D. Then p is an *endpoint of D in the classical sense* if p is not the only intersection point of any two different arcs. The point p is an *ordinary* point of D if $D \setminus \{p\}$ has exactly two components, and p is a *ramification* point of D if $D \setminus \{p\}$ has more than two components. The *order* of a point p in a dendrite D is n, $n \in \mathbb{N} \cup \{\omega\}$, if $D \setminus \{p\}$ has exactly n components. We denote the set of endpoints of D by $E(D)$, the set of ordinary point of D by $O(D)$ and the set of ramification points of D by $R(D)$. The Gehman dendrite G_n is the dendrite having all ramification points of order n and the set of endpoints is homeomorphic to the Cantor set [1, Theorem 4.1].

2. Main Theorem

Let A be a non-empty finite subset of $(0,1)$ of cardinality $|A| \geq 2$, $C = \{0,1\}$ and $\alpha \in (0,1)$ such that $\alpha \notin A$ and there exist $\beta_1, \beta_2 \in A$ such that $\beta_1 < \alpha < \beta_2$. Given A, α, and $r \in C$ define the upper semi-continuous function $f = f_{A\alpha r} : [0,1] \to 2^{[0,1]}$ by $f(x) = A \cup \{r\}$ if $x \in [0, \alpha)$, $f(x) = [0,1]$ if $x = \alpha$, and $f(x) = A \cup (C \setminus \{r\})$ if $x \in (\alpha, 1]$. Let $\mathscr{Q}$ be the family of all such upper semi-continuous functions. The main theorem of the paper is the following.

Theorem 2.1. *If $f \in \mathscr{Q}$ then $\varprojlim\{I, f\} = G_4$.*

Proof. Let $D = \varprojlim\{I, f\}$. As in [3, Theorem 2.1], D is a dendrite. To prove the D is homeomorphic to G_4, all that is left to show is that all of the ramification points are of order 4 and the set of endpoints is a Cantor set. Because $f(C \cup A) \subseteq C \cup A$ and $f^{-1}(\alpha) = \{\alpha\}$, points in D must be of one of the following three forms: (R): $(t_1, t_2, ..., t_n, \alpha, \alpha, \ldots)$, where $t_i \in C \cup A$, $i \neq n$ and $t_n \in A$, (O): $(t_1, t_2, ..., t_n, y, \alpha, \alpha, ...)$ where $y \in [0,1] \setminus A$, $t_i \in C \cup A$, $n \in \mathbb{N} \cup \{0\}$, and (E): $(t_1, t_2, \ldots)$ where $t_i \in C \cup A$.

We claim that any point of the form (O) is an ordinary point of D. To see this, let y be a fixed value not in A and let $p = (t_1, t_2, ..., t_n, y, \alpha, \alpha, ...)$ where $t_i \in C \cup A$, $n \in \mathbb{N} \cup \{0\}$. First, if $y \notin C$ then the set $K = \{t_1\} \times \{t_2\} \times \ldots \times \{t_n\} \times (t_{y_m}, t_{y_M}) \times \{\alpha\} \times \{\alpha\} \times \ldots$ where $t_{y_m} = \max\{t_i : t_i < y \text{ and } t_i \in C \cup A \cup \{\alpha\}\}$ and $t_{y_M} = \min\{t_i : t_i > y \text{ and } t_i \in C \cup A \cup \{\alpha\}\}$ is an open arc in D containing

