NON-PERIODICITY OF CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In this paper we show that a non-constant periodic function cannot have a periodic Caputo fractional derivative by relaxing the conditions appearing in previous works.

1. Preamble

Consider, for $a \in \mathbb{R}$, $\alpha > 0$ and a continuous function $f : [a, \infty) \to \mathbb{R}$, the Riemann–Liouville fractional integral of order α,

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds.$$

For a sufficiently smooth function f and $n-1 < \alpha < n$ ($n \in \mathbb{N}$), the Riemann–Liouville fractional derivative and the Caputo fractional derivative of order α of f are defined, respectively, by

$$D_{a^+}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n [I_{a^+}^{n-\alpha} f](t),$$

and

$${}^C D_{a^+}^{\alpha} f(t) = D_{a^+}^{\alpha} \left[f(\cdot) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\cdot - a)^k \right](t).$$

In recent times engineers and scientists have developed new models that involve fractional differential equations. These models have been applied successfully, e.g., in mechanics (theory of viscoelasticity and viscoplasticity), (bio-)chemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads), etc... (cf. [3]). The mathematical theory of fractional calculus has also been evolving, in several different research directions [3, 5, 6].

In the past decade some works appeared in the literature showing that a non-constant $T-$periodic ($T > 0$) function cannot have a $T-$periodic fractional derivative (whether it's the Riemann–Liouville or the Caputo one) [1, 4]. One application of such a result is the following: Consider the dynamical system

$${}^C D_{0^+}^{\alpha} y(t) = f(t, y(t)), \ t > 0, \ y(0) = y_0, \tag{1}$$

where f is $T-$periodic with respect to its first argument. Then, there are no non-constant $T-$periodic solutions to (1). For, if it exists such a solution, then

$${}^C D_{0^+}^{\alpha} y(t+T) = f(t+T, y(t+T)) = f(t, y(t)) = {}^C D_{0^+}^{\alpha} y(t),$$

2010 *Mathematics Subject Classification.* 26A33; 34C25.
Key words and phrases. Caputo fractional derivative, periodic function, dynamical system.

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i.e., ${}^C D^{\alpha}_{0+} y$ is $T-$periodic, which contradicts the aforementioned result (cf. [1, 4]).

The (non-)periodicity result proved in [1, 4] is obtained imposing harsh restrictions on the functional space or on the order of the derivative. Concretely, in [1], the authors consider only $\alpha \in (0,1)$ and in [4] the authors consider functions $y \in C^n[0,\infty)$, with $n-1<\alpha<n$. With respect to the latter we should remind the reader that, under the continuity assumption of f in (1), a solution $y \in C^{n-1}[0,\delta]$ ($\delta>0$) to (1) exists. However, in general, there are no solutions in $C^n[0,\delta]$ (cf. [3, Section 6.4]).

Motivated by the reasoning in the previous paragraph, in this work we will prove the following result.

Theorem 1. *Let $n-1<\alpha<n$ ($n\in\mathbb{N}$). Consider the functional space $\mathcal{F}=\{f:[0,\infty):\to\mathbb{R}\}$ s.t. $f\in C^{n-1}[0,\infty), {}^C D^{\alpha}_{0+} f \in C[0,\infty)\}$.*

If $f\in\mathcal{F}$ is a non-constant $T-$periodic function ($T>0$), then ${}^C D^{\alpha}_{0+} f$ is not a $T-$periodic function.

For the proof of Theorem 1, which we postpone to the next section, we will make use of the following (key) result:

Proposition 1. [2, Lemma A.2] *Let $c>a$ be a real number and $u\in L^1[a,c]$. Consider the function $\Psi:(c,\infty)\to\mathbb{R}$ defined by*

$$\Psi(t)=\int_a^c (t-s)^{\mu} u(s)ds, \quad \mu\in\mathbb{R}\backslash\mathbb{N}.$$

If Ψ is a polynomial over a subinterval $I\subset(c,\infty)$ with a nonempty interior, then $u=0$.

We close this section emphasizing that Theorem 1 furnishes a remarkable difference between the classical (integer order derivative) and the fractional calculus.

