On impulsive p-Laplacian differential equations

Nemat Nyamoradi[1] [*], Fatemeh Sahami[1]

[1] Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.

Abstract

In this article, we discuss a p-Laplacian fractional differential equation involving instantaneous and non-instantaneous impulses. We obtain variational structure for the stated problem. Under this framework, using the critical point theory, we prove the existence result of solutions.

Keywords: Fractional differential equations, non-instantaneous impulses, Solutions.

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1 Introduction

In the past few decades, there has been shown a considerable interest in studying fractional calculus and fractional differential equations, for instance, see [1, 2, 3, 4, 5, 6] and the references cited therein. Recently, many authors studied the impulsive fractional differential equations and impulsive fractional differential equations by using variational methods [7, 8, 9, 10, 11, 12, 13]. Agarwal et al. in [14] and Hernádez et al. in [15] introduced non-instantaneous impulses differential equations. In [16], the authors firs used the variational method and the Lax-Milgram theorem to study the existence of weak solutions to not-instantaneous impulsive differential equations. Also, Khaliq and Rehman [17] by the Lax-Milgram theorem studied not-instantaneous impulsive fractional differential equations. Finally, Tian and Zhang [18] studied the existence of solutions to the following equation:

$$\begin{cases} -\nu''(z) = f_i(z, \nu(z)), & z \in [\zeta_i, \xi_{i+1}],\ i = 2, \dots, L, \\ \triangle \nu'(\xi_i) = I_i(\nu(\xi_i)), & i = 1, 2, \dots, L, \\ \nu'(t) = \nu'(\xi_i^+), & z \in (\xi_i, \zeta_i],\ i = 1, 2, \dots, L, \\ \nu'(\zeta_i^-) = \nu'(\zeta_i^+), & i = 1, 2, \dots, L, \\ \nu(0) = \nu(T) = 0. \end{cases} \tag{1}$$

By motivation from above works, we study the following p-Laplacian fractional differential equation with not-instantaneous impulses:

$$\begin{cases} {}_zD_Z^{\vartheta}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z)) = f_i(z, \nu(z)), & z \in [\zeta_i, z_{i+1}],\ i = 2, \dots, L, \\ \Delta\left({}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_i)^{p-2}}\Phi_p(\mu(\xi_i){}_0D_z^{\vartheta}\nu(\xi_i))\right)\right) = \varpi_i(\nu(\xi_i)), & i = 1, 2, \dots, L, \\ {}_zD_Z^{\vartheta-1}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z)) & \\ \quad = {}_zD_Z^{\vartheta-1}(\frac{1}{\mu(\xi_i{}^+)^{p-2}}\phi_p(\mu(\xi_i{}^+){}_0^cD_z^{\vartheta}\nu(\xi_i{}^+)), & z \in (\xi_i, \zeta_i],\ i = 1, 2, \dots, L, \\ {}_zD_Z^{\vartheta-1}(\frac{1}{\mu(\zeta_i{}^-)^{p-2}}\phi_p(\mu(\zeta_i{}^-){}_0^cD_z^{\vartheta}\nu(\zeta_i{}^-)) & \\ \quad = {}_tD_Z^{\vartheta-1}(\frac{1}{\mu(\zeta_i{}^+)^{p-2}}\phi_p(\mu(\zeta_i{}^+){}_0^cD_z^{\vartheta}\nu(\zeta_i{}^+)), & i = 1, 2, \dots, L, \\ \nu(0) = \nu(Z) = 0, & \end{cases} \tag{2}$$

*Correspond Author E-mail address: nyamoradi@razi.ac.ir; neamat80@yahoo.com.

