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SOME SOLUTIONS TO q-STEP NONLINEAR RECURRENCE EQUATIONS

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ABSTRACT. Let $\mathscr{C}$ be the complex numbers, and q any positive integer. Let $F(x_1,\dots,x_q): \mathscr{C}^q \to \mathscr{C}$ be a given function. Let w be any solution to $F(w,\dots,w)=w$. Suppose that F is analytic in a neighbourhood of $(w,\dots,w)$. For each such w, we give a solution to

$$x_n = F(x_{n-1},\dots,x_{n-q})$$

of the form

$$x_n(\alpha,w,r(w)) = w + \sum_{i=1}^{\infty} A_i(w)\, \alpha^i\, [r(w)]^{ni},$$

where α is arbitrary and $r(w)$ is any root of a certain polynomial that is not a root of 1.

1. Introduction

Withers and Nadarajah [3] gave solutions to linear recurrence equations. Withers and Nadarajah [4, 5] gave solutions to nonlinear recurrence equations. Withers and Nadarajah [6] gave solutions to vector nonlinear recurrence equations. The aim of this note is to give solutions to q-step nonlinear recurrence equations.

Let $\mathscr{C}$ denote the complex numbers. It is well known that the linear recurrence equation in $\mathscr{C}$, $x_n = \sum_{j=0}^{p} c_j x_{n-j}$, has a solution of the form $x_n = \sum_{i=1}^{p} a_i r_i^n$, where $\{r_i\}$ are the roots of $1 = \sum_{j=0}^{p} c_j r^{-j}$ if distinct. Less known is its solution in terms of the Bell polynomials below, as given in Withers and Nadarajah [3]. In contrast there has been no theory giving exact solutions to non-linear recurrence equations until Withers and Nadarajah [4] gave solutions to the *recurrence equation of order* 1,

$$x_{n+1} = F(x_n).$$

These are of the form

$$x_n(\alpha,w,r) = w + z_n, \tag{1}$$

where

$$z_n = \sum_{i=1}^{\infty} A_i\, \alpha^i r^{ni},\ A_1 = 1,$$

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$r = F_{.1}(w)$, $F_{.j}(z)$ is the jth derivative of $F(z)$, and w is any *fixed point* of F, that is, $w = F(w)$. The solutions holds for a given x_0 if α can be chosen so that

$$x_0 - w = \sum_{i=1}^{\infty} A_i\, \alpha^i, \tag{2}$$

that is, if $x_0 - w$ is small enough to obtain α by Lagrange inversion, see Section 5 of Withers and Nadarajah [4].

In Section 3, we extend this to the *additive recurrence equation of order q,*

$$x_{n+1} = \sum_{k=0}^{q-1} F_k\left(x_{n-k}\right). \tag{3}$$

The solution has the form (1) with w any fixed point of $F(z) = \sum_{k=0}^{q-1} F_k(z)$, and $r = r(w)$ any root of a certain polynomial of degree q, excluding roots of 1. So, if there are N fixed points w, say w_i, and for each w there are q such $r(w)$, then we have qN solutions, say $r_j(w_i)$. One can then plot each $x_n\left(\alpha, w_i, r_j(w_i)\right)$ versus α for $n = 0, 1, \ldots, q-1$ to see which $\left(x_0, \ldots, x_{q-1}\right)$ are possible. This seems better than obtaining α from a given x_0 by Lagrange inversion as above. A solution need not diverge if $|r(w)| > 1$, see Examples 2.1 and 3.4 of Withers and Nadarajah [4]. A_i is given by *an easily programmed* recurrence equation in terms of w and the derivatives of F at $(w, \ldots, w)$. Our examples give the first few A_i explicitly, but this can be a distraction. For each example, one can plot x_0 or $(x_1, \ldots, x_q)$ against α for each of the qN roots $r(w)$. Our method excludes the special cases

$$r = 0 \Rightarrow x_n \equiv w, \; r^I = 1 \Rightarrow x_{n+I} = x_n. \tag{4}$$

Section 2 deals with (3) for $q = 2$. Section 5 extends (3) to the *general recurrence equation of order q,*

$$x_n = F\left(x_{n-1}, \ldots, x_{n-q}\right),$$

beginning in Section 4 with $q = 2$.

Example 4.1 is an example of a *multiplicative recurrence equation of order q,*

$$x_n = \prod_{k=1}^{q} F_k\left(x_{n-k}\right). \tag{5}$$

These results may be extended to a wider class of solutions, as in Withers and Nadarajah [5]. If $q \geq 2$, the solutions are special as they only have one free variable α, to match x_0, but not x_1.