$p=(t_1,t_2,\dots,t_n,y,\alpha,\alpha,\dots)$. Moreover, if for each i, $1\le i\le n$, we let $\varepsilon_i=\min\{|t_i-k|:k\in(C\cup A\cup\{\alpha\})\setminus\{t_i\}\}$, then $H=(t_1-\varepsilon_1,t_1+\varepsilon_1)\times\dots\times(t_n-\varepsilon_n,t_n+\varepsilon_n)\times(t_{y_m},t_{y_M})\times\Pi_{i=1}^{\infty}I$, is an open neighborhood of p in the product $\Pi_{i=1}^{\infty}I$ such that $K\cap H=K$. If $y\in C$, then y is either 0 or 1. In both cases we have $f^{-1}(y)=[0,\alpha]$ or $f^{-1}(y)=[\alpha,1]$. We consider the case $y=1$ and $f^{-1}(y)=[\alpha,1]$ and remaining cases are similar. Let $K_1=\{t_1\}\times\{t_2\}\times\dots\times\{t_n\}\times\{1\}\times[\alpha,t_{y_M})\times\{\alpha\}\times\{\alpha\}\times\dots$ where $t_{y_M}=\min\{t_i:t_i>\alpha \text{ and } t_i\in A\}$. Then K_1 is a segment in D and $p\in E(K_1)$. Let $K_2=\{t_1\}\times\{t_2\}\times\dots\times\{t_n\}\times(t_{y_m},1]\times\{\alpha\}\times\{\alpha\}\times\dots$ where $t_{y_m}=\max\{t_i:t_i<y \text{ and } t_i\in A\}$. Then K_2 is a segment in D with $p\in E(K_2)$. So $K=K_1\cup K_2$ in an open neighborhood of p in D. Moreover, if for each i, $1\le i\le n$, we let $\varepsilon_i=\min\{|t_i-k|:k\in(A\cup\{\alpha\})\setminus\{t_i\}\}$, then $H=(t_1-\varepsilon_1,t_1+\varepsilon_1)\times\dots\times(t_n-\varepsilon_n,t_n+\varepsilon_n)\times(1-\varepsilon_{n+1},1]\times(t,t_{y_M})\times\Pi_{i=1}^{\infty}I$, where $t=\max\{t_i:t_i<\alpha \text{ and } t_i\in A\}$, is an open neighborhood of p such that $K\cap H=K$.

Next, we claim that any point of the form (R) is a ramification point of D. Let $p=(t_1,t_2,\dots,t_n,\alpha,\alpha,\dots)\in D$ and where $t_i\in C\cup A$, $i\ne n$ and $t_n\in A$. Note that $e_1=\{t_1\}\times\{t_2\}\times\dots\times\{t_n\}\times(t_{y_m},t_{y_M})\times\{\alpha\}\times\{\alpha\}\times\dots$ where $t_{y_m}=\max\{t_j:t_j<\alpha \text{ and } t_j\in C\cup A\}$ and $t_{y_M}=\min\{t_j:t_j>\alpha \text{ and } t_j\in C\cup A\}$ is a segment in D containing p. Also, if we let $\varepsilon_i=\min\{|t_i-k|:k\in(C\cup A\cup\{\alpha\})\setminus\{t_i\}\}$ then $e_2=\{t_1\}\times\{t_2\}\times\dots\times((t_n-\varepsilon_n,t_n+\varepsilon_n)\cap I)\times\{\alpha\}\times\{\alpha\}\times\dots$ is also a segment in D containing p. Let $X=e_1\cup e_2$. Moreover, then $U=(t_1-\varepsilon_1,t_1+\varepsilon_1)\times\dots\times(t_n-\varepsilon_n,t_n+\varepsilon_n)\times(t_{y_m},t_{y_M})\times\Pi_{i=1}^{\infty}I$, is an open neighborhood of p such that $X\cap U=X$ and contains no ramification points of D other than p.

To show that the above ramification point is of order at least four, we have two cases: Case 1: If $t_{n-1}\in C$ and $t_n\in A$. Suppose without loss of generality $t_{n-1}=0$. The point $p=(t_1,t_2,...,0,t_n,\alpha,\alpha,...)$ is an interior ordinary point of the segment $e_2=(t_1,t_2,...,t_{n-1}=0,t,\alpha,\alpha,...)$, t is either in $[0,\alpha]$ or in $[\alpha,1]$ and the segment $e_1=(t_1,t_2,...,0,t_n,s,\alpha,\alpha,...)$, $s\in[0,1]$. Case 2: If $t_{n-1},t_n\in A$. The point $p=(t_1,t_2,...,t_{n-1},t_n,\alpha,\alpha,...)$ is the interior ordinary point of the segment $e_1=(t_1,t_2,...,t_{n-1},t_n,s,\alpha,\alpha,...)$, where $s\in[0,1]$ and the ordinary interior point of the segment $e_2=(t_1,t_2,...,t_{n-1},t,\alpha,\alpha,\alpha,...)$, where $t\in[0,1]$.

