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A PROBABILISTIC APPROACH FOR CONVERGENCE OF AN OPERATOR BASED UPON HERMITE POLYNOMIALS

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ABSTRACT. In this article we study some properties of an operator, which is based on Hermite polynomials. We find the estimates of convergence for such operators in the light of probabilistic approach.

1. Introduction

Krech [8] introduced an operator based on Hermite polynomials, which for $f \in C[0,\infty)$ is defined as follows

$$(1) \qquad (G_n^\alpha f)(x) = e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) f\left(\frac{k}{n}\right), \qquad \alpha, x \geq 0, n \in N,$$

where $H_k(n,\alpha)$ are the Hermite polynomials depending on two variables given by

$$H_k(n,\alpha) := k! \sum_{m=0}^{[\frac{k}{2}]} \frac{\alpha^m}{m!} \frac{n^{k-2m}}{(k-2m)!}, n, k \in N.$$

As a special case when $\alpha = 0$, then $H_k(n,0) = n^k$ and we get the classical Szász-Mirakyan operators defined by

$$(2) \qquad (G_n f)(x) := (G_n^0 f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad \alpha, x \geq 0, n \in N,$$

Also, for negative values the two variable Hermite polynomials are connected with standard Hermite polynomial $H_k(\alpha) := k! \sum_{s=0}^{[\frac{k}{2}]} \frac{(-1)^s}{s!} \frac{(2\alpha)^{k-2s}}{(k-2s)!}$, by the relation $H_k(2\alpha,-1) = H_k(\alpha)$.

Remark 1. *As the Hermite distribution is a combination of two independent Poisson distributions, the moment generating function of the operators G_n^α for A real may be evaluated as*

$$\begin{aligned}(G_n^\alpha e^{At})(x) &= e^{-nx-\alpha x^2} \sum_{m=0}^{[k/2]} \frac{(nx)^{k-2m}}{(k-2m)!} \frac{(\alpha x^2)^m}{m!} e^{Ak/n} \\ &= e^{(e^{A/n}-1)nx} e^{\alpha x^2 (e^{2A/n}-1)}.\end{aligned}$$

The proof of this can also be obtained by using generating function of Hermite polynomials (see [6, Lemma 2.1]).

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The above generating function may be used to find moments of the operators G_n^α as discussed in [7] for several other operators. The commendable work related to probabilistic distributions of many operators was discussed by Adell and collaborators see for instance [1], [2], [3] etc.

In the present article, we discuss some convergence estimates for the operators G_n^α using probabilistic approach.

2. Probabilistic representation

We start with the original Szász-Mirakyan operators:

Proposition 1. *Suppose that $\{N_i := N_i(x)\}_{i=1}^{\infty}$ be a sequence of identically distributed and independent Poisson random variables with the parameter $x \in [0,\infty)$, that is*

$$P(N(x)=k) = e^{-x}\frac{x^k}{k!}.$$

Taking $N(nx) = N_1(x)+N_2(x)+.....N_n(x)$. The distribution $N(nx)$ is also Poisson with

$$(G_n f)(x) = Ef\left(\frac{N(nx)}{n}\right), \tag{3}$$

where E denotes the expectation. Obviously

$$E[N(nx)] = nx \qquad E\left[(N(nx))^2\right] = n^2x^2+nx.$$

Also, mean and variance of G_n are given by

$$E\left(\frac{N(nx)}{n}-x\right) = 0, \quad E\left(\left(\frac{N(nx)}{n}-x\right)^2\right) = \frac{x}{n}.$$

Proof. The proof follows by simple analysis. □

Proposition 2. *Let $x>0$ and $n=1,2,\ldots$ with $N(nx)$ and $N(\alpha x^2)$ are two independent Poisson variables with parameters nx and αx^2. Also, the function $e^{-(nx+\alpha x^2)}\sum_{m=0}^{[\frac{k}{2}]}\frac{(\alpha x^2)^m}{m!}\frac{(nx)^{k-2m}}{(k-2m)!}$ defined in (1) is the probability density of the random variable*

$$V_n^\alpha(x) = N(nx)+2N(\alpha x^2). \tag{4}$$

As a consequence,

$$(G_n^\alpha f)(x) = Ef\left(\frac{N(nx)+2N(\alpha x^2)}{n}\right). \tag{5}$$

Proof. Consider the independent Poisson process $N(nx)$ and $N(\alpha x^2)$ given by

$$N(nx) = e^{-nx}\frac{(nx)^r}{r!}, r=0,1,2,\ldots; \quad N(\alpha x^2) = e^{-\alpha x^2}\frac{(\alpha x^2)^m}{m!}, m=0,1,2,\ldots$$

