Existence results of a nonlinear fractional integral equation via measure of noncompactness

Monkhum Khilak[a], Saifur Rahman[b,c], Bipan Hazarika[d,*]

[a]Department of Mathematics, Jawaharlal Nehru College Pasighat, East Siang-791103, Arunachal Pradesh, India

[b]Department of Mathematics, Jamia Millia Islamia University, Jamia Nagar, Okhla, New Delhi, Delhi 110025, India

[c]Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791112, Arunachal Pradesh, India

[d]Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India

Email: [a]monkhum.khilak@rgu.ac.in; [b]saifur.rahman@rgu.ac.in; [d,*]bh_rgu@yahoo.co.in; [d,*]bh_gu@gauhati.ac.in

Abstract. In this paper, we generalize the Darbo's fixed point theorem via measure of noncompactness. Moreover, the existence result of nonlinear fractional integral equation has been studied, and proposed some concrete examples thereof.

Key Words: Measure of noncompactness; fractional integral equations; fixed point theorem.

MSC subject classification No: 45G10; 45P05; 47H10.

1. Introduction and basic results

The theory of measure of noncompactness(MNC) has various applications especially in the study of nonlinear analysis, fixed point theory, optimizations, integral equations, differential equations and many more. In the year 1930, Kuratowski provided the first description of it. Gohberg et al. [13], Sadovskii [26], and Geobel [12] proposed the Hausdorff's measure of noncompactness. One can refer [4] for more details of MNC. The concept of MNC was used by Darbo [7] and he developed the famous Darbo fixed point theorem which generalize the Schauder fixed point theorem. Since then many researcher generalized the Darbo fixed point theorem and present their applications in verifying the existence solution for a class of fractional integral equations, integral equation, differential equation, hybrid differential equation in Banach spaces, see [6, 8, 9, 10, 16, 15, 22, 27, 29].

The integral equations is helpful in modelling and defining real life problems in the field of physics, biology and economics. Many authour have extensively studied and contributed in solving equations involving integral and differential equations with the help of fixed point

*Corresponding author.

19 Feb 2024 10:14:25 PST
230912-Hazarika Version 3 - Submitted to Rocky Mountain J. Math.

theory and measure of noncompactness. Arab et al. [3] introduced a new μ contraction to generalized Darbo's fixed point theorem and verified the result on functional integral equation. Alqahtani et al. [2] studied the solvability of volterra type fractional equation by using hybrid type contraction in metric space and also merged several known fixed point theorems. Recently Haque et al. [14] generalized the Darbo's fixed point theorem and investigate the solvability of infinite system of integral equation of fractional order integral equations. Which enthused us to study fractional integral equation and in this paper we checked the solvability of fractional integral equation (3.1).

At the outset we will familiarize the reader with the notations that will be use in this paper. Let $\mathbb{R}_+$ denote the set $[0, \infty)$ and $\mathbb{N}$ denotes the set of all natural number. Let $(E, \| \, . \, \|)$ be a Banach space and C be a nonempty subset of E. $\overline{C}$ and $conC$ are also used to represent the closure and convex closure of C, respectively. Further $\mathcal{M}_E$ is the family of nonempty and bounded subset of E, while $\mathcal{N}_E$ denotes its subfamily of all relatively compact sets.

Definition 1.1. [4] *A function $\nu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness in the space E if it satisfies the following conditions:*

(i) *the family* $ker\, \nu = \{Q_1 \in \mathfrak{M}_E : \nu(Q_1) = 0\}$ *is nonempty and* $ker\, \nu \subset \mathcal{N}_E$;
(ii) $Q_1 \subset Q_2 \implies \nu(Q_1) \le \nu(Q_2)$;
(iii) $\nu(\bar{Q_1}) = \nu(Q_1)$;
(iv) $\nu(conQ_1) = \nu(Q_1)$;
(v) $\nu(\lambda Q_1 + (1-\lambda) Q_2) \le \lambda \nu(Q_1) + (1-\lambda)\nu(Q_2)$ *for* $\lambda \in [0,1]$;
(vi) *if* $Q_n \in \mathfrak{M}_E$, $Q_n = \overline{Q_n}$, $Q_{n+1} \subset Q_n$ *for* $n = 1, 2, 3, \ldots$ *and* $\lim_{n\to\infty} \nu(Q_n) = 0$, *then* $\bigcap_{n=1}^{\infty} Q_n$ *is nonempty.*

In addition to the above conditions, if ν satisfies the following conditions then, ν is said to be sublinear

(i) $\nu(\lambda Q_1) = |\lambda| \nu(Q_1)$ for $\lambda \in \mathbb{R}$.
(ii) $\nu(Q_1 + Q_2) \le \nu(Q_1) + \nu(Q_2)$.

Definition 1.2. [4, 17] *For a bounded subset D of a metric space X the Kuratowski measure of noncompactness is defined as*

$$\alpha(D) = \inf \left\{ \varepsilon > 0 : D \subset \bigcup_{i=1}^{n} D_i,\ diam\,(D_i) < \varepsilon \, for\ \ i = 1, 2, 3, \ldots, n; n \in \mathbb{N} \right\}.$$

Then the function α is known as Kuratowski's MNC.

Definition 1.3. [5] *For a bounded subset D of a metric space Q the Hausdorff measure of noncompactness $\chi(D)$ is defined as*

$$\chi(D) = \inf \left\{ \varepsilon > 0 : D \subset \bigcap_{i=1}^{n} \Delta(\xi_i, \overline{r}_i), \xi_i \in Q, \overline{r}_i < \varepsilon, 1 \le i \le n; n \in \mathbb{N} \right\}.$$

Here $\Delta(\xi_i, \overline{r}_i)$ denotes the open ball whose center is at ξ_i and of radius $\overline{r}_i$.

Theorem 1.4. [1] *There is at least one fixed point for every continuous mapping $T : \mathcal{D} \to \mathcal{D}$ in a nonempty, convex, and compact subset $\mathcal{D}$ of the Banach space E.*

Definition 1.5. [7] *Consider D to be a nonempty, bounded, closed, and convex subset of the Banach space E. If for a continuous mapping $T : D \to D$, a constant $\kappa \in [0, 1)$ exists such that*

$$\nu(TQ) \leq \kappa\nu(Q),\ Q \subseteq D$$

Then T has a fixed point.