We use the *partial ordinary Bell polynomial* $B_{i,j} = \widehat{B}_{i,j}(A)$. It is tabled on page 309 of Comtet [1] for $1 \leq i \leq 10$ and defined as follows. For r in $\mathscr{C}$ and $A = (A_1, A_2, \ldots)$ any sequence in $\mathscr{C}$, set

$$S(r, A) = \sum_{i=1}^{\infty} A_i r^i. \tag{6}$$

Then $B_{i,j}$ is defined by

$$[S(r, A)]^j = \sum_{i=j}^{\infty} B_{i,j}\, r^i \tag{7}$$

for $j = 0, 1, \ldots$. So,

$$z_n^j = \sum_{i=j}^{\infty} B_{i,j}\ \alpha^i r^{ni}$$

for z_n of (1). Taking the coefficient of r^i in $[S(r,A)]^j = [S(r,A)]^{j-1} S(r,A)$ gives

$$B_{i,j} = \sum_{k=j-1}^{i-1} B_{k,j-1}\, A_{i-k}$$

for $i \geq j \geq 1$. For subroutines for $B_{i,k}$ for various packages, see Note 1.1 of Withers and Nadarajah [4]. In our case, $A_1 = 1$ so that $B_{i,i} = 1$.

The results here can be extended to x_n a vector, as done in Withers and Nadarajah [6]. Through, $I(A)$ denotes the indicator function.

2. The additive recurrence equation of order 2

In this section, we give solutions to (3) for $q = 2$. Let us write (3) as

$$x_{n+1} = F\left(x_n\right) + G\left(x_{n-1}\right). \tag{8}$$

Choose any w such that $w = F(w) + G(w)$. For $j = 0, 1, \ldots$, set

$$f_j = F_{\cdot j}(w)/j!,\ g_j = F_{\cdot j}(w)/j!,\ h_{i,j} = f_j + g_j r^{-i}, \tag{9}$$

assuming that F and G are analytic at w. We seek a solution of the form $x_n = x_n(\alpha, w, r)$ of (1). We can write this as $z_n = S\left(\alpha r^n, A\right)$ for $S(r,A)$ of (6). So, by (7) and Taylor's expansion,

$$\begin{aligned} \sum_{i=1}^{\infty} A_i\ \left(\alpha r^{n+1}\right)^i &= z_{n+1} = x_{n+1} - w = F\left(x_n\right) - F(w) + G\left(x_{n-1}\right) - G(w) \\ &= \sum_{j=1}^{\infty} \left(f_j z_n^j + g_j z_{n-1}^j\right) = \sum_{i=1}^{\infty} C_i\ \left(\alpha r^n\right)^i, \end{aligned} \tag{10}$$

where

$$C_i = \sum_{j=1}^{i} B_{i,j} h_{i,j} = A_i h_{i,1} + E_i,\ E_i = \sum_{j=2}^{i} B_{i,j} h_{i,j},\ E_1 = 0. \tag{11}$$

The coefficient of $\left(\alpha r^n\right)^i$ in (10) is

$$A_i r^i = C_i = A_i h_{i,1} + E_i.$$

For $i = 1$, this gives $r = h_{1,1} = f_1 + g_1 r^{-1}$, implying

$$r^2 = f_1 r + g_1,\ r = f_1/2 \pm \delta^{1/2} = r_1 \text{ and } r_2 \tag{12}$$

say, for $\delta = f_1^2/4 + g_1$. For $i \geq 2$, it gives the recurrence equation for A_i, in terms of $h_{i,j}$ of (9),

$$A_1 = 1,\ A_i = E_i/R_i = \sum_{j=2}^{i} B_{i,j} H_{i,j},\ i \geq 2,$$

where

$$H_{i,j} = h_{i,j}/R_i, \tag{13}$$

$$R_i = r^i - h_{i,1} = U_i S_i,\ U_i = r^{i-1} - 1, \tag{14}$$

$$S_i = f_1 + g_1\left(r^{-1} + r^{-i}\right) = h_{1,1} + g_1 r^{-i} = r + g_1 r^{-i}. \tag{15}$$

This proves

Theorem 2.1. *Let F, G be analytic functions. Let w be any root of $w = F(w) + G(w)$. Define f_j, g_j, $h_{i,j}$ by (9). Then for $r = r_1$ or r_2 of (12) not a root of 1, (1) is a solution of the recurrence equation (8), where A_i is given by the recurrence equation (13) in terms of R_i of (14) and $\alpha \in \mathscr{C}$ is arbitrary.*

If $r^I = 1$, then $R_{I+1} = R_1 = 0$ and the method fails. As noted in Section 1, if $|r| \geq 1$, the series is likely to diverge. If $F(z)$ and $G(z)$ are polynomials of degree p or less, then $f_j = g_j = h_{i,j} = H_{i,j} = 0$ for $j > p$. In Examples 2.1 to 2.4 and the second part of Example 2.5, $F(z)$ and $G(z)$ are polynomials of degree 2 or 1, so that $f_j = g_j = h_{i,j} = H_{i,j} = 0$ for $j > 2$, and for S_i of (15),

$$\begin{aligned} &h_{i,1} = f_1 + g_1 r^{-i},\ h_{i,2} = f_2 + g_2 r^{-i},\ H_{i,2} = h_{i,2}/R_i,\ R_i = r^i - h_{i,1} = \left(r^{i-1} - 1\right) S_i, \\ &A_1 = 1,\ A_i = B_{i,2} H_{i,2}, \end{aligned}$$

where

$$B_{i,2} = \sum_{j=1}^{i-1} A_j A_{i-j}. \tag{16}$$

There are two choices of w, and for each there are two choices of r, giving four solutions. For each of these one can plot x_0, x_1 versus α, to see which are possible.