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where $0=\zeta_0<\xi_1<\zeta_1<\xi_2<\zeta_2<\ldots<\xi_L<\zeta_L<\xi_{L+1}=Z$, $L\in\mathbb{N}$, $L>2$, ${}_zD_Z^{\vartheta}$ and ${}_0^cD_z^{\vartheta}$ are right Riemann-Lioville and left Caputo fractional derivatives of the order $0<\vartheta\leq 1$ respectively (see [2]), $\mu(z)\in L^{\infty}([0,Z])$ with $\mu_0=\text{ess inf}_{[0,Z]}\mu(z)>0$, $\mu^0=\text{ess sup}_{[0,Z]}\mu(z)$, $\phi_p(\sigma)=|\sigma|^{p-2}\sigma$ for $p>1$, $\varpi_i\in C(\mathbb{R},\mathbb{R})$, $f_i\in C((\zeta_i,\xi_{i+1}]\times\mathbb{R},\mathbb{R})$,

$$
{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\zeta_i{}^{\pm})^{p-2}}\phi_p(\mu(\zeta_i{}^{\pm}){}_0^cD_z^{\vartheta}u(\zeta_i{}^{\pm}))\right)=\lim_{r\to\zeta_i^{\pm}}{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(r)^{p-2}}\phi_p(\mu(r){}_0^cD_z^{\vartheta}\nu(r))\right)
$$

and

$$
\begin{aligned}
&\Delta\left({}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_j)^{p-2}}\Phi_p(\mu(\xi_j){}_0D_z^{\vartheta}\nu(\xi_j))\right)\right)\\
&={}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_j^{+})^{p-2}}\Phi_p(\mu(\xi_j^{+}){}_0D_z^{\vartheta}\nu(\xi_j^{+}))\right)-{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_j^{-})^{p-2}}\Phi_p(h(\xi_j^{-}){}_0D_z^{\vartheta}\nu(\xi_j^{-}))\right),\\
&{}_zD_Z^{\vartheta-1}\left(\frac{1}{h(\xi_j^{+})^{p-2}}\Phi_p(\mu(\xi_j^{+}){}_0D_z^{\vartheta}\nu(\xi_j^{+}))\right)=\lim_{z\to\xi_j^{+}}{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\Phi_p(\mu(z){}_0D_z^{\vartheta}\nu(z))\right),\\
&{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_j^{-})^{p-2}}\Phi_p(\mu(\xi_j^{-}){}_0D_z^{\vartheta}\nu(\xi_j^{-}))\right)=\lim_{z\to\xi_j^{-}}{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\Phi_p(\mu(z){}_0D_z^{\vartheta}\nu(z))\right).
\end{aligned}
$$

To state our result, we need the assumptions:

(H_1) There exists a constant $\gamma_i\in[0,p)$ for any $i=1,\ldots,L$, such that

$$
\gamma_i\int_0^s\varpi_i(\tau)d\tau\leq\varpi_i(s)s,\quad\text{for every}\ \ s\in\mathbb{R}.
$$

(H_2) ϖ_i satisfy $\mathcal{H}_i:=\inf_{|\tau|=1}\int_0^s\varpi_i(\tau)d\tau>0$.

(H_3) There exists positive constants $\beta_i\in[0,p)$ for any $i=1,\ldots,L$, such that

$$
F_i(z,\tau)\leq\beta_i\tau^q,\ \ \forall\,q\in[0,p),z\in[0,Z].
$$

The our main result is as follows:

Theorem 1. *Assume that $\frac{1}{p}<\vartheta\leq 1$, $1<p<+\infty$ and (H_1)-(H_3) hold, then the problem (2) has a weak solution.*

2 Preliminaries

In this section, we introduce some basic definitions and lemmas.

Definition 1. *([13]) Let $p\in[1,\infty)$ and $\vartheta\in(0,1]$. Define the following space*

$$
E^{\vartheta,p}=\overline{C_0^{\infty}([0,Z],\mathbb{R})}^{\|\nu\|_{\vartheta,p}}
$$

with the norm

$$
\|\nu\|_{\vartheta,p}=\left(\int_0^Z|\nu(z)|^pdz+\int_0^Z\mu(z)|{}_0^cD_z^{\vartheta}\nu(z)|^pdz\right)^{\frac{1}{p}}. \tag{3}
$$

Therefore,

$$
E^{\vartheta,p}=\{\nu\in L^p[0,Z]|\ {}_0^cD_z^{\vartheta}\nu(z)\in L^p[0,Z],\ \nu(0)=\nu(Z)=0\}.
$$

Also, we know that $E_0^{\vartheta,p}$ for $0<\vartheta\leq 1$ is a separable and reflexive Banach space (See [12, 19]).