Definition 1.6. *Let T_F be the set of continuous and increasing functions of each variables F from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$ such that*

$$\lim_{m\to\infty} F(x_m, y_m) = 0 \Leftrightarrow \lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = 0.$$

Definition 1.7. *Let T_B be the set of functions $b : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1)$. Also, let T_ϕ be the set of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous and increasing.*

Example 1.8. *Consider the function $b(\bar{\vartheta}, \vartheta) = |\bar{\vartheta} - \vartheta|\ \forall \bar{\vartheta}, \vartheta \in (n, n+1)$, where $n \in \mathbb{R}_+$. Then $b \in T_B$.*

2. New results

Theorem 2.1. *Let $\hat{B}$ be a nonempty bounded, convex and closed subset of a Banach space E and $T : \hat{B} \to \hat{B}$ is a continuous function such that*

$$F(\nu TQ, \phi(\nu TQ)) \leq b(\nu Q, \phi(\nu Q))F(\nu Q, \phi\nu Q),$$

where Q is a nonempty subset of $\hat{B}$, $F \in T_F$, $b \in T_B$, $\phi \in T_\phi$ and ν is an arbitrary MNC. Then T has at least one fixed point.

Proof. First we construct a sequence (Q_n), where $Q_1 = Q$ and $Q_{n+1} = \mathrm{con}(TQ_n)$ for $n \geq 1$. Then $TQ_1 = TQ \subseteq Q = Q_1, Q_2 = \mathrm{con}(TQ_1) \subseteq Q = Q_1$ and proceeding in same way we get $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \cdots \supseteq Q_n \supseteq Q_{n+1} \supseteq \ldots$.
If there exists $\hat{n} \in \mathbb{N}$ such that $\nu(Q_{\hat{n}}) = 0$ then the theorem is proved. Let $\nu(Q_n) > 0,\ \forall n \in \mathbb{N}$,

then the sequence $\{\nu Q_n\}$ is nonnegative, decreasing and bounded below sequence.

$$\begin{aligned}
F(\nu Q_{n+1}, \phi(\nu Q_{n+1})) &= F(\nu conTQ_n, \phi\nu(\mathrm{con}TQ_n)) \\
&= F(\nu TQ_n, \phi(\nu TQ_n)) \\
&\leq b(\nu Q_n, \phi(\nu Q_n))F(\nu Q_n, \phi(\nu Q_n)) \\
&\leq b(\nu Q_n, \phi(\nu Q_n))b(\nu Q_{n-1}, \phi(\nu Q_{n-1}))F(\nu Q_{n-1}, \phi(\nu Q_{n-1})) \\
&\vdots \\
&\leq \left(\prod_{i=1}^{n} b(\nu Q_i, \phi(\nu Q_i))\right)F(\nu Q_1, \phi(\nu Q_1)).
\end{aligned}$$

The above inequality suggest that

$$\lim_{n\to\infty} F(\nu Q_n, \phi(\nu Q_n)) = 0.$$

By the definition of F we conclude that $\lim_{n\to\infty}(\nu Q_n) = 0$. Now $Q_n \supseteq Q_{n+1}$ and by the definition of ν it is proved that $Q_\infty = \bigcap_{n=1}^{\infty} Q_n$ is nonempty, convex and closed subset of Q and also under T, Q_∞ is invariant. So applying Schauder's theorem we get that T has atleast one fixed point in $Q_\infty \subseteq Q$. □

Corollary 2.2. *Let $T : \Omega \to \Omega$ be a continous operator. Let $F \in T_F$,$b \in T_B$ and also $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$F(\nu TQ, \phi(\nu TQ)) \leq \kappa F(\nu Q, \phi(\nu Q)).$$

Then T has a fixed point.

Proof. Putting $b(\nu Q, \phi(\nu Q)) = \kappa \in [0, 1)$ in Theorem 2.1 then we get the desired result. □

Corollary 2.3. *Let $T : \Omega \to \Omega$ be a continous operator. Let $F \in T_F$, $b \in T_B$ and also $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\nu TQ + \phi(\nu TQ) \leq \kappa(\nu Q + \phi(\nu Q))$$

Then T has a fixed point

Proof. Putting $F(x, y) = x + y$ in the the Corollary 2.2, we get the desired result □

Remark 2.4. *Putting $\phi(\zeta) \equiv 0$ in the Corollary 2.3 we get Darbo's fixed point theorem*

Theorem 2.5. *Let $T : \Omega \to \Omega$ be a continuous operator. Also $F \in T_F$, $b \in T_B$, $\phi \in T_\phi$ and if $diam(T\Omega) > 0$ implies that*

$$F(diam(T\Omega), \phi(diamT\Omega)) \leq b(diam\Omega, \phi(diam\Omega))F(diam\Omega, \phi(diam\Omega)).$$

Then T has a fixed point.

Proof. Consider the function $\sigma(\Omega) = \mathrm{diam}(\Omega)$, where $\mathrm{diam}(\Omega) = \sup\{\| \omega - \varpi \|; \omega, \varpi \in \Omega\}$, then σ is a MNC on the space E in the sense of Definition 1.1. By the same argument as given in [11, Proposition 3.2], the Theorem 2.1 insured a closed convex and nonempty subset of Ω which is invariant under T such that $\mathrm{diam}(\Omega_\infty) = 0$. Then we infer that the set Ω_∞ contains a single element.

If possible there exist two elements say ζ_1, ζ_2 and let $\Omega = \{\zeta_1, \zeta_2\}$.
Here $\mathrm{diam}(\Omega) = \mathrm{diam}(T\Omega) = \| \zeta_1 - \zeta_2 \| > 0$. Then

$$F(diamT\Omega, \phi(diamT\Omega)) \leq b(diam\Omega, \phi(diam\Omega))F(diam\Omega, \phi(diam\Omega)).$$

This implies that $b(\mathrm{diam}\Omega, \phi(\mathrm{diam}\Omega)) \geq 1$, which is a contradiction. Hence $\zeta_1 = \zeta_2$. □

Corollary 2.6. *Let $T : \Omega \to \Omega$ be an operator. If there exist $F \in T_F$, $b \in T_B$ and $\phi \in T_\phi$ such that $\| T\omega - T\varpi \| > 0$ implies that*

$$F(\| T\omega - T\varpi \|, \phi(\| T\omega - T\varpi \|)) \leq b(\| \omega - \varpi \|, \phi(\| \omega - \varpi \|))F(\| \omega - \varpi \|, \phi(\| \omega - \varpi \|)).$$

Then T has a fixed point.