2. Proof of Theorem 1 and some observations

Proof. (of Theorem 1) Assume that $f\in\mathcal{F}$ is a non-constant $T-$periodic function such that

$${}^C D^{\alpha}_{0+} f(t) = {}^C D^{\alpha}_{0+} f(t+T), \quad \forall t\geq 0.$$

Then (cf. the proof of [5, Theorem 2.1]),

$$\frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{n-\alpha-1}[f^{(n-1)}(s)-f^{(n-1)}(0)]ds = \frac{1}{\Gamma(n-\alpha)}\left(\int_0^{t+T}(t+T-s)^{n-\alpha-1}[f^{(n-1)}(s)-f^{(n-1)}(0)]ds\right)'$$

where we also have used the chain rule on the right hand side. Integrating both sides of the previous equality, we obtain

$$\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}[f^{(n-1)}(s)-f^{(n-1)}(0)]ds = \frac{1}{\Gamma(n-\alpha)}\int_0^{t+T}(t+T-s)^{n-\alpha-1}[f^{(n-1)}(s)-f^{(n-1)}(0)]ds + c, \quad c\in\mathbb{R}.$$

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Performing the change of variable $s = r - T$ on the left hand side of the previous equality and recalling that f^{n-1} is also a non-constant $T-$periodic function on $[0,\infty)$, we deduce that

$$\frac{1}{\Gamma(n-\alpha)}\int_T^{t+T}(t+T-r)^{n-\alpha-1}[f^{(n-1)}(r)-f^{(n-1)}(0)]dr = \frac{1}{\Gamma(n-\alpha)}\int_0^{t+T}(t+T-s)^{n-\alpha-1}[f^{(n-1)}(s)-f^{(n-1)}(0)]ds + c,$$

or

$$\frac{1}{\Gamma(n-\alpha)}\int_0^{T}(t+T-s)^{n-\alpha-1}[f^{(n-1)}(s)-f^{(n-1)}(0)]ds = -c, \quad t \geq 0.$$

By Proposition 1 we conclude that $f^{(n-1)}$ is constant on $[0,T]$, property that extends to $[0,\infty)$ by periodicity. Therefore, f is a polynomial function which, by hypothesis, must be constant. This is absurd and, therefore, the theorem is proved. □

Remark 1. *It is pertinent to highlight the following:*

(1) *The main result of* [1] *and* [4] *relies on the usage of the Laplace transform and the Mellin transform, respectively. The proof of Proposition 1 does not make use of such methods, therefore, being of different nature of the previous known ones in the literature.*

(2) *In* [1, Section 4] *the authors actually show that the Caputo fractional derivative of a $T-$periodic function cannot be $\tilde{T}-$periodic for any period $\tilde{T}$. It is unclear for us if one can directly apply Proposition 1 to show such result without making use of integral transforms.*

(3) *Consider the function,*

$$S(x) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}, \quad x \geq 0.$$

This function is continuous and $2\pi-$periodic on $[0,\infty)$. It has continuous first order derivatives for all $x > 0$ except at the points $x = 2m\pi$, $m = 1, 2, \ldots$. But, for $0 < \alpha < 1$, ${}^CD^{\alpha}_{0+}S \in C[0,\infty)$ (cf. [7, Theorem 3] *and recall that $S(0) = 0$, hence, ${}^CD^{\alpha}_{0+}S = D^{\alpha}_{0+}S$). Therefore, S is a function for which we can apply Theorem 1 (hence, concluding that ${}^CD^{\alpha}_{0+}S$ is not $2\pi-$periodic on $[0,\infty)$) but not* [4, Theorem 2].

(4) *Let $1 < \alpha < 2$. We may show that the function $f_\alpha(x) = E_\alpha(-x^\alpha)$, $x \geq 0$, where $E_\alpha(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+1)}$ is the Mittag-Leffler function, is not periodic*[1] *(note that $E_2(-x^2) = \cos(x)$). Indeed, the function $f_\alpha \in \mathcal{F}$ solves the following fractional initial value problem (cf.* [5, Example 4.10]*),*

$$ {}^CD^{\alpha}_{0+}y(t) = -y(t), \quad y(0) = 1, \; y'(0) = 0, \tag{2}$$

which proves the claim by Theorem 1. Observe that, since $\alpha \notin (0,1)$ and $f_\alpha \notin C^2[0,\infty)$, we could not use the results of [1] *and* [4] *to* (2).

[1] We note that this result follows easilly from, e.g., [3, Theorem 7.5].

Acknowledgments

Rui A. C. Ferreira was supported by the "Fundação para a Ciência e a Tecnologia (FCT)" through the program "Stimulus of Scientific Employment, Individual Support-2017 Call" with reference CEECIND/00640/2017.

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13 Oct 2023 03:08:34 PDT
230907-Ferreira Version 2 - Submitted to Rocky Mountain J. Math.