In view of Proposition 3.2 in [19], we have the following Lemma:

Lemma 1. *Let $p \in [1,\infty)$ and $0 < \vartheta \leq 1$. For every $\nu \in E_0^{\vartheta,p}$, we have*

$$\|\nu\|_{L^p} \leq \frac{Z^\vartheta}{\Gamma(\vartheta+1)\mu_0^{\frac{1}{p}}} \left(\int_0^Z \mu(z) |{}_0D_z^\vartheta \nu(z)|^p dz \right)^{\frac{1}{p}}, \quad \textit{for}\, 0 < \vartheta \leq 1, \tag{4}$$

also, when $\vartheta > \frac{1}{p}$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\|\nu\|_\infty \leq \frac{Z^{\vartheta - \frac{1}{p}}}{\Gamma(\vartheta)((\vartheta-1)p'+1)^{\frac{1}{p'}} \mu_0^{\frac{1}{p}}} \left(\int_0^Z \mu(z) |{}_0D_z^\vartheta \nu(z)|^p dz \right)^{\frac{1}{p}}. \tag{5}$$

Remark 1. *By (4), the norm of (3) is equivalent of*

$$\|\nu\|_{\vartheta,p} = \left(\int_0^Z \mu(z) |{}_0D_z^\vartheta \nu(z)|^p dz \right)^{\frac{1}{p}}, \quad \forall \nu \in E_0^{\vartheta,p}. \tag{6}$$

Proposition 1. *Let $p \geq 1, p' \geq 1, \frac{1}{p} + \frac{1}{p'} \leq 1 + \eta$ or $p \neq 1, p' \neq 1, \frac{1}{p} + \frac{1}{p'} = 1 + \eta$ and $\nu \in L^p([0,Z]), v \in L^{p'}([0,Z])$. Then,*

$$\int_0^T ({}_0D_z^{-\eta} \nu(z)) v(t) dt = \int_0^Z ({}_zD_Z^{-\eta} v(t)) \nu(z) dt, \quad \textit{for}\, \eta > 0.$$

Now, by similar methods in [4], one can get the following lemma:

Lemma 2. *Let $m-1 < \vartheta \leq m$, $\nu_2 \in AC[0,Z], \nu_2' \in L^p[0,Z]$, ${}_0^cD_z^\vartheta \in L^p[0,Z]$ and ${}_zD_Z^\vartheta \left(\frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu_1(z)) \right) \in AC[0,Z]$. Then*

$$\begin{aligned}
&\int_{a_0}^{b_0} \frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu_1(z)) ({}_0^cD_z^\vartheta \nu_2(z)) dz \\
&= \int_{a_0}^{b_0} \frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu_1(t)) ({}_0^cD_z^{\vartheta-1} \nu_2'(z)) dz \\
&= \int_{a_0}^{b_0} \left[{}_zD_Z^{\vartheta-1} \left(\frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu_1(z)) \right) \right] \nu_2'(z) dz \\
&= {}_zD_Z^{\vartheta-1} \left(\frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu_1(z)) \right) \nu_2(z) \Big|_{a_0}^{b_0} \\
&\quad - \int_{a_0}^{b_0} \frac{d}{dz} \left[{}_zD_Z^{\vartheta-1} \left(\frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu_1(z)) \right) \right] \nu_2(z) dz.
\end{aligned} \tag{7}$$

3 Proof of the main result

We now prove the variational structure to the equation.

Lemma 3. *For $\nu \in E_0^{\vartheta,p}$, the problem (2) is equivalent of the following form:*

$$\begin{aligned}
&\int_0^Z \frac{1}{\mu(z)^{p-2}} \phi_p(\mu(z) {}_0^cD_z^\vartheta \nu(z)) ({}_0^cD_z^\vartheta \Upsilon(z)) dz \\
&\quad = \sum_{i=0}^L \int_{\zeta_i}^{\xi_{i+1}} f_i(z,\nu) \phi(z) dz - \sum_{i=1}^L \varpi_i(\nu(z)) \Upsilon(\xi_i), \quad \forall\, \Upsilon \in E_0^{\vartheta,p}.
\end{aligned} \tag{8}$$