Proof. If $\mathrm{diam}(T\Omega) > 0$ then

$$\begin{aligned}
&F(\sup_{\omega,\varpi\in\Omega} \| T\omega - T\varpi \|, \sup_{\omega,\varpi\in\Omega} \phi(\| T\omega - T\varpi \|)) \\
&= \sup_{\omega,\varpi\in\Omega} F(\| T\omega - T\varpi \|, \phi(\| T\omega - T\varpi \|)) \\
&\leq \sup_{\omega,\varpi\in\Omega} b(\| \omega - \varpi \|, \phi(\| \omega - \varpi \|))F(\| \omega - \varpi \|, \phi(\| \omega - \varpi \|)) \\
&\leq b(\sup_{\omega,\varpi\in\Omega} \| \omega - \varpi \|, \phi(\sup_{\omega,\varpi\in\Omega} \| \omega - \varpi \|))F(\sup_{\omega,\varpi\in\Omega} \| \omega - \varpi \|, \phi(\sup_{\omega,\varpi\in\Omega} \| \omega - \varpi \|)).
\end{aligned}$$

Thus we get that if $\mathrm{diam}(T\Omega) > 0$ then

$$F(diam(T\Omega), \phi(diamT\Omega)) \leq b(diam\Omega, \phi(diam\Omega))F(diam\Omega, \phi(diam\Omega)).$$

Thus by Theorem 2.5, T has a fixed point. □

Theorem 2.7. *If E is a Banach space and C is a closed convex subset of E. Let T_1 and T_2 be two operators on C such that*

(1) $(T_1 + T_2)\hat{C} \subseteq C, \quad \forall \hat{C} \subseteq C$.
(2) *There exist $F \in T_F$, $b \in T_B$ and $\phi \in T_\phi$ such that $\| T_1\omega - T_1\varpi \| > 0$ implies that*

$$F(\| T_1\omega - T_1\varpi \|, \phi(\| T_1\omega - T_1\varpi \|)) \leq b(\| \omega - \varpi \|, \phi(\| \omega - \varpi \|))F(\| \omega - \varpi \|, \phi(\| \omega - \varpi \|))$$

(3) *T_2 is a continuous and compact operator.*

Then $T = T_1 + T_2 : C \to C$ has a fixed point.

Proof. Suppose $Q \subseteq E$ with $\alpha(Q) > 0$, where α is the Kuratowski's MNC. Then corresponding to every $n \in \mathbb{N}$, $Q_1, Q_2, \ldots, Q_n$ are all bounded sets such that $Q \subseteq \bigcup_{i=1}^{n} Q_i$ and $\mathrm{diam}(Q_i) \leq \alpha(Q) + \frac{1}{n}$. If $\alpha(T_1 Q) > 0$ and since $T_1(Q) \subseteq \bigcup_{i=1}^{n} T_1(Q_1)$ then there exist $\hat{i} \in \{1, 2, 3, ..., n\}$ such that $\alpha(T_1 Q) \leq \mathrm{diam} T_1(Q_{\hat{i}})$.
Then

$$\begin{aligned}
F(\alpha(T_1Q), \phi(\alpha T_1 Q) &\leq F(diam T_1 Q_{\hat{i}}, \phi(\alpha T_1 Q_{\hat{i}})) \\
&\leq b(diam Q_{\hat{i}}, \phi(diam Q_{\hat{i}})) F(diam Q_{\hat{i}}, \phi(diam Q_{\hat{i}})) \\
&\leq b\Big(\alpha(Q) + \frac{1}{n}, \phi(\alpha(Q)) + \frac{1}{n}\Big) F\Big(\alpha(Q) + \frac{1}{n}, \phi(\alpha(Q)) + \frac{1}{n}\Big)
\end{aligned}$$

as $n \to \infty$ we get

$$F(\alpha(T_1Q), \phi(\alpha T_1 Q)) \leq b(\alpha(Q), \phi(\alpha(Q))) F(\alpha Q, \phi(\alpha Q)).$$

Using assumption (3) and by using the definition of α

$$\begin{aligned}
F(\alpha(TQ), \phi(\alpha TQ)) &= F(\alpha(T_1 + T_2)Q, \phi(\alpha(T_1 + T_2)Q)) \\
&\leq F(\alpha T_1 Q + \alpha T_2 Q, \phi(\alpha T_1 Q + \alpha T_2 Q)) \\
&= F(\alpha T_1 Q, \phi(\alpha T_1 Q)) \\
&\leq b(\alpha Q, \phi(\alpha Q)) F(\alpha Q, \phi(\alpha Q)).
\end{aligned}$$

Thus T has a fixed point. □

3. Applications

In this section we will deal with the existence of the solution of the equation (3.1). Let $(E, \| \,.\, \|)$ be a Banach space and $C(I, E)$ be the collection of continuous fuction from $I \to E$, where $\gamma, \beta > 0$ $I = [0, D]$. Let $\vartheta(\zeta) \in C(I, E)$ and consider the fractional integral equation

$$\vartheta(\zeta) = f(\zeta, \vartheta(\zeta)) + \frac{H\vartheta(\zeta)}{\Gamma\gamma} \int_0^{\zeta} \frac{s^{\beta-1} u(\zeta, s, \vartheta(s))}{(\zeta^\beta - s^\beta)^{1-\gamma}} ds. \tag{3.1}$$

We consider the following assumptions

(a1) $f : I \times E \to E$ is continuous and there exist $F \in T_F, b \in T_B$ and $\phi \in T_\phi$ such that

$$\begin{aligned}
&\| f(\zeta, \vartheta(\zeta)) - f(\zeta, \varrho(\zeta)) \| > 0 \\
&\Rightarrow F(\| f(\zeta, \vartheta(\zeta)) - f(\zeta, \varrho(\zeta)) \|, \phi(\| f(\zeta, \vartheta(\zeta)) - f(\zeta, \varrho(\zeta)) \|) \\
&\leq b(\| \vartheta - \varrho \|, \phi(\| \vartheta - \varrho \|)) F(\| \vartheta - \varrho \|, \phi(\| \vartheta - \varrho \|)).
\end{aligned}$$

Also let $\| f(\zeta, \vartheta(\zeta)) \| \leq \psi_1(\| \vartheta \|)$ and $\hat{M} = \sup\{\psi_1(\| \vartheta \|); \vartheta \in C(I, E)\} < \infty$, where $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $I = [0, D]$.