Example 2.1. *An extension of the Mandelbrot equation. Take $F(z) = z^2 + c_0$, $G(z) = z^2 + c_1$. Set $c = c_0 + c_1$. Then*

$$\begin{aligned} &w = 2w^2 + c,\ w = \left(1 \pm \Delta^{1/2}\right)/4,\ \Delta = 1 - 8c,\ f_1 = g_1 = 2w,\ f_2 = g_2 = 1, \\ &r^2 = 2w(r+1),\ r = w + \nu, \\ &h_{i,1}/2w = h_{i,2} = 1 + r^{-i},\ H_{i,2} = \left(1 + r^{-i}\right)/R_i,\ S_i = 2w\left(1 + r^{-1} + r^{-i}\right), \\ &R_i = r^i - 2w\left(1 + r^{-i}\right) = \left(s_i + t_i \nu\right)/d_i, \end{aligned}$$

where

$$\begin{aligned} &\nu = \pm\delta^{1/2},\ \delta = w^2 + 2w = (5w - c)/2, \\ &s_2 = 2w^2 - w - 1,\ t_2 = 4w^3 - 8w^2 + 2w + 1,\ d_2 = 2w^2 + 4w + 1, \\ &s_3 = (2w-1)w(2w+3),\ t_3 = (2w+1)\left(2wd_3 + 1\right),\ d_3 = 4w^4 + 6w^3 + 13w^2 + 4w + 1. \end{aligned}$$

Example 2.2. *An extension of the logistic map. Take* $F(z) = c_0\left(z - z^2\right)$, $G(z) = c_1\left(z - z^2\right)$. *Then*

$$w = c\left(w - w^2\right),$$
$$f_1/c_0 = g_1/c_1 = 1 - 2w,\ 1 - 2w_1 = 2c^{-1} - 1,\ f_2/c_0 = g_2/c_1 = -1,$$
$$r = c_0(1-2w)/2 \pm \delta^{1/2},\ h_{i,1}/(1-2w) = -h_{i,2} = c_0 + c_1 r^{-i},\ H_{i,2} = -\left(c_0 + c_1 r^{-i}\right)/R_i,$$
$$R_i = r^i - (1-2w)\left(c_0 + c_1 r^{-i}\right),\ S_i = (1-2w)\left[c_0 + c_1\left(r^{-1} + r^{-i}\right)\right]$$

for $c = c_0 + c_1$, $w = 0$ *or* $w = 1 - c^{-1} = w_1$ *say and* $\delta = c_0^2(1-2w)^2/(4c_1) + 1 - 2w$.

The case $w = 0$**.** *Then*

$$\delta = c_0^2/(4c_1) + 1,\ r = c_0/2 \pm \delta^{1/2},\ R_i = r^i - c_0 - c_1 r^{-i}.$$

The case $w = 1 - c^{-1}$**.** *Then*

$$\delta = c_0^2\left(2c^{-1} - 1\right)^2/(4c_1) + 2c^{-1} - 1,\ r = c_0\left(2c^{-1} - 1\right)/2 \pm \delta^{1/2},$$
$$R_i = r^i - \left(2c^{-1} - 1\right)\left(c_0 + c_1 r^{-i}\right).$$

Example 2.3. *Take* $F(z) = c_0\left(z - z^2\right)$, $G(z) = z^2 + c_1$. *If* $c = 1 - c_0 \neq 0$ *then*

$$w = cw^2 + c_0 w + c_1,\ w^2 - w + c_2 = 0,$$
$$w = 1/2 \pm (1/4 - c_2)^{1/2},\ f_1 = c_0(1-2w),\ g_1 = 2w,\ f_2 = -c_0,\ g_2 = 1,$$
$$r = f_1/2 \pm \delta^{1/2} = r_1 \text{ and } r_2 \text{ say},$$
$$h_{i,1} = c_0(1-2w) + 2wr^{-i},\ R_i = r^i - h_{i,1},\ h_{i,2} = -c_0 + r^{-i},\ H_{i,2} = \left(-c_0 + r^{-i}\right)/R_i$$

for $c_2 = c_1/c$ *and* $\delta = f_1^2/4 + g_1$. *If* $c_0 = 1$ *then* w *is not defined so the method fails.*

Example 2.4. *Take* $F(z) = z^2 + c_1$, $G(z) = c_0\left(z - z^2\right)$. *Then, if* $c_0 \neq 1$, w *is given by Example 2.3,*

$$f_1 = 2w,\ g_1 = c_0(1-2w),\ f_2 = 1,\ g_2 = -c_0,$$
$$r = w \pm \delta^{1/2},$$
$$h_{i,1} = 2w + c_0(1-2w)r^{-i},\ R_i = r^i - h_{i,1},\ h_{i,2} = 1 - c_0 r^{-i},\ H_{i,2} = \left(-c_0 + r^{-i}\right)/R_i$$

for $\delta = w^2 + c_0(1-2w)$.