Proof. For any $\nu, \Upsilon \in E_0^{\vartheta,p}$, In view of Proposition 1 and (7), we have

$$\int_0^T \frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))(_0^cD_z^\vartheta\Upsilon(z))dt = \int_0^Z \frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))(_0^cD_z^{\vartheta-1}\Upsilon'(z))dz$$
$$= \int_0^Z \left[{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))\right)\right]\Upsilon'(z)dz$$
$$= \int_0^{t_1} \left[{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))\right)\right]\Upsilon'(z)dz$$
$$+\sum_{i=1}^{L}\int_{\xi_i}^{\zeta_i} \left[{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))\right)\right]\Upsilon'(z)dz$$
$$+\sum_{i=1}^{L-1}\int_{\zeta_i}^{\xi_{i+1}} \left[{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))\right)\right]\Upsilon'(z)dz$$
$$+\int_{\zeta_n}^{Z} \left[{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))\right)\right]\Upsilon'(z)dz$$
$$= {}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_1^-)^{p-2}}\phi_p(h(\xi_1^-)_0^cD_z^\vartheta\nu(\xi_1^-)\right)\Upsilon(\xi_1)$$
$$-\int_0^{\xi_1}\frac{d}{dz}\left[{}_zD_Z^{\vartheta-1}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z)))\right]\phi(z)dz$$
$$+\sum_{i=1}^{L}\Bigg\{{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\zeta_i^-)^{p-2}}\phi_p(\mu(\zeta_i^-)_0^cD_z^\vartheta\nu(\zeta_i^-))\right)\Upsilon(\zeta_i)$$
$$-{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_i^+)^{p-2}}\phi_p(\mu(\xi_i^+)_0^cD_z^\vartheta\nu(\xi_i^+))\right)\Upsilon(\xi_i)\Bigg\}$$
$$-\sum_{i=1}^{L}\int_{\xi_i}^{\zeta_i}\frac{d}{dz}\left[{}_zD_Z^{\vartheta-1}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z)))\right]\Upsilon(z)dz$$
$$+\sum_{i=1}^{L-1}\Bigg\{{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_{i+1}^-)^{p-2}}\phi_p(\mu(\xi_{i+1}^-)_0^cD_z^\vartheta\nu(\xi_{i+1}^-))\right)\Upsilon(\xi_{i+1})$$
$$-{}_tD_Z^{\vartheta-1}\left(\frac{1}{\mu(\zeta_i^+)^{p-2}}\phi_p(\mu(\zeta_i^+)_0^cD_z^\vartheta\nu(\zeta_i^+))\right)\Upsilon(\zeta_i)\Bigg\}$$
$$-\sum_{i=1}^{L-1}\int_{\zeta_i}^{\xi_{i+1}}\frac{d}{dz}\left[{}_zD_Z^{\vartheta-1}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z)))\right]\Upsilon(z)dz$$
$$-{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\zeta_n^+)^{p-2}}\phi_p(\mu(\zeta_n^+)_0^cD_z^\vartheta\nu(\zeta_n^+))\right)\Upsilon(\zeta_n)$$
$$-\int_{\zeta_n}^{Z}\frac{d}{dz}\left[{}_zD_Z^{\vartheta-1}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z)))\right]\Upsilon(z)dz$$
$$= \int_0^Z {}_zD_Z^{\vartheta}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z)_0^cD_z^\vartheta\nu(z))\right)\Upsilon(z)dz$$
$$+\sum_{i=1}^{L}\Bigg\{{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_i^+)^{p-2}}\phi_p(\mu(\xi_i^+)_0^cD_z^\vartheta\nu(\xi_i^+))\right)$$
$$-{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\xi_i^-)^{p-2}}\phi_p(\mu(\xi_i^-)_0^cD_z^\vartheta)\nu(\xi_i^-)\right)\Bigg\}\Upsilon(\xi_i)$$
$$+\sum_{i=0}^{L}\Bigg\{{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(\zeta_i^+)^{p-2}}\phi_p(\mu(\zeta_i^+)_0^cD_z^\vartheta\nu(\zeta_i^+))\right)$$
$$-{}_tD_Z^{\vartheta-1}\left(\frac{1}{\mu(\zeta_i^-)^{p-2}}\phi_p(\mu(\zeta_i^-)_0^cD_z^\vartheta)\nu(\zeta_i^-)\right)\Bigg\}\Upsilon(\zeta_i),$$