(a2) H is an operator on $C(I, E)$ which is continuous and there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\| H\vartheta(\zeta) \| \leq \psi(\| \vartheta \|)$.

(a3) $u : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $u(I \times I \times \times \mathbb{R}) \subseteq \mathbb{R}_+$ and $\hat{U} = \sup\{|u(\zeta, s, \vartheta(s))|; \zeta, s \in I, u(s) \in C_+(I)\}$.

(a4) $\lim\limits_{\xi\to\infty} \inf \frac{\psi(\xi)\hat{U}D^{\beta\gamma}}{\xi\beta\Gamma(\gamma+1)} < 1$.

Theorem 3.1. *Under the above assumtions $(a1)-(a4)$ the equation (3.1) has a solutions in $C(I,E)$, where $I = [0, D]$*

Proof. Consider the operator on $C(I,E)$ defined by

$$T\vartheta(\zeta) = f(t, \vartheta(\zeta)) + \frac{H\vartheta(\zeta)}{\Gamma\gamma}\int_0^{\zeta} \frac{s^{\beta-1}u(\zeta, s, \vartheta(s))}{(\zeta^\beta - s^\beta)^{1-\gamma}}ds.$$

Let us divide the operator T into two parts T_1 and T_2 such that $T = T_1 + T_2$, where $T_1\vartheta(\zeta) = f(\zeta, \vartheta(\zeta))$ and $T_2 = H\vartheta(\zeta)F_1\vartheta(\zeta)$ such that $F_1\vartheta(\zeta) = \frac{1}{\Gamma\gamma}\int_0^\zeta \frac{s^{\beta-1}u(t,s,\vartheta(s))}{(t^\beta-s^\beta)^{1-\gamma}}ds$.

Now we will first show that T is well defined on $C(I,E)$. It is very easy to see that T_1 is well defined. We will show that T_2 is well defined. Let $\varepsilon > 0$ be arbitrary and $\vartheta(\zeta) \in C(I,E)$ be fixed such that $\hat{r} = \| \vartheta \|$. Also let $\zeta_1, \zeta_2 \in I$, without the loss of generality we can assume that $\zeta_2 > \zeta_1$. Then

$$\begin{aligned}
&|T_2\vartheta(\zeta_2) - T_2\vartheta(\zeta_1)| \\
&= \frac{1}{\Gamma\gamma}\left|H\vartheta(\zeta_2)\int_0^{\zeta_2} \frac{s^{\beta-1}u(\zeta_2, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds - H\vartheta(\zeta_1)\int_0^{\zeta_1} \frac{s^{\beta-1}u(\zeta_1, s, \vartheta(s))}{(\zeta_1^\beta - s^\beta)^{1-\gamma}}ds\right| \\
&\le \frac{1}{\Gamma\gamma}\left|H\vartheta(\zeta_2)\int_0^{\zeta_2} \frac{s^{\beta-1}u(\zeta_2, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds - H\vartheta(\zeta_2)\int_0^{\zeta_1} \frac{s^{\beta-1}u(\zeta_2, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds\right| \\
&+ \frac{1}{\Gamma\gamma}\left|H\vartheta(\zeta_2)\int_0^{\zeta_1} \frac{s^{\beta-1}u(\zeta_2, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds - H\vartheta(\zeta_1)\int_0^{\zeta_1} \frac{s^{\beta-1}u(\zeta_1, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds\right| \\
&+ \frac{1}{\Gamma\gamma}\left|H\vartheta(\zeta_1)\int_0^{\zeta_1} \frac{s^{\beta-1}u(\zeta_1, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds - H\vartheta(\zeta_1)\int_0^{\zeta_1} \frac{s^{\beta-1}u(\zeta_1, s, \vartheta(s))}{(\zeta_1^\beta - s^\beta)^{1-\gamma}}ds\right| \\
&\le \frac{1}{\Gamma\gamma}\left|H\vartheta(\zeta_2)\int_{\zeta_1}^{\zeta_2} \frac{s^{\beta-1}u(\zeta_2, s, \vartheta(s))}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds\right| \\
&+ \frac{|H\vartheta(\zeta_2) - H\vartheta(\zeta_1)|}{\Gamma\gamma}\int_0^{\zeta_1} \frac{s^{\beta-1}|u(\zeta_2, s, \vartheta(s)) - u(\zeta_1, s, \vartheta(s))|}{(\zeta_2^\beta - s^\beta)^{1-\gamma}}ds \\
&+ \frac{1}{\Gamma\gamma}\left|H\vartheta(\zeta_1)\int_0^{\zeta_1} s^{\beta-1}u(\zeta_1, s, \vartheta(s))\left[\frac{1}{(\zeta_2^\beta - s^\beta)^{1-\gamma}} - \frac{1}{(\zeta_1^\beta - s^\beta)^{1-\gamma}}\right]ds\right. \\
&\le \frac{\psi(\| \vartheta \|)\hat{U}}{\beta\Gamma(\gamma+1)}(\zeta_2^\beta - \zeta_1^\beta)^\gamma + \frac{\| H\vartheta(\zeta_2) - H\vartheta(\zeta_1)) \| \, \omega(u,\varepsilon)}{\beta\Gamma(\gamma+1)}\zeta_1^{\beta\gamma} + \frac{\psi(\| \vartheta \|)\hat{U}}{\beta\Gamma(\gamma+1)}[\zeta_2^{\beta\gamma} - \zeta_1^{\beta\gamma} - (\zeta_2^\beta - \zeta_1^\beta)^\gamma] \\
&= \frac{\| H\vartheta(\zeta_2 - H\vartheta(\zeta_1)) \| \, \omega(u,\varepsilon)}{\beta\Gamma(\gamma+1)}\zeta_1^{\beta\gamma} + \frac{\psi(\| \vartheta \|)\hat{U}}{\beta\Gamma(\gamma+1)}[\zeta_2^{\beta\gamma} - \zeta_1^{\beta\gamma}],
\end{aligned}$$

where $\omega(u, \varepsilon) = \sup\{|u(\zeta_2, s, \vartheta(s)) - u(\zeta_1, s, \vartheta(s))|; \zeta_1, \zeta_2 \in I, \vartheta \in [-\hat{r}, \hat{r}]$.