Example 2.5. *Take* $F(z) = z$. *Then* w *is any root of* $G(w) = 0$. $r = 1/2 \pm \delta^{1/2}$ *for* $\delta = 1/4 + g_1$. $r \neq 0, 1$ *implies that* $g_1 \neq 0$. *So, the method does not cover* $G(x) = cx^d$ *with* $d > 1$, *but it does allow for* $G(x)$ *any quadratic with non-zero discriminant. In that case,* A_i *is given by (16) with* $H_{i,2} = g_2 r^{-i}/R_i$, $R_i = r^i - 1 - g_1 r^{-i}$.

3. The additive recurrence equation (3)

For $k,\ j \geq 0$, let $F_k(x)$ be an analytic function with jth derivative $F_{k.j}(x)$. Set

$$f_{k,j} = F_{k.j}(w)/j!,\ h_{i,j} = \sum_{k=0}^{q-1} f_{k,j}\, r^{-ik},\ F(x) = \sum_{k=0}^{q-1} F_k(x), \tag{17}$$

where $q \geq 1$. Let w be any solution of $F(w) = w$. By (7) and Taylor's expansion, $x_n = w + z_n$ is a solution to (3) for $z_n = \sum_{i=1}^{\infty} A_i \alpha^i r^{in}$ if

$$\sum_{i=1}^{\infty} a_i r^{in+i} = z_{n+1} = x_{n+1} - F(w) = \sum_{k=0}^{q-1} \left[F_k(x_{n-k}) - F_k(w)\right] = \sum_{i=1}^{\infty} r^{in} C_i$$

for C_i, E_i of (11) and $h_{i,j}$ of (17). For $i = 1$ and $\alpha \neq 0$ this gives

$$r = h_{1,1} = \sum_{k=0}^{q-1} f_{k,1}\, r^{-k}. \tag{18}$$

Multiplying by r^{q-1} gives a polynomial of degree q for r with roots $r_1, \ldots, r_q$ say. For $i \geq 2$, it gives the recurrence equation (13) for A_i in terms of

$$H_{i,j} = h_{i,j}/R_i,\ R_i = r^i - h_{i,1},$$

where

$$h_{i,1} = \sum_{k=0}^{q-1} f_{k,1}\, r^{-ik}.$$

If $r^I = 1$, then $R_{I+1} = R_1 = 0$ and the method fails. This proves

Theorem 3.1. *For $k = 0, 1, \ldots, q-1$, let F_k be any function. Choose any w such that $F(w) = w$, where $F(x) = \sum_{k=0}^{q-1} F_k(x)$. Suppose that $\{F_k\}$ are analytic at w. Define $f_{k,j}$, $h_{i,j}$ by (17), R_i by (14), and $H_{i,j}$ by (13). Then for r any root of (18) that is not a root of 1, the additive recurrence equation (8) has solution (3), where A_i is given by the recurrence equation (13).*

Again, α can be obtained from x_0 by Lagrange inversion of (2), but doing that will fix the value of x_1. When $q = \infty$, $F(x)$ must be finite at w. If each $F_k(x)$ is a polynomial of degree p or less, then $h_{i,j} = H_{i,j} = 0$ for $j > p$. For the case $q = 1$, see Withers and Nadarajah [4].

Example 3.1. *Take $q = \infty$, $F_k(x) = [G(x)]^k = c_k(G(x))$ for $c_k(G) = G^k$. So,*

$$F(x) = [1 - G(x)]^{-1}$$

when $|G(x)| < 1$, and the fixed points are the roots of

$$w[1 - G(w)] = 1$$

when $\left|1 - w^{-1}\right| = |G(w)| < 1$. If w is real, this holds if and only if $w > 1/2$. If $w = w_0 e^{i\gamma}$ for $w_0 > 0$ and $i = \sqrt{-1}$, this holds if and only if $w_0 \cos\gamma > 1/2$. $h_{i,j}$ of (17) needs the derivatives of $F_k(x)$ at w. These are given in terms of those of $G(x)$ at w by Faa di Bruno's chain rule, equation [4c], page 137 of Comtet [1]:

$$j!\, f_{k,j} = F_{k\cdot j}(w) = \sum_{i=1}^{j} B_{j,i}(G) c_{k,i}$$

for $j \geq 1$, where

$$c_{k,i} = c_{k\cdot i}(G(w)) = (k)_i\, [G(w)]^{k-i},$$

where $(k)_i = k(k-1)\cdots(k-i+1)$, $B_{j,i}(G)$ *is the partial* exponential *Bell polynomial in* $G=(G_1,G_2,\dots)$ *and* $G_i = G_{.i}(w)$. *These polynomials are tabled on pages 307-308 of Comtet* [1] *for* $1 \le j \le 12$. *We now solve (18):*

$$f_{k,1} = kG(w)^{k-1}G_{.1}(w) = k\left(1-w^{-1}\right)^{k-1}G_{.1}(w),$$

which implies $r^2 = w^2 G_{.1}(w)$ *which implies* $r = 1-w^{-1} \pm [G_{.1}(w)]^{1/2}$.