As the probability distribution of random variable $N(nx)+N(\alpha x^2)$ is the Hermite distribution, the probability function is given by

$$\begin{aligned} P(V_n^\alpha = k) &= e^{-nx-\alpha x^2} \sum_{m=0}^{[k/2]} \frac{(nx)^{k-2m}}{(k-2m)!} \frac{(\alpha x^2)^m}{m!} \\ &= e^{-nx-\alpha x^2} \sum_{\substack{r+2m=k \\ r,m\geq 0}} \frac{(nx)^r}{r!} \frac{(\alpha x^2)^m}{m!}, \end{aligned}$$

implying

$$(G_n^\alpha f)(x) = Ef\left(\frac{N(nx)+2N(\alpha x^2)}{n}\right).$$

□

Let us consider $W = \frac{V_n^\alpha(x)}{n}$, then it can be observed that

$$E(W) = x + \frac{2\alpha x^2}{n} \tag{6}$$

and

$$Var(W) = \frac{x}{n} + \frac{4\alpha x^2}{n^2}. \tag{7}$$

Remark 2. *In view of (4), we can define $V_n^\alpha(0) = 0$. Thus, the operator G_n^α acts on real measurable functions f defined on positive real axis for which the sum in (1) makes sense, because $H_0(n,\alpha) = 1$ and $(G_n^\alpha f)(0) = f(0)$. In other words, G_n^α interpolates f at the origin. On the other hand, representation (5) immediately implies that*

$$(G_n^\alpha f)(x) \to f(x), \quad x \geq 0, \ \ as \ \ n \to \infty,$$

whenever we can apply dominated convergence.

Corollary 1. *Suppose that f is increasing. For any $n = 1,2,...$ and $\beta \geq 0$, we have*

$$\begin{aligned} (G_n^\alpha f)(x) &\leq (G_n^\alpha f)(y), \ \ 0 \leq x \leq y \\ (G_{n+1}^\alpha f)(x) &\leq (G_n^\alpha f)(x), \ \ 0 \leq x. \end{aligned} \tag{8}$$

In addition,

$$(G_n^{\alpha_1} f)(x) \leq (G_n^{\alpha_2} f)(x), \ \ 0 \leq \alpha_1 \leq \alpha_2, \ \ 0 \leq x. \tag{9}$$

Proof. The Eq. (8), follows from the fact that the Poisson pdf is a member of the monotone likelihood ratio family, hence is stochastically ordered (increasing).
As the Poisson processes $(N_t)_{t\geq 0}$ have non decreasing paths, this implies that $V_n(\alpha_1,x) \leq V_n(\alpha_2,x)$, whenever $0 \leq \alpha_1 \leq \alpha_2$ i.e. (9) follows. □

By $C_B[0,\infty)$ we denote the class of bounded and continuous functions on $x \geq 0$. The Peetre's K-functional can be defined as

$$K_2(f,\delta) = \inf\{||f-g|| + \delta||g''|| : g \in W^2\},$$

where

$$W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

This Peetre's K-functional will be needed in the proof of Theorem 1.

Theorem 1. *Let $f \in C_B[0,\infty)$, $\alpha \geq 0$ and $n = 1,2,\ldots$ Then,*

$$|(G_n^\alpha f)(x) - f(x)| \leq C\omega_2\left(f; \frac{x^{1/2}}{\sqrt{n}}\right) + 2\omega_1\left(f; \frac{2\alpha x^2}{n}\right),$$

where C is an absolute constant.

Proof. Since G_n is centered, following [4, Theorem 1] we have

$$|(G_n^\alpha f)(x) - f(x)| \leq |(G_n f)(x) - f(x)| + |(G_n^\alpha f)(x) - (G_n f)(x)|. \tag{10}$$

To estimate the first term in right side of (10), we proceed as follows:

Consider a function $g \in W^2$, and $x, t \in [0,\infty)$. Using the integral form of the Taylor series

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

and by Proposition 1 in the following steps, we have

$$\begin{aligned} (G_n g)(x) - g(x) &= E\left[g\left(\frac{N(nx)}{n}\right) - g(x)\right] \\ &= E\left[g'(x)\left(\frac{N(nx)}{n} - x\right) + \int_x^{N(nx)/n}\left(\frac{N(nx)}{n} - v\right)g''(v)dv\right] \\ &\leq ||g''||E\left[\frac{1}{2}\left(\frac{N(nx)}{n} - x\right)^2\right] = \frac{x}{2n}||g''||. \end{aligned}$$