So by the continuity of H and u we can say that $\| T_2\vartheta(\zeta_2) - T_2\vartheta(\zeta_1) \| \to 0$, as $\zeta_2 \to \zeta_1$. Thus $T_2 \in C(I,E)$.

Now we will show that T_2 is continuous operator let $\vartheta(\zeta) \in C(I,E)$ be fixed. Since H is continuous operator on $C(I,E)$ so there exists $\hat{\varepsilon}_1 > 0$ such that

$$\| H\vartheta(\zeta) - H\varrho(\zeta) \| < \hat{\varepsilon}_1, \ \forall \| \vartheta(\zeta) - \varrho(\zeta) \| < \hat{\delta_1}.$$

Also

$$\begin{aligned}
|F_1\vartheta(\zeta) - F_1\varrho(\zeta)| &= \frac{1}{\Gamma\gamma}\left| \int_0^\zeta \frac{s^{\beta-1}u(\zeta,s,\vartheta(s))}{(\zeta^\beta - s^\beta)^{1-\gamma}} ds - \int_0^\zeta \frac{s^{\beta-1}u(\zeta,s,\varrho(s))}{(\zeta^\beta - s^\beta)^{1-\gamma}} ds \right| \\
&= \frac{1}{\Gamma\gamma} \int_0^\zeta \frac{s^{\beta-1}|u(\zeta,s,\vartheta(s)) - u(\zeta,s,\varrho(s))|}{(\zeta^\beta - s^\beta)^{1-\gamma}} ds \\
&\le \frac{\omega(u,\hat{\varepsilon}_2)D^{\beta\gamma}}{\beta\Gamma(\gamma+1)},
\end{aligned}$$

where $\omega(u,\hat{\varepsilon}_2) = \sup\{|u(\zeta,s,\vartheta) - u(\zeta,s,\varrho)|; \zeta, s \in I, \| \vartheta(\zeta) - \varrho(\zeta) \| < \hat{\delta_2}\}$.
Thus, $\| F_1\vartheta(\zeta) - F_1\varrho(\zeta) \| \le \frac{\omega(u,\varepsilon_2)D^{\beta\gamma}}{\beta\Gamma(\gamma+1)}$.
Also

$$\begin{aligned}
|F_1\vartheta(\zeta)| &= \frac{1}{\Gamma\gamma}\left| \int_0^\zeta \frac{s^{\beta-1}u(\zeta,s,\vartheta(s))}{(\zeta^\beta - s^\beta)^{1-\gamma}} ds \right| \\
&\le \frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)}.
\end{aligned}$$

Therefore $\| F_1\vartheta(\zeta) \| \le \frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)}$
For $\delta = \min\{\hat{\delta_1}, \hat{\delta_2}\}$ and $\| \vartheta(\zeta) - \varrho(\zeta) \| < \delta$. We have

$$\begin{aligned}
\| T_2\vartheta(\zeta) - T_2\varrho(\zeta) \| &= \| H\vartheta(\zeta)F_1\vartheta(\zeta) - H\varrho(\zeta)T\varrho(\zeta) \| \\
&\le \| H\vartheta(\zeta) - Hy(\zeta) \| \| F_1\vartheta(\zeta) \| + \| H\vartheta(\zeta) \| \| F_1\vartheta(\zeta) - F_1\varrho(\zeta) \| \\
&\le \hat{\varepsilon_1} \frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)} + \frac{\psi \| \vartheta \| D^{\beta\gamma}\omega(u,\hat{\varepsilon}_2)}{\beta\Gamma(\gamma+1)}.
\end{aligned}$$

As $\| \vartheta(\zeta) - \varrho(\zeta) \| \to 0$ implies that $\hat{\varepsilon_1}, \omega(u,\hat{\varepsilon_2}) \to 0$, therefore from the above relation we get that T_2 is continuous on $C(I,E)$.

Now we will show that T_2 is compact. Let $\hat{B} = \{\vartheta \in C(I,E) : \| \vartheta \| \underline{< 1}\}$ be an open ball in $C(I,E)$. In order to prove T_2 is compact, we are required to show that $\overline{T_2\hat{B}}$ is compact.

Let $\vartheta(\zeta) \in \hat{B}$

$$\begin{aligned}\| T_2\vartheta(\zeta) \| &= \| H\vartheta(\zeta)F_1\vartheta(\zeta) \| \\ &\leq \| H\vartheta(\zeta) \| \| F_1\vartheta(\zeta) \| \\ &\leq \psi(\| 1 \|)\frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)} \\ &= \overline{M}(say).\end{aligned}$$

Thus $T_2\hat{B}$ is uniformly bounded. Let $\vartheta \in \hat{B}$ and $\varepsilon > 0$ be given. Since $H\vartheta(\zeta)$ and $F_1\vartheta(\zeta)$ are uniformly continuous so there exist $\hat{\delta}_1(\varepsilon), \hat{\delta}_2(\varepsilon) > 0$ such that for all $\zeta_1, \zeta_2 \in I$

(i) $\| \zeta_2 - \zeta_1 \| < \hat{\delta}_1(\varepsilon) \Rightarrow \| H\vartheta(\zeta_2) - H\vartheta(\zeta_1) \| < \varepsilon_1$
(ii) $\| \zeta_2 - \zeta_1 \| < \hat{\delta}_2(\varepsilon) \Rightarrow \| H\vartheta(\zeta_2) - H\vartheta(\zeta_1) \| < \varepsilon_2$.

Let $\hat{\delta}(\varepsilon) = \min\{\hat{\delta}_1(\varepsilon), \hat{\delta}_2(\varepsilon), \varepsilon_1, \varepsilon_2\}$, where $\varepsilon_1, \varepsilon_2$ depends on ε. So for $\zeta_1, \zeta_2 \in I$ and $\| (\zeta_2) - \zeta_1 \| < \hat{\delta}(\varepsilon)$ we have

$$\begin{aligned}\| T\vartheta(\zeta_2) - T\vartheta(\zeta_1) \| &= \| H\vartheta(\zeta_2)F_1\vartheta(\zeta_2) - H\vartheta(\zeta_1)F_1\vartheta(\zeta_1) \| \\ &\leq \| H\vartheta(\zeta_2) - H\vartheta(\zeta_1) \| \| F_1\vartheta(\zeta_2) \| + \| H\vartheta(\zeta_1) \| \| F_1\vartheta(\zeta_2) - F_1\vartheta(\zeta_1) \| \\ &< \varepsilon_1\frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)} + \psi(\| \vartheta \|)\varepsilon_2 \\ &< \varepsilon.\end{aligned}$$

For $\varepsilon_1 = \frac{\varepsilon\beta\Gamma(\gamma+1)}{2\hat{U}T^{\beta\gamma}}$ and $\varepsilon_2 = \frac{\varepsilon}{2\psi(\|1\|)}$. Thus $T_2(\hat{B})$ is uniformly bounded and equcontinuous subset of $C(I, E)$. By Arzelá-Ascoli's theorem we conclude that T_2 is compact.