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which together by (2), we obtain

$$\int_0^Z {}_zD_Z^{\vartheta}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))\right)\Upsilon(z)dz = \int_0^Z \frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z)){}_0^cD_z^{\vartheta}\Upsilon(z)dz + \sum_{i=1}^{L}\varpi_i(u(\xi_i))\Upsilon(\xi_i). \tag{9}$$

Also from problem (2), one can get

$$\begin{aligned}
&\int_0^Z {}_zD_Z^{\vartheta}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))\Upsilon(z)dt \\
&= \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} {}_zD_Z^{\vartheta-1}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))\Upsilon(z)dt \\
&\quad + \sum_{i=1}^{L}\int_{\xi_i}^{\zeta_i} {}_zD_Z^{\vartheta}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))\Upsilon(z)dt \\
&= \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} f_i(z,\nu)\Upsilon(z)dz + \sum_{i=1}^{L}\int_{\xi_i}^{\zeta_i} -\frac{d}{dz}\left[{}_zD_Z^{\vartheta-1}\left(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))\right)\right]\Upsilon(z)dz \\
&= \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} f_i(z,\nu)\Upsilon(z)dz + \sum_{i=1}^{L}\int_{\xi_i}^{\zeta_i} -\frac{d}{dz}(\varpi_i(\nu))\Upsilon(z)dz \\
&= \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} f_i(z,\nu)\Upsilon(z)dz.
\end{aligned} \tag{10}$$

So, by (9) and (10), we get

$$\begin{aligned}
&\int_0^Z \frac{1}{\mu(z)^{p-2}}\Phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))({}_0^cD_z^{\vartheta}\Upsilon(z))dz \\
&\quad = \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} f_i(z,\nu)\Upsilon(z)dz - \sum_{i=1}^{L}\varpi_i(\nu(z))\Upsilon(\xi_i).
\end{aligned}$$

So, we have the conclusion. □

Now, we can define the weak solution of (2).

Definition 2. *Let $\upsilon \in E_0^{\vartheta,p}$, then ν is called weak solution of (2) if (8) is satisfied for every $\phi \in E_0^{\vartheta,p}$.*

Define the functional $\boldsymbol{\psi} : E_0^{\vartheta,p} \to \mathbb{R}$ as

$$\boldsymbol{\psi}(\nu) = \frac{1}{p}\int_0^Z \mu(z)|{}_0^cD_z^{\vartheta}\nu(z))|^p dz - \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} F_i(z,\nu)dz + \sum_{i=1}^{L}\int_0^{\nu(\xi_i)}\varpi_i(\tau)d\tau, \tag{11}$$

where $F_i(z,\nu) = \int_0^{\nu} f_i(z,\tau)d\tau$.

Obviously, $\boldsymbol{\psi}$ is continuously differentiable on $E_0^{\vartheta,p}$ and

$$\begin{aligned}
\langle\boldsymbol{\psi}'(\nu),\phi\rangle \quad = \quad & \int_0^Z \frac{1}{\mu(z)^{p-2}}\Phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu(z))({}_0^cD_z^{\vartheta}\phi(z))dz - \sum_{i=0}^{L}\int_{\zeta_i}^{\xi_{i+1}} f_i(z,\nu)\phi dz \\
& + \sum_{i=1}^{L}\varpi_i(\nu(\xi_i))\phi(\xi_i).
\end{aligned} \tag{12}$$

Clearly, the critical points of ψ are equivalent by weak solutions of (2).

To prove the main result (Theorem 1), we bring the following theorem.

Theorem 2. *(Theorem 4.2 ([21])) Let E be a Banach space, $\Theta : E \to \mathbb{R}$ be a differentiable on E and bounded from below function. Then, for every $\epsilon > 0$ and for each $\nu \in E$ such that*

$$\Theta(\nu) \leq \inf_E \Theta + \epsilon$$

there exists $\phi \in E$ such that $\Theta(\phi) \leq \Theta(\nu), |\nu - \phi| \leq \epsilon^{\frac{1}{2}}$ and $|\Theta'(\phi)| \leq \epsilon^{\frac{1}{2}}$.

Now, we can prove the main result (Theorem 1).