The case $G(x) = gx$**, where** $g \neq 0$**,** 1**.** *That is,*

$$x_{n+1} = \sum_{k=0}^{\infty} (gx_{n-k})^k .$$

The fixed points are the roots of $w(1-gw)=1$, *that is,* $w = \left(1 \pm \Delta^{1/2}\right)/(2g)$, *where* $\Delta = 1-4g$, *and we require that* $|gw|<1$. *Also*

$$f_{k,j} = \binom{k}{j} g^k w^{k-j},\ h_{i,j} = w^{-j} H_j\left(gwr^{-i}\right),$$

where

$$H_j(x) = \sum_{k=j}^{\infty} \binom{k}{j} x^k = x^j(1-x)^{-j-1}$$

for $|x|<1$. *So, by (18),*

$$r = h_{1,1} = H_1\left(gwr^{-1}\right)w^{-1},$$

where $H_1(x) = x(1-x)^{-2}$, *which implies* $g=(r-gw)^2$ *which implies* $r = gw \pm g^{1/2} = r_1, r_2$ *say. Set* $D_i = r^i - gw$, $N_i = D_i^2 - g$. *Then* $R_i = r^i N_i D_i^{-2}$, $h_{i,1} = gr^i D_i^{-2}$. *If* $g = 1/4$ *then* $r=1$.

Example 3.2. *Take* $q=\infty$, $F_0(x) = b + c_0 x$, $F_k(x) = c_k x$ *for* $k \ge 1$. *So,* $F(x) = b+cx$ *for finite* $c = \sum_{k=0}^{\infty} c_k \neq 1$. $w = b/(1-c)$, $f_{k,1} = c_k$, $f_{k,j} = h_{i,j} = H_{i,j} = 0$ *for* $j \ge 2$, $R_i = r^i - h_{i,1}$, $h_{i,1} = \sum_{k=0}^{\infty} c_k r^{-ik}$, $A_i = 0$ *for* $i \ge 2$ *and* $x_n = w + \alpha r^n$, *where* $\alpha = x_0 - w$ *and* r *is any solution of*

$$r = h_{1,1} = \sum_{k=0}^{\infty} c_k\, r^{-k}.$$

Example 3.3. *Take* $q=\infty$, $F_k(x) = bI(k=0) + c_k x + d_k x^2$ *for* $k \ge 0$. *So,* $F(x) = b+cx+dx^2$ *for finite* $c = \sum_{k=0}^{\infty} c_k$, $d = \sum_{k=0}^{\infty} d_k$. $dw^2 + (c-1)w + b = 0$ *implies* $w = \left(1-c \pm \delta^{1/2}\right)/(2d)$, *where* $\delta = (1-c)^2 - 4bd$ *and* r *is any solution of* $r = h_{1,1}$, *where*

$$h_{1,1} = \sum_{k=0}^{\infty} f_{k,1}\, r^{-ik},\ f_{k,1} = c_k + 2wd_k.$$

For $i \ge 2$, $A_i = B_{i,2} h_{i,2}/R_i$, *where* $R_i = r^i - h_{i,1}$, $h_{i,2} = \sum_{k=0}^{\infty} f_{k,2} r^{-ik}$, $f_{k,2} = 2d_k$ *and* $B_{i,2} = \sum_{j=1}^{i-1} A_j A_{i-j}$. *So,* $A_2 = h_{2,2}/R_2$, $A_3 = 2A_2 h_{3,2}/R_3$, *and so on.*

4. The general two step recurrence equation

Let $F(x_1,x_2): \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ be a given function. In this section, we extend Section 2 by finding solutions to

$$x_n = F(x_{n-1},x_{n-2}). \tag{19}$$

Let w be any root of $F(w,w)=w$. Suppose that $F(x_1,x_2)$ is analytic in a neighbourhood of (w,w). For $j_1, j_2 = 0,1,\ldots$, set

$$F_{j_1,j_2}(x_1,x_2) = \partial_1^{j_1}\partial_2^{j_2}F(x_1,x_2)$$

for

$$\partial_i = \partial/\partial x_i,\ f_{j_1,j_2} = F_{j_1,j_2}(w,w)/j!k!.$$

Let us try again for a solution of the form (1). By (7),

$$\sum_{i=1}^{\infty}(\alpha r^n)^i A_i = z_n = x_n - w = F(x_{n-1},x_{n-2}) - F(w,w) = \sum_{j_1,j_2=0}^{\infty} z_{n-1}^{j_1} z_{n-2}^{j_2} f_{j_1,j_2}$$
$$= \sum_{i_1,i_2=1}^{\infty}\left(\alpha r^{n-1}\right)^{i_1}\left(\alpha r^{n-2}\right)^{i_2} C(i_1,i_2),$$

where

$$C(i_1,i_2) = \sum_{j_1=0}^{i_1}\sum_{j_2=0}^{i_2} B_{i_1,j_1}B_{i_2,j_2} f_{j_1,j_2},$$

excluding $j_1=j_2=0$. For $i\geq 1$, the coefficient of $(\alpha r^n)^i$ is

$$A_i = C_i,$$

where

$$C_i = \sum_{i_1+i_2=i} r^{-i_1-2i_2} C_{i_1,i_2},$$

implying

$$1 = r^{-1}f_{1,0} + r^{-2}f_{0,1},\ r = \left(f_{1,0}\pm\delta^{1/2}\right)/2 = r_1, r_2 \text{ say,} \tag{20}$$