Let $\vartheta, \varrho \in C(I, E)$ be such that $\| T_1\vartheta - T_1\varrho \| > 0$. Since every continuous functions attains its maximum on a compact set, there exists $\zeta \in I$ such that

$$0 < \| T_1\vartheta - T_1\varrho \| = \| f(\zeta, \vartheta(\zeta)) - f(\zeta, \varrho(\zeta)) \| .$$

Now by using assumption (a1) and for $F \in T_F, b \in T_B$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\begin{aligned}F(\| T_1\vartheta - T_1\varrho \|, \phi(\| T_1\vartheta - T_1\varrho \|)) &= F(\| f(\zeta, \vartheta(\zeta)) - f(\zeta, \varrho(\zeta)) \|, \phi(\| f(\zeta, \vartheta(\zeta)) - f(\zeta, \varrho(\zeta)) \|) \\ &\leq b(\| \vartheta - \varrho \|, \phi(\| \vartheta - \varrho \|)F(\| \vartheta - \varrho \|, \phi(\| \vartheta - \varrho \|)).\end{aligned}$$

Since by assumption (a1)

$$\| T_1\vartheta \| = \| f(\zeta, u(\zeta)) \| \leq \psi_1(\| \vartheta \|) \leq \hat{M}.$$

So T_1 is bounded. Finally we will show that there exists $\hat{r} > 0$ such that $T(\hat{B}_{\hat{r}}) \subseteq \hat{B}_{\hat{r}}$, where $\hat{B}_{\hat{r}} = \{\vartheta \in C(I, E); \| \vartheta \| \leq \hat{r}\}$. Let if possible suppose there exists $\xi > 0$ such that $\vartheta_\xi \in \hat{B}_{\hat{r}}$ such that $\| T\vartheta_\xi \| > \xi$. Then we have

$$\lim_{\xi\to\infty} \inf \frac{1}{\xi} \| T\vartheta_\xi \| \geq 1.$$

Then,

$$\begin{aligned}\| T\vartheta_\xi \| &\leq \| f(\zeta,\vartheta_\xi(\zeta)) \| + \| H\vartheta_\xi(\zeta)F\vartheta_\xi(\zeta) \| \\ &\leq \| T_1\vartheta_\xi \| + \| H\vartheta_\xi(\zeta) \|\| F\vartheta_\xi(\zeta) \| \\ &\leq \hat{M} + \psi(\| \vartheta_\xi \|)\frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)} \\ &\leq \hat{M} + \psi(\| \xi \|)\frac{\hat{U}D^{\beta\gamma}}{\beta\Gamma(\gamma+1)}.\end{aligned}$$

Thus

$$\liminf_{\xi\to\infty} \frac{1}{\xi} \| T\vartheta_\xi \| \leq \liminf_{\xi\to\infty} \frac{\psi(\xi)\hat{U}D^{\beta\gamma}}{\xi\beta\Gamma(\gamma+1)} < 1,$$

which is a contradiction. Thus from the given above discussion and using Theorem 2.7 we conclude that T has at least one fixed point □

Example 3.2. *Consider the fractional integral equation*

$$\vartheta(\zeta) = \frac{e^{-\alpha(\zeta+2)}}{1+\zeta^2}\cos(|\vartheta(\zeta)|) + \frac{\sqrt{|\vartheta(\zeta)|}}{9\Gamma(\frac{1}{2})(1+|\vartheta(\zeta)|^2)}\int_0^\zeta \frac{s^3\sin(|\vartheta(\zeta)|)}{\sqrt{\zeta^4-s^4}(1+\zeta^2+s^2)}ds. \tag{3.2}$$

Proof. Here $\gamma = \frac{1}{2}, \beta = 4$ and $I = [0, D],\ D < \infty$. The function $f : I \times \mathbb{R} \to \mathbb{R}$ is defined by $f(\zeta, \vartheta(\zeta)) = \frac{e^{-\alpha(\zeta+2)}}{1+\zeta^2}\cos(|\vartheta(\zeta)|)$ is continuous which can be easily seen as follows

$$|f(\zeta,\vartheta(\zeta)) - f(\zeta,\varrho(\zeta)| \leq \frac{e^{-\alpha(\zeta+2)}}{1+\zeta^2}|\cos(\vartheta) - \cos(\varrho)| \leq e^{-2\alpha}|\vartheta-\varrho|.$$

Also $\psi_1(\zeta) : \mathbb{R}_+ \to \mathbb{R}$ by $\psi_1(\zeta) = \cos(\zeta)$ with $\hat{M} = 1$ such that

$$|f(\zeta,\vartheta(\zeta))| = \left|\frac{e^{-\alpha(\zeta+2)}}{1+\zeta^2}\cos(|\vartheta(\zeta)|)\right| \leq \cos(|\vartheta(\zeta)|) = \psi_1(|\vartheta(\zeta)|).$$

The operator H on $C(I,\mathbb{R})$ given by $H\vartheta(\zeta) = \frac{\sqrt{|\vartheta(\zeta)|}}{3(1+|\vartheta(\zeta)|^2)}$. And if we define $\psi(\zeta) = \frac{\sqrt{\zeta}}{3}$ which is an increasing function then

$$|H\vartheta(\zeta)| = \left|\frac{\sqrt{|\vartheta(\zeta)|}}{3(1+|\vartheta(\zeta)|^2)}\right| \leq \frac{\sqrt{|\vartheta(\zeta)|}}{3} = \psi(|\vartheta|).$$

Now by considering the function $F \in T_F$ by $F(x_1, x_2) = x_1 + x_2,\ \phi(x_1) = e^{-2\alpha}x_1$ and $b \in T_B$ by $b(x_1, x_2) = e^{-2\alpha}$.