Also, $|(G_n f)(x)| \leq ||f||$, thus

$$\begin{aligned} |(G_n f)(x) - f(x)| &\leq |(G_n f - g)(x) - (f-g)(x)| + |(G_n g)(x) - g(x)| \\ &\leq 2||f-g|| + \frac{x}{2n}||g''|| \leq C\omega_2\left(f, \sqrt{\frac{x}{2n}}\right), \end{aligned} \tag{11}$$

where in the last step, we have used the K-functional and moduli of continuity connection (see [5]). Next, using the subadditivity of $\omega_1(f;.)$, (3) and Proposition 2, we obtain for any $\delta > 0$

$$\begin{aligned} |(G_n^\alpha f)(x) - (G_n f)(x)| &= \left|Ef\left(\frac{N(nx) + 2N(\alpha x^2)}{n}\right) - Ef\left(\frac{N(nx)}{n}\right)\right| \\ &\leq E\omega_1\left(f; \frac{2N(\alpha x^2)}{n}\right) \\ &\leq \left(1 + \frac{1}{\delta}E\left(\frac{2N(\alpha x^2)}{n}\right)\right)\omega_1(f;\delta), \end{aligned} \tag{12}$$

as $E\left(\frac{2N(\alpha x^2)}{n}\right) = 2\alpha x^2/n$. The conclusion follows from (10), (11) and (12) by choosing $\delta = 2\alpha x^2/n$.

□

Let $s = 1,2,...$ be fixed. We consider the operator $\widetilde{G}^{\alpha}_{n,s}$ defined as

$$(\widetilde{G}^{\alpha}_{n,s}f)(x) = (G^{\alpha}_{sn}f(nt))\left(\frac{x}{n}\right).$$

By Propositions 1 and 2, with $f_n(t) = f(nt)$ this operator can be represented in probabilistic terms as

$$\begin{aligned}(\widetilde{G}^{\alpha}_{n,s}f)(x) &= Ef_n\left(\frac{N(sx)+2N(\alpha x^2/n^2)}{sn}\right)\\ &= Ef\left(\frac{N(sx)}{s}+2\frac{N((\alpha x^2)/n^2)}{s}\right).\end{aligned} \tag{13}$$

We simply denote by

$$\widetilde{Y}(x) = \frac{N(sx)}{s},\ \widetilde{\widetilde{Y}} = 2\frac{N((\alpha x^2)/n^2)}{s}. \tag{14}$$

Thus following Propositions 1 and 2, we get

$$E\widetilde{Y}(x) = x,\ E\widetilde{\widetilde{Y}} = \frac{2\alpha x^2}{sn^2}. \tag{15}$$

Observe that $E\widetilde{\widetilde{Y}}$ is much less than $2\alpha x^2/n$ as given in (6). Next following Proposition 2 the variance is given by

$$\widetilde{\sigma}^2(x) = \frac{x^2}{s}+\frac{4\alpha x^2}{n^2s^2}. \tag{16}$$

Again this variance for large values of s is much less than that i.e. (7) as given in Proposition 2. As we will see in the following result, these two facts imply that the rate of convergence for the operator $\widetilde{G}^{\alpha}_n$ is much faster than that for G^{α}_n.

Theorem 2. *Let $f \in C_B[0,\infty)$, $x \geq 0$, $\alpha \geq 0$ and $s = 1,2,...$ Then,*

$$|(\widetilde{G}^{\alpha}_{n,s}f)(x) - f(x)| \leq C\omega_2\left(f;\sqrt{\frac{x}{s}}\right) + 2\omega_1\left(f;\frac{\alpha x^2}{sn^2}\right).$$

Proof. By (13) and (14), we can write

$$(\widetilde{G}^{\alpha}_{n,s}f)(x) - f(x) = Ef(\widetilde{Y}(x)) - f(x) + Ef(\widetilde{Y}(x)+\widetilde{\widetilde{Y}}) - Ef(\widetilde{Y}(x)). \tag{17}$$

Recalling Proposition 1 and (16), we can apply Theorem 1 to obtain

$$|Ef(\widetilde{Y}(x)) - f(x)| \leq C\omega_2\left(f;\sqrt{\frac{x}{s}}\right). \tag{18}$$

As in the proof of Theorem 1, we have

$$|Ef(\widetilde{Y}(x)+\widetilde{\widetilde{Y}}) - Ef(\widetilde{Y}(x)| \leq E\omega_1(f;\widetilde{\widetilde{Y}}) \leq \omega_1(f;E\widetilde{\widetilde{Y}}) = 2\omega_1\left(f;\frac{\alpha x^2}{sn^2}\right),$$

where the last equality follows from (15). This, together with (17) and (18), completes the proof. □

Remark 3. *The above quantitative estimate justifies the point-wise convergence recently estimated in* [6, Theorem 3.1]*, for sufficiently large n. Because for n large enough we have the only first term in the right hand side of the statement of Theorem 2, which is true for Szász-Mirakyan operator of index s. Also from Theorem 1, which may be verified by taking* $\alpha = 0$*.*

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