where $\delta = f_{1,0}^2 + 4f_{0,1}$. This holds since $B_{1,1}=A_1=1$. So, for r not a root of 1,

$$A_i = R_i^{-1}E_i \tag{21}$$

for $i \geq 2$, where

$$R_i = 1 - r^{-i} f_{1,0} - r^{-2i} f_{0,1},$$

(22)
$$E_i = r^{-i} E_{i,0} + r^{-2i} E_{0,i} + J_i,\ E_{i,0} = \sum_{j=2}^{i} B_{i,j} f_{j,0},\ E_{0,i} = \sum_{j=2}^{i} B_{i,j} f_{0,j},$$

$$J_i = \sum \left[r^{-i_1 - 2i_2} C_{i_1,i_2} :\ i_1 + i_2 = i,\ i_1 \geq 1, i_2 \geq 1 \right] = \sum_{i_1=1}^{i-1} r^{-2i+i_1} C_{i_1,i-i_1}.$$

This proves

Theorem 4.1. *Given $F(x_1,x_2) : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ let w be any root of $F(w,w) = w$. Suppose that $F(x_1,x_2)$ is analytic in a neighbourhood of (w,w), and that r is either root of (20) but not a root of 1. Then a solution of (19) is (1), where A_i is given by (21) in terms of E_i of (22).*

Example 4.1. *Suppose that (5) holds with $q = 2$, and for $k = 1,2$, $F_k(x_k) = x_k^{a_k}$. So,*

$$f_{j_1,j_2} = \prod_{k=1}^{2} f_{k.j_k},$$

where

$$f_{k.j_k} = F_{k.j_k}(w_k) / j_k! = \binom{a_k}{j_k} w^{a_k - j_k},$$
$$w = w^a,\ a = a_1 + a_2,\ w = 0 \text{ or } 1.$$

Then r is given by (20) in terms of

$$f_{1,0} = a_1 w^{a_1 - 1},\ f_{0,1} = a_2 w^{a_2 - 1}.$$

The case $w = 0$. *Suppose that $a_1, a_2 \in \mathscr{N}$ so that both $F_k(x_k)$ are analytic at 0. Then*

$$f_{1,0} = I(a_1 = 1),\ f_{0,1} = I(a_2 = 1),\ \delta = I(a_1 = 1) + 4I(a_2 = 1).$$

There are four subcases:

(i) *If $F(x) = x_1 x_2$, then $\delta = 5$, $r = \left(1 \pm 5^{1/2}\right)/2$, $R_i = 1 - r^{-i} - r^{-2i}$.*
(ii) *If $F(x) = x_1$, then $\delta = 1$, $r = 0$ or 1, which are not allowed, see (4).*
(iii) *If $F(x) = x_2$, then $\delta = 4$, $r = \pm 1$, and $R_i = 1 - r^{-2i}$.*
(iv) *Otherwise $r = \delta = 0$, $R_i = 1$, see (4).*

In each case, $f_{j_1,j_2} = 0$ unless $(j_1,j_2) = (0,0)$, $(1,0)$ or $(0,1)$, so that $A_i = E_i = 0$ for $i \geq 2$, and $x_n = \alpha r^n$, where $\alpha = x_0$.

The case $w=1$. *Then*

$$f_{j_1,j_2}=\prod_{k=1}^{2}\binom{a_k}{j_k},\ f_{1,0}=a_1,\ f_{0,1}=a_2,\ R_i=1-r^{-i}a_1-r^{-2i}a_2,$$
$$r=\left(a_1\pm\delta^{1/2}\right)/2,$$
$$E_2=r^{-2}\binom{a_1}{2}+r^{-3}a_1a_2+r^{-4}\binom{a_2}{2},$$
$$E_3=r^{-3}\left[2A_2\binom{a_1}{2}+\binom{a_1}{3}\right]+r^{-4}\left[A_2a_1+\binom{a_1}{2}\right]a_2+r^{-5}a_1\left[A_2a_2+\binom{a_2}{2}\right]$$
$$+r^{-6}\left[2A_2\binom{a_2}{2}+\binom{a_2}{3}\right],$$
$$E_4=r^{-4}\left[B_{4,2}\binom{a_1}{2}+3A_2\binom{a_1}{3}+\binom{a_1}{4}\right]+r^{-5}\left[A_3a_1+2A_2\binom{a_1}{2}+\binom{a_1}{3}\right]a_2$$
$$+r^{-6}\left[A_2^2a_1a_2+A_2\left(a_1+a_2-2\right)a_1a_2/2+\binom{a_1}{2}\binom{a_2}{2}\right]+r^{-7}a_1\left[A_3a_2+2A_2\binom{a_2}{2}+\binom{a_2}{3}\right]$$
$$+r^{-8}a_1\left[B_{4,2}\binom{a_2}{2}+3A_2\binom{a_2}{3}+\binom{a_2}{4}\right]$$

for $\delta=a_1^2+4a_2$ *and* $B_{4,2}=2A_3+A_2^2$. *Ocalan and Duman* [2] *gave a solution when* $-a_1=a_2=p>0$.