If $|f(\zeta,\vartheta(\zeta))-f(\zeta,\varrho(\zeta))|>0$

$$\begin{aligned}
&F(|f(\zeta,\vartheta(\zeta))-f(\zeta,\varrho(\zeta))|,\phi(|f(\zeta,\vartheta(\zeta))-f(\zeta,\varrho(\zeta))|))\\
&=F((|f(\zeta,\vartheta(\zeta))-f(\zeta,\varrho(\zeta))|),e^{-2\alpha}|f(\zeta,\vartheta(\zeta))-f(\zeta,\varrho(\zeta))|)\\
&\le F(e^{-2\alpha}|\vartheta-\varrho|,e^{-4\alpha}|\vartheta-\varrho|)\\
&=e^{-2\alpha}|\vartheta-\varrho|+e^{-4\alpha}|\vartheta-\varrho|\\
&=e^{-2\alpha}(|\vartheta-\varrho|+e^{-2\alpha}|\vartheta-\varrho|)\\
&=b(|\vartheta-\varrho|,\phi(|\vartheta-\varrho|))F(|\vartheta-\varrho|,e^{-2\alpha}|\vartheta-\varrho|)\\
&=b(|\vartheta-\varrho|,\phi(|\vartheta-\varrho|))F(|\vartheta-\varrho|,\phi(|\vartheta-\varrho|)).
\end{aligned}$$

Also $u: I\times I\times\mathbb{R}\to\mathbb{R}$ is defined as $u(\zeta,s,\vartheta(s))=\frac{sin(|\vartheta(s)|)}{3(1+\zeta^2+s^2)}$, then $\hat{U}=\frac{1}{3}$.

Finally

$$\liminf_{\xi\to\infty}\frac{\psi(\xi)\hat{U}D^{\beta\gamma}}{\xi\beta\Gamma(\gamma+1)}=\liminf_{\xi\to\infty}\frac{\sqrt{\xi}D^2}{36\xi\Gamma(\frac{3}{2})}=0<1.$$

Thus the equation (3.2) satisfies all the assumptions of Theorem 3.1. So we conclude that there exists a solution for equation (3.2) in $C(I,\mathbb{R})$. □

Example 3.3. *Consider the fractional integral equation*

$$\mu(t)=\frac{\sin|\mu(t)|}{(2+t^2)\ln(3+\alpha)}+\frac{\sqrt[3]{|\upsilon(t)|}}{2\Gamma(\frac{2}{3})e^{1+|\mu(t)|}}\int_0^t\frac{s^5\cos(1+|\mu(t)|)}{4(t^6-s^6)^{\frac{1}{3}}\sqrt{1+t^\alpha s^\rho}}ds,\ \text{where } \alpha,\rho>0. \tag{3.3}$$

Proof. Here $\gamma=\frac{2}{3}$, $\beta=6$ and $I=[0,D], D<\infty$. The function $f: I\times\mathbb{R}\to\mathbb{R}$ which is defined as $f(t,\mu(t))=\frac{\sin|\mu(t)|}{(2+t^2)\ln(3+\alpha)}$ is continuous, because

$$|f(t,\mu(t))-f(t,\vartheta(t))|\le\frac{1}{(2+t^2)\ln(3+\alpha)}|\sin(\mu)-\sin(\vartheta)|\le\frac{1}{\ln(3+\alpha)}|\mu-\vartheta|.$$

Also $\psi_1:\mathbb{R}_+\to\mathbb{R}_+$ is given by $\psi_1(t)=\sin(t)$ with $\hat{M}=1$ such that

$$|f(t,\mu(t))|=\left|\frac{\sin|\mu(t)|}{(2+t^2)ln(3+\alpha)}\right|\le\sin(|\mu(t)|)=\psi_1(|\mu(t)|).$$

The operator H on $C(I,\mathbb{R})$ given by $H(\mu(t))=\frac{\sqrt[3]{|\mu(t)|}}{2\Gamma(\frac{2}{3})e^{1+|\mu(t)|}}$. Then

$$|H\mu(t)|=\left|\frac{\sqrt[3]{|\mu(t)|}}{2\Gamma(\frac{2}{3})e^{1+|\mu(t)|}}\right|\le\frac{\sqrt[3]{|\mu(t)|}}{2}=\psi(|\mu(t)|),$$

where $\psi(t)=\frac{\sqrt[3]{|t|}}{2}$ is increasing function which can be easily seen.

Now by considering the functions $F\in T_F$ by $F(x_1,x_2)=x_1+x_2$, $\phi(\mu)=\frac{\mu}{\ln(3+\alpha)}$ and $b\in T_B$ by $b(x_1,x_2)=\frac{1}{\ln(3+\alpha)}$.

Indeed if $|f(t,\mu(t))-f(t,\vartheta(t))|>0$

$$\begin{aligned}
&F(|f(t,\mu(t))-f(t,\vartheta(t))|,\phi(|f(t,\mu(t))-f(t,\vartheta(t))|))\\
&=F\bigg(|f(t,\mu(t))-f(t,\vartheta(t))|,\frac{1}{\ln(3+\alpha)}|f(t,\mu(t))-f(t,\vartheta(t))|\bigg)\\
&\leq F\bigg(\frac{1}{\ln(3+\alpha)}|\mu-\vartheta|,\frac{1}{(\ln(3+\alpha))^2}|\mu-\vartheta|\bigg)\\
&=\frac{1}{\ln(3+\alpha)}|\mu-\vartheta|+\frac{1}{(\ln(3+\alpha))^2}|\mu-\vartheta|\\
&=\frac{1}{\ln(3+\alpha)}\bigg(|\mu-\vartheta|+\frac{1}{\ln(3+\alpha)}|\mu-\vartheta|\bigg)\\
&=b(|\mu-\vartheta|,\phi(|\mu-\vartheta|))F\bigg(|\mu-\vartheta|,\frac{1}{\ln(3+\alpha)}|\mu-\vartheta|\bigg)\\
&=b(|\mu-\vartheta|,\phi(|\mu-\vartheta|))F(|\mu-\vartheta|,\phi(|\mu-\vartheta|)).
\end{aligned}$$

Also $u:I\times I\times\mathbb{R}\to\mathbb{R}$ is defined by $u(t,s,\mu(t))=\frac{\cos(1+|\mu(t)|)}{4\sqrt{1+t^\alpha s^\rho}}$, then $\hat{U}=\frac{1}{4}$.