Prof of the Theorem 1. We will use Theorem 2 to prove this theorem. In view of (H_1), (H_2) and similar methods the formula (36) in [22], one can get

$$\int_0^z \varpi_i(\tau)d\tau \geq \mathcal{H}_i \, |z|^{\gamma_i}\,, \tag{13}$$

where $\mathcal{H}_i = \inf_{|z|=1} \int_0^z \varpi_i(\tau)d\tau > 0$. Then by (5), (13) and (H_3), we have

$$\begin{aligned} \psi(\nu) &= \frac{1}{p}\|u\|^p_{\vartheta,p} - \sum_{i=0}^{L} \int_{\zeta_i}^{\xi_{i+1}} F_i(z,\nu)dz + \sum_{i=1}^{L} \int_0^{\nu(\xi_i)} \varpi_i(\tau)d\tau \\ &\geq \frac{1}{p}\|\nu\|^p_{\vartheta,p} - \|\nu\|^q_{\vartheta,p} \left(\frac{Z^{\vartheta-\frac{1}{p}}}{\Gamma(\vartheta)((\vartheta-1)p'+1)^{\frac{1}{p'}} \mu_0^{\frac{1}{p}}} \right)^q \sum_{i=0}^{L} \beta_i(\xi_{i+1} - \zeta_i) \\ &\quad - \sum_{i=1}^{L} \mathcal{H}_i \left(\frac{Z^{\vartheta-\frac{1}{p}}}{\Gamma(\vartheta)((\vartheta-1)p'+1)^{\frac{1}{p'}} \mu_0^{\frac{1}{p}}} \right)^{\gamma_i} \|\nu\|^{\gamma_i}_{\vartheta,p}. \end{aligned} \tag{14}$$

So, there exists $\rho > 0$ such that $\psi(\nu) > 0$ for all $\nu \in E^{\vartheta,p}$ with $\|\nu\|_{\vartheta,p} = \rho$, which by define $E = \overline{B_\rho(0)} \subset E^{\vartheta,p}$, since $\gamma_i, q < p$ then $\psi(\nu)$ is bounded from below.
By similar argument in the proof of Theorem 2.1 in [18], for each $\epsilon > 0$, one can get

$$\inf_{\nu\in E} \psi(\nu) - \epsilon < \psi(\phi) \leq \psi(z) \leq \inf_{\nu\in E} \psi(\nu) + \epsilon. \tag{15}$$

Also, by Theorem 2, we have

$$\|\psi'(\phi)\|_{E^*} \leq \epsilon^{\frac{1}{2}}. \tag{16}$$

By (15) and (16) there exists sequence $\{\nu_n\} \subset B_\rho(0)$ such that

$$\psi(\nu_n) \to \inf_{\nu\in E} \psi(\nu), \quad \psi'(\nu_n) \to 0.$$

Obviously, $\{\nu_n\}$ is bounded. Since E is a close subset of the reflexive space $E^{\vartheta,p}$, then E by the restrict norm $\|\cdot\|_{\vartheta,p}$ on E is reflexive. So the sequence $\{\nu_n\}$ weakly converges to ν^* in E. Also, we claim that $\{\nu_n\}$ strongly converges to ν^* in E. From (12), we get

$$\begin{aligned}\left\langle \boldsymbol{\psi}'(\nu_n) - \boldsymbol{\psi}'(\nu^*), \nu_n - \nu^* \right\rangle &= \int_0^Z \frac{1}{\mu(z)^{p-2}} \Bigg(\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu_n) \\ &\quad - \phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu^*) \Bigg) ({}_0^cD_z^{\vartheta}\nu_n(z) - {}_0^cD_z^{\vartheta}\nu^*(z))dz \\ &\quad - \sum_{i=0}^{L} \int_{\zeta_i}^{\xi_{i+1}} (f_i(z,\nu_n) - f_i(z,\nu^*))(\nu_n - \nu^*)dz \\ &\quad - \sum_{i=1}^{L} (\varpi_i(\nu_n(\xi_i)) - \varpi_i(\nu^*(\xi_i)))(\nu_n(\xi_i) - \nu^*(\xi_i)). \end{aligned} \tag{17}$$