5. General q step recurrence

Let $F\left(x_1,\ldots,x_q\right):\ \mathscr{C}^q\to\mathscr{C}$ be any function. We give solutions to

$$x_n=F\left(x_{n-1},\ldots,x_{n-q}\right). \tag{23}$$

Let w be any root of $F(w,\ldots,w)=w$. Suppose that F is analytic in a neighbourhood of $(w,\ldots,w)$. For $j_1,\ldots,j_q=0,1,\ldots,$ set

$$F_{j_1,\ldots,j_q}\left(x_1,\ldots,x_q\right)=\partial_1^{j_1}\cdots\partial_q^{j_q}F\left(x_1,\ldots,x_q\right)$$

for

$$\partial_i=\partial/\partial x_i,\ f\left(j_1,\ldots,j_q\right)=f_{j_1,\ldots,j_q}=F_{j_1,\ldots,j_q}(w,\ldots,w)/j_1!\cdots j_q!.$$

Let us try again for a solution of the form (1). Since

$$z_{n-k}^{j_k}=\sum_{i_k=j_k}^{\infty}\left(\alpha r^{n-k}\right)^{i_k}B_{i_k,j_k},$$

we have

$$\sum_{i=1}^{\infty}(\alpha r^n)^i A_i = z_n = x_n - F(w,\dots,w)$$
$$= F(x_{n-1},\dots,x_{n-q}) - F(w,\dots,w) = \sum_{j_1,\dots,j_q=0}^{\infty} f(j_1,\dots,j_q)\, z_{n-1}^{j_1}\cdots z_{n-q}^{j_q}$$
$$(24)\qquad = \sum_{i_1,\dots,i_q=1}^{\infty} \left(\alpha r^{n-1}\right)^{i_1}\cdots\left(\alpha r^{n-q}\right)^{i_q} C(i_1,\dots,i_q),$$

where

$$C(i_1,\dots,i_q) = \sum_{j_1=0}^{i_1}\cdots\sum_{j_q=0}^{i_q} B_{i_1,j_1}\cdots B_{i_q,j_q}\, f(j_1,\dots j_q),$$

excluding $j_1 = \cdots j_q = 0$. Let e_k be the kth unit vector in $\mathscr{C}^q$. Set $|i| = i_1 + \cdots + i_q$. For $I \geq 1$, the coefficient of $(\alpha r^n)^I$ in (24) is

$$A_I = C_I,$$

where

$$(25)\qquad C_I = \sum_{|i|=I} r^{-i_1-2i_2-\cdots-qi_q}\, C(i_1,\dots,i_q).$$

In particular,

$$1 = A_1 = C_1 = \sum_{k=1}^{q} r^{-k} C(e_k),\ C(e_k) = f(e_k),$$

implying

$$(26)\qquad 1 = \sum_{k=1}^{q} r^{-k} f(e_k),$$

a polynomial of degree q in r^{-1} with solutions $r_1,\dots,r_q$ say. Let $\sum_k^{s'}$ denote summation over $1 \leq k_1 < \cdots < k_s \leq q$, and $\sum_k^{s}$ denote summation over $1 \leq k_1 \leq \cdots \leq k_s \leq q$. For $J \geq 1$, set

$$(27)\qquad S_J = \sum_{k=1}^{q} r^{-Jk} f(e_k),\ R_J = 1 - S_J.$$

If $r^J = 1$, then $R_{J+1} = R_1 = 0$ and the method fails. Suppose that r is not a root of 1. If $J = 2$, then $i = e_{k_1} + e_{k_2}$ say, and $\sum_{k=1}^{q} k i_k = k_1 + k_2$. So,

$$A_2 = C_2 = \sum_{1\leq k_1\leq k_2\leq q} r^{-k_1-k_2} C\left(e_{k_1}+e_{k_2}\right),$$
$$C\left(e_{k_1}+e_{k_2}\right) = B_{1,k_1}B_{1,k_2} f\left(e_{k_1}+e_{k_2}\right) = f\left(e_{k_1}+e_{k_2}\right),\ k_1 < k_2,$$
$$(28)\qquad C\left(Je_{k_1}\right) = \sum_{j=1}^{J} B_{J,j} f(je_k),\ C\left(2e_{k_1}\right) = A_2 f(e_k) + f(2e_k)$$

which implies

$$C_2 = E_2 + A_2 S_2$$

for

$$E_2 = \sum_{1 \le k_1 \le k_2 \le q} r^{-k_1-k_2} f\left(e_{k_1} + e_{k_2}\right)$$

and $R_2 A_2 = E_2$ implies $A_2 = R_2^{-1} E_2$. If $J = 3$, then $i = e_{k_1} + e_{k_2} + e_{k_3}$ say, where $k_1 \le k_2 \le k_3$, and $\sum_{k=1}^{3} k i_k = k_1 + k_2 + k_3$. So,

$$A_3 = C_3 = \sum_k^3 r^{-k_1-k_2-k_3} C\left(e_{k_1} + e_{k_2} + e_{k_3}\right) = C^{1,1,1} + \sum^2 C^{2,1} + C^3,$$

where

$$C^{1,1,1} = \sum_k^{3'} r^{-k_1-k_2-k_3} f\left(e_{k_1} + e_{k_2} + e_{k_3}\right),$$