Finally

$$\liminf_{\xi\to\infty}\frac{\psi(\xi)\hat{U}D^{\beta\gamma}}{\xi\beta\Gamma(\gamma+1)}=\liminf_{\xi\to\infty}\frac{\sqrt[3]{|\xi|}D^2}{54\xi\Gamma(\frac{5}{4})}=0<1.$$

Thus the equation (3.3) satisfies all the assumptions of Theorem 3.1. So we conclude that there exists a solution for equation (3.3) in $C(I,\mathbb{R})$. □

Declaration

Conflict of Interest/Competing interests: The authors declare that they have no computing interests.

Author's Contributions: All the authors have equal contribution for the preparation of the article. All authors read and approved the final manuscript.

References

[1] R. P. Agarwal, D. O'Regan, Fixed point theory and applications. Cambridge University Press, 2001.

[2] B. Alqahtani, H. Aydi, E. Karapınar, V. Rakočević, A solution for Volterra fractional integral equations by hybrid contractions. Mathematics, 7(8)(2019), 694.

[3] R. Arab, H.K. Nashine, N.H. Can, T.T. Binh, Solvability of functional-integral equations (fractional order) using measure of noncompactness. Adv. Diff. Equ. 2020(1), 1–13.

[4] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, vol. 60, 1980.

[5] J. Banaś, M. Mursaleen. Sequence spaces and measures of noncompactness with applications to differential and integral equations. New Delhi: Springer, 2014.

[6] S. Beloul, M. Mursaleen, A.H. Ansari, A generalization of Darbo's fixed point theorem with an application to fractional integral equations. J. Math. Ineq, 15(2021), 911–921.

[7] G. Darbo. Punti uniti in trasformazioni a codominio non compatto. Rendiconti del Seminario matematico della Università di Padova, 24(1995), 84–92.

[8] A. Das, B. Hazarika, P. Kumam, Some new generalization of Darbo's fixed point theorem and its application on integral equations. Mathematics, 7(3)(2019), 214.

[9] A. Das, B. Hazarika, S.K. Panda, V. Vijayakumar, An existence result for an infinite system of implicit fractional integral equations via generalized Darbo's fixed point theorem. Comput. Appl. Math. 40(4)(2021), 1–17.

[10] B. C. Deuri, M. V. Paunović, A. Das, V. Parvaneh, Solution of a Fractional Integral Equation Using the Darbo Fixed Point Theorem. J. Mathematics, 2022, Article ID 8415616, https://doi.org/10.1155/2022/8415616.

[11] J.G. Falset, K. Latrach, On Darbo-Sadovskii's fixed point theorems type for abstract measures of (weak) noncompactness. Bulletin of the Belgian Mathematical Society-Simon Stevin, 22(5)(2015), 797–812.

[12] K. Goebel, Thickness of sets in metric spacea and its applicationa to the fixed point theory, Habilit. Thesia, Lublin, 1970.

[13] I. T. Gohberg, L.S. Goldenstein, A.S. Markus, Investigation of some properties of bounded linear operators in connection with their q-norms. Ucen. Zap. Kishinevsk. Un-ta, 29(1957), 29–36.

[14] I. Haque, J. Ali, M. Mursaleen, Solvability of Implicit Fractional Order Integral Equation in $\ell_p\,(1 < p < \infty)$ Space via Generalized Darbo's Fixed Point Theorem, J. Function Spaces, 2022.

[15] B. Hazarika, H.M. Srivastava, R. Arab, M. Rabbani, Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it. J. Comput. Appl. Math. 343(2018), 341–352.

[16] B. Hazarika, H.M. Srivastava, R. Arab, M. Rabbani, Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution. Appl. Math. Comput. 360(2019), 131–146.

[17] K. Kuratowski, Sur les espaces complets, Fundamenta mathematicae, 1(15)(1930), 301–309.

[18] B. Mohammadi, A.A. Shole Haghighi, M. Khorshidi, M. De la, V. Parvaneh, Existence of solutions for a system of integral equations using a generalization of Darbo's fixed point theorem. Mathematics, 8(4)(2020), 492.

[19] H. Monfared, M. Asadi, A. Farajzadeh, New Generalization of Darbo's Fixed Point Theorem via *alpha*-admissible Simulation Functions with Application. Sahand Communications in Mathematical Anal. 17(2)(2020), 161–171.

[20] H.K. Nashine, R. Arab, Existence of solutions to nonlinear functional-integral equations via the measure of noncompactness. Journal of Fixed Point Theory and Applications, 20(2)(2018), 1–17.

[21] H.K. Nashine, A. Das, Extension of Darbo's fixed point theorem via shifting distance functions and its application. Nonlinear Analysis: Modelling and Control, 27(2)(2022), 275–288.

[22] H.K. Nashine, R.W. Ibrahim, R. Arab, M. Rabbani, Generalization of Darbo-Type Fixed Point Theorem and Applications to Integral Equations, In Advances in Metric Fixed Point Theory and Applications, Springer, Singapore, 2021, 333–364.

[23] V. Parvaneh, M. Khorshidi, M. De La Sen, H. Isik, M. Mursaleen, Measure of noncompactness and a generalized Darbo's fixed point theorem and its applications to a system of integral equations. Adv. Dif. Equ. 2020(1), 1–13.

[24] R. Pietkun, On some generalizations of the Mönch, Sadovskii and Darbo fixed point theorems. J. Fixed Point Theory Appl. 20(3)(2018), 1–11.

[25] H.U. Rehman, D. Gopal, P. Kumam, Generalizations of Darbo's fixed point theorem for new condensing operators with application to a functional integral equation. Demonstratio Mathematica, 52(1)(2019), 166–182.

[26] B.N. Sadovskii, On a fixed point principle. Funkc. Analiz i ego Pilo., 1(2)(1967), 74–76 (in Russian).

[27] A. Seemab, M. ur Rehman, Existence of solution of an infinite system of generalized fractional differential equations by Darbo's fixed point theorem. J. Comput. Appl. Math. 364(2020), 112355.

[28] Y. Touail, A. Jaid, D. El Moutawakil, Fixed Point Results for Condensing Operators via measure of non-compactness. Vestnik St. Petersburg University, Mathematics, 55(3), 2022, 347–352.

[29] M. B. Zada, M. Sarwar, T, Abdeljawad, F. Jarad, Generalized Darbo-Type-Contraction an expansion and Its Applications to a Nonlinear Fractional-Order Differential Equation, J. Function Spaces, 2020.