To show that ramification points are of order four, note that if $0<r<\min\{|\pi_k(p)-t|: t\in C\cup A\cup\{\alpha\}$ and $\pi_k(p)\ne t\}$ and $\varepsilon=r/2^{n+2}$ then for any $x\in D\cap B_\varepsilon(p)$, $\pi_{n+2}(x)=\alpha$ and $\pi_k(x)=\pi_k(p)$ if $k\ne n,n+1$. Thus if $x\in D\cap B_\varepsilon(p)$ then x must lie on e_1 or e_2.

We next prove that any point of the form (E) is an endpoint of D. Since D is a dendrite, there is a unique arc between any two points in D. If z is a point in D of the form (O) or (R) then for some n, $z=(t_1,t_2,\dots,t_n,y_{n+1},\alpha,\dots)$ where $y_{n+1}\in[0,1]\setminus\{\alpha\}$. Then $[(\alpha,\alpha,\dots),(t_1,\alpha,\alpha,\dots)]\cup[(t_1,\alpha,\alpha,\dots),(t_1,t_2,\alpha,\alpha,\dots)]\cup\dots\cup[(t_1,t_2,\dots,t_n,\alpha,\dots),(t_1,t_2,\dots,t_n,y_{n+1},\alpha,\dots)]$ is the unique arc joining z to a. If z is a point of the form (E) then $[(\alpha,\alpha,...),(t_1,\alpha...)]\cup[(t_1,\alpha,...),(t_1,t_2,\alpha,...)]\cup...\cup\{(t_1,t_2,t_3,...)\}$ is the unique arc from a to z. Suppose there is a point $p=(t_1,t_2,...)\in(E)$ and p is not an endpoint of D. Let J be the unique arc from a to p. Since p is not an endpoint of D, there is a second arc K starting at p and disjoint from J except for p. Then there is a point y in $K\setminus J$. If y is of the form (E) then $y=(t'_1,t'_2,\dots)$ where $t'_i\in C\cup A$ for all i. Then for some n , $t_n\ne t'_n$. It follows that if L is the unique arc from a to y then $L\not\subseteq J$ and $J\not\subseteq L$. Thus there are two different arcs from a to y in D, a contradiction. If y is of the form (O) or (R) then $y=(t'_1,t'_2,\dots,t'_n,y_{n+1},\alpha\dots)$. If $t_i=t'_i$ for all i, $1\le i\le n$ then $y\in J$, a contradiction. If $t_i\ne t'_i$ for some i, $1\le i\le n$ then again there are two different arcs from a to y in D, a contradiction.

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To finish the proof, we need to show that $E(D)$ is a Cantor Set. The idea is similar to the proof given in [3, Theorem 2.1]. We include it to show why there needed to be the β_i's in the definition of the bonding functions. It is enough to prove that $E(D)$ is closed, perfect and totally disconnected. First, we show that $E(D)$ is closed. Let $p = (p_1, p_2, \ldots)$ be in $D \setminus E(D)$. Then there is n such that $p_n \notin A \cup C$. If d is the minimum distance from p_n to $A \cup C$ then the distance from p to $E(D)$ is at least $d/2^n$. So p is not a limit point of $E(D)$. Hence $E(D)$ is closed. Next, we show that $E(D)$ is perfect, i.e., every point in $E(D)$ is a limit point of $E(D)$. Let $p = (p_1, p_2, \ldots) \in E(D)$ and $\varepsilon > 0$ be given. Then there is an n such that $1/2^n < \varepsilon$. Since $f^{-1}(p_{n-1}) \in \{[0,\alpha], [\alpha,1], [0,1]\}$, so $f^{-m}(p_{n-1}), m \geq 2$ contains at least two points of $A \cup C$. Let $p'_{n+m} \in (A \cup C) \setminus \{p_{n+m}\}$. Let $p' = (p'_1, p'_2, \ldots) \in E(D)$ be such that $p_i = p'_i$ for $i \neq n+m$ and $p_{n+m} \neq p'_{n+m}$. Then the distance between p and p' is less than ε so p is a limit point of $E(D)$. Finally, to prove that $E(D)$ is totally disconnected, let p and q be elements in $E(D)$ such that $p \neq q$. There exits $n \in \mathbb{N}$ such that $\pi_n(p) \neq \pi_n(q)$. Since $A \cup C$ is finite, there exist disjoint open sets U and V in I such that U contains $\pi_n(p)$ and V contains $(A \cup C) \setminus \{\pi_n(p)\}$. So $\pi_n^{-1}(U)$ and $\pi_n^{-1}(V)$ are disjoint open sets in D containing p and q respectively and their union contains $E(D)$. We obtain that $E(D)$ is totally disconnected. Hence $E(D)$ is a Cantor set and D is homeomorphic to G_4. □