Since $\nu_n \rightharpoonup \nu^*$ in E, we get $\{\nu_n\}$ uniformly converges to ν^* in E. Thus

$$\begin{cases} \sum_{i=0}^{L} \int_{\zeta_i}^{\xi_{i+1}} (f_i(z,\nu_n) - f_i(z,\nu^*))(\nu_n - \nu^*)dz \to 0 \ \ as \ \ n \to \infty, \\ \sum_{i=1}^{L} (\varpi_i(\nu_n(\xi_i)) - \varpi_i(\nu^*(\xi_i)))(\nu_n(\xi_i) - \nu^*(\xi_i)) \to 0 \ \ as \ \ n \to \infty. \end{cases} \tag{18}$$

By (17) and (18) we get

$$\int_0^Z \frac{1}{\mu(z)^{p-2}} \left(\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu_n) - \phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu^*) \right) ({}_0^cD_z^{\vartheta}\nu_n(z) - {}_0^cD_z^{\vartheta}\nu^*(z))dz \to 0,$$

which yields that

$$\int_0^Z \left(\phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu_n) - \phi_p(\mu(z){}_0^cD_z^{\vartheta}\nu^*) \right) ({}_0^cD_z^{\vartheta}\nu_n(z) - {}_0^cD_z^{\vartheta}\nu^*(z))dz \to 0.$$

Then, by similar methods of the proof of Theorem 16 in [12], we can get $\|\nu_n - \nu^*\|_{\vartheta,p} \to 0$ as $n \to \infty$, $\{\nu_n\}$ strongly converges to ν^* in E. Then

$$\boldsymbol{\psi}(\nu^*) = \inf_{\nu \in E} \boldsymbol{\psi}(\nu), \quad \boldsymbol{\psi}'(\nu^*) = 0$$

Therefore, ν^* is a weak solution of (2). □

Example 1. Consider the following boundary value problem,

$$\begin{cases} {}_zD_Z^{\frac{3}{4}}(\frac{1}{\mu(z)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\frac{3}{4}}\nu(z)) = f_i(z,\nu), \qquad t \in [\zeta_i, \xi_{i+1}], \ i = 2, \dots, L, \\ \Delta \left({}_zD_Z^{-\frac{1}{4}} \left(\frac{1}{\mu(\xi_i)^{p-2}} \Phi_p(\mu(\xi_i){}_0D_z^{\frac{3}{4}}\nu(\xi_j)) \right) \right) = \varpi_i(\nu(\xi_i)), \qquad i = 1, 2, \dots, L, \\ {}_zD_Z^{-\frac{1}{4}}(\frac{1}{\mu(t)^{p-2}}\phi_p(\mu(z){}_0^cD_z^{\frac{3}{4}}\nu(z)) \\ \qquad = {}_zD_Z^{-\frac{1}{4}}(\frac{1}{\mu(\xi_i{}^+)^{p-2}}\phi_p(\mu(\xi_i{}^+){}_0^cD_z^{\frac{3}{4}}\nu(\xi_i{}^+)), \qquad z \in (\xi_i, \zeta_i], \ i = 1, 2, \dots, L, \\ {}_zD_Z^{-\frac{1}{4}}(\frac{1}{\mu(\zeta_i{}^-)^{p-2}}\phi_p(\mu(\zeta_i{}^-){}_0^cD_z^{\frac{3}{4}}\nu(\zeta_i{}^-)) \\ \qquad = {}_zD_Z^{-\frac{1}{4}}(\frac{1}{\mu(\zeta_i{}^+)^{p-2}}\phi_p(\mu(\zeta_i{}^+){}_0^cD_z^{\frac{3}{4}}\nu(\zeta_i{}^+)), \qquad i = 1, 2, \dots, L, \\ \nu(0) = \nu(Z) = 0, \end{cases} \tag{19}$$

where $\varpi_i(\nu) = \nu$ and $f_i(z,\nu) = \nu z^{\frac{i}{4}}$ for $i = 1, 2, \dots, L$. Direct computation shows that (H_1)-(H_3) holds with $\gamma_i = \frac{1}{2}$, $q = 1$ and $\beta_i = Z^{\frac{i}{4}}$. According to Theorem 1, the above non-instantaneous impulsive problem of fractional order has a unique weak solution.

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