$$C^{2,1} = \sum_k^{2'} r^{-2k_1-k_2} C\left(2e_{k_1} + e_{k_2}\right),$$

$$C\left(2e_{k_1} + e_{k_2}\right) = \sum_{j_1=1}^{2} B_{2,j_1} \sum_{j_2=1} B_{1,j_2} f\left(j_1 e_{k_1} + j_2 e_{k_2}\right) = A_2 f\left(e_{k_1} + e_{k_2}\right) + f\left(2e_{k_1} + e_{k_2}\right),$$

$$C^{1,2} = \sum_k^{2'} r^{-k_1-2k_2} \left[A_2 f\left(e_{k_1} + e_{k_2}\right) + f\left(e_{k_1} + 2e_{k_2}\right)\right],$$

where $\sum^2 C^{2,1} = C^{2,1} + C^{1,2}$. Further,

$$C^3 = \sum_{k=1}^{q} r^{-3k}\, C\left(3e_k\right),\ C\left(3e_k\right) = A_3 f\left(e_k\right) + 2A_2 f\left(2e_k\right) + f\left(3e_k\right)$$

by (28), implying

$$C_3 = E_3 + A_3 S_3$$

for

$$E_3 = \sum_k^3 r^{-k_1-k_2-k_3} f\left(e_{k_1} + e_{k_2} + e_{k_3}\right) + A_2 \sum^2 \sum_k^{2'} r^{-2k_1-k_2} f\left(e_{k_1} + e_{k_2}\right) + 2A_2 \sum_{k=1}^{q} r^{-3k}\, f\left(2e_k\right).$$

$R_3 A_3 = E_3$ implies $A_3 = R_3^{-1} E_3$. Further,

$$A_4 = C_4 = \sum_k^4 = C^{1,1,1,1} + \sum^3 C^{2,1,1} + C^{2,2} + \sum^2 C^{3,1} + C^4,$$

where

$$C^{1,1,1,1} = \sum_k^{4'} r^{-k_1-k_2-k_3-k_4} f\left(\sum_{a=1}^{4} e_{k_a}\right),$$

$$C^{2,1,1} = \sum_k^{3'} r^{-2k_1-k_2-k_3} C\left(2e_{k_1}+e_{k_2}+e_{k_3}\right),$$

$$C\left(2e_{k_1}+e_{k_2}+e_{k_3}\right) = A_2 f\left(e_{k_1}+e_{k_2}+e_{k_3}\right) + f\left(2e_{k_1}+e_{k_2}+e_{k_3}\right),$$

$$C^{2,2} = \sum_k^{2'} r^{-2k_1-2k_2} C\left(2e_{k_1}+2e_{k_2}\right),$$

$$C\left(2e_{k_1}+2e_{k_2}\right) = \sum_{j_1,j_2=1}^{2} B_{2,j_1} B_{2,j_2} f\left(j_1 e_{k_1}+j_2 e_{k_2}\right)$$

$$= A_2^2 f\left(e_{k_1}+e_{k_2}\right) + A_2 \sum^{2} f\left(2e_{k_1}+e_{k_2}\right) + f\left(2e_{k_1}+2e_{k_2}\right),$$

$$C^{3,1} = \sum_k^{2'} r^{-3k_1-k_2} C\left(3e_{k_1}+e_{k_2}\right),$$

$$C\left(3e_{k_1}+e_{k_2}\right) = \sum_{j_1=1}^{3} B_{3,j_1} f\left(j_1 e_{k_1}+e_{k_2}\right)$$

$$= A_3 f\left(e_{k_1}+e_{k_2}\right) + 2A_2 f\left(2e_{k_1}+e_{k_2}\right) + f\left(3e_{k_1}+e_{k_2}\right),$$

$$C(4e_k) = A_4 f(e_k) + B_{4,2} f(2e_k) + 3A_2 f(3e_k) + f(4e_k).$$

Further,

$$C^4 = \sum_{k=1}^{q} r^{-4k} C(4e_k) = A_4 S_4 + E^4$$

say, implying

$$A_4 = R_4^{-1} E_4,$$

where

$$E_4 = C^{1,1,1,1} + \sum^{3} C^{2,1,1} + C^{2,2} + \sum^{2} C^{3,1} + E^4.$$

Similarly, for $J \geq 2$,

$$A_J = R_J^{-1} E_J, \tag{29}$$

where $E_J = C_J - A_J S_J$. This proves

Theorem 5.1. *Given $F(x_1,\ldots,x_q): \mathscr{C}^q \to \mathscr{C}$, let w be any root of $F(w,\ldots,w) = w$. Suppose that $F(x_1,\ldots,x_q)$ is analytic in a neighbourhood of $(w,\ldots,w)$. Then a solution of (23) is (1), where A_J and R_J are given by (29) and (27), and r is any solution of (26) but not a root of 1.*

(25) is a polynomial in r^{-1}. If we started from $x_{n+1} = F\left(x_n,\ldots,x_{n-q+1}\right)$, as in Sections 2-3, rather than from (23), then the left hand side of (25) would be $r^J A_I$. Starting with (23) gives simpler equations.

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