Example 2.2. *Let $f : [0,1] \to 2^{[0,1]}$ be an upper semi-continuous function defined as in $\mathcal{Q}$, where $r = 0$, $A = \{\beta_1, \beta_2\}$ and $0 < \beta_1 < \alpha < \beta_2 < 1$. Using Theorem 2.1, $\varprojlim\{I, f\}$ homeomorphic to G_4 and the representation of the inverse limit is shown in Figure 1. In the Figure 1 we have labeled some ordinary points such as $(\alpha,\alpha,\alpha,\ldots)$, $(0,0,\alpha,\ldots)$, and $(\beta 1,1,\alpha,\ldots)$ and we labeled some ramification points such as $(0,\beta 1,\alpha,\ldots)$ and $(1,1,\beta 2,\alpha,\ldots)$.*

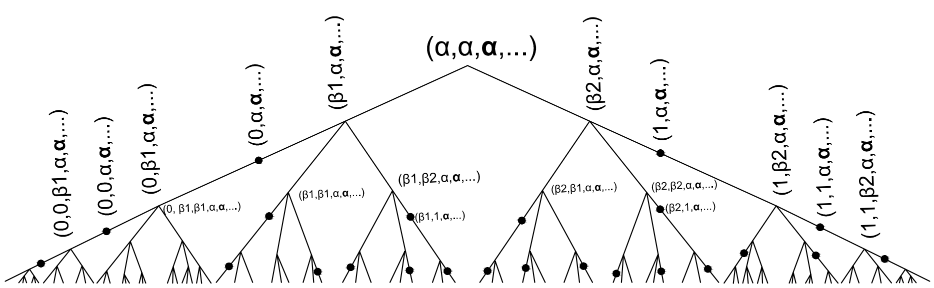

FIGURE 1. G_4 as an inverse limit space

We may generalize Theorem 2.1 by considering a sequence of bonding functions (f_i) instead of having just a single bonding function. In particular, let $\Gamma = (\alpha_i)$ be a sequence of numbers in $(0,1)$, (A_i) be a sequence of finite subsets of $(0,1)$ such that for all i, $2 \leq |A_i| < \infty$, $C = \{0,1\}$, $\alpha_j \notin \bigcup A_i$, and there exist $\beta_{1i}, \beta_{2i} \in A_i$ such that $\beta_{1i} < \alpha_i < \beta_{2i}$. Given $r_i \in C$, define the upper semi-continuous function $f_i = f_{A_i \alpha_i r_i}$ in the same manner as $f_{A\alpha r}$ was defined before Theorem 2.1.

Theorem 2.3. *For any sequence of (f_i) defined above, $\varprojlim\{I, f_i\} = G_4$.*

Proof. Note that $f_i(C \cup A_{i+1}) = C \cup A_i$ and $f^{-1}(\alpha_i) = \{\alpha_{i+1}\}$. So, except for notation, the proof is essentially the same as that of Theorem 2.1. □

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