Solvability of Ψ-Hilfer hybrid fractional differential equations in Banach space

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Abstract. Fractional differential equations extend ordinary differential equations by including derivatives of non-integer order. This article investigates the existence of solutions for Ψ-Hilfer hybrid fractional differential equations using a measure of noncompactness technique to establish a fixed point of the sum of an $\mathcal{L}$-contraction and a compact operator. The contraction condition proposed in this article is not a *generalized contraction*. Furthermore, a number of existing results in the literature are also improved by the presented results.

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1. Introductions

The fixed point methods serve as an indispensable and versatile tool in actualizing the solution of wide range of problems arising in engineering, biology, economy, physical, chemical, mathematical and social sciences. These problems can be represented with an appropriate mathematical model. The investigation of fixed points plays a vital role in examination of existence and uniqueness of solutions of the model determined from the formulated problem. The Banach contraction principle [4] and Schauder fixed point theorem [34] are most celebrated and readily applicable results to several problems involving differential, integral or other functional equations. There are several issues coming from different branches of natural science, when modelled under the mathematical point of view, involve the study of solutions of

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nonlinear equations of the form

$$\mathcal{T}\varkappa + \mathcal{S}\varkappa = \varkappa \text{ for all } \varkappa \in \mathcal{C}, \tag{1.1}$$

where $\mathcal{C}$ is a nonempty closed, convex subset of a Banach space $(\mathcal{X}, \|\cdot\|)$ and $\mathcal{T}, \mathcal{S} : \mathcal{C} \to \mathcal{X}$. Especially, many problems in differential and integral equations can be formulated in terms of Eq. (1.1). The Krasnosel'skii's fixed point theorem is a powerful method for solving equations of the form (1.1). It combines two key concepts in fixed point theory, namely the Banach contraction principle and the Schauder fixed point theorem, to provide an effective pattern for finding fixed points. This theorem is named after mathematician Mark Krasnosel'skii, and the original statement of the theorem can be found in his published work [22] and we may read it as follows:

Theorem 1.1. *Let $\mathcal{C} \neq \varnothing$ be a convex and closed set in a Banach space $\mathcal{X}$. Suppose $\mathcal{T}, \mathcal{S} : \mathcal{C} \to \mathcal{M}$ such that*

(i) $\mathcal{T}$ is continuous and compact,
(ii) $\mathcal{S}$ is contraction,
(iii) $\mathcal{T}\varkappa + \mathcal{S}\vartheta \in \mathcal{C}\ \forall\ \varkappa, \vartheta \in \mathcal{C}$. Then $\mathcal{T} + \mathcal{S}$ admits fixed points.

However, numerous enhancements have been published in the literature over the years, that modify the underlying assumptions. These improvements occurred in different directions. The first direction in achieving improvement is to weaken the condition (iii) in Theorem 1.1. A major breakthrough occurred when the condition $\mathcal{T}(\mathcal{C}) + \mathcal{S}(\mathcal{C}) \subset \mathcal{C}$ is replaced by $(\mathcal{T} + \mathcal{S})(\mathcal{C}) \subset \mathcal{C}$. This was made possible due to use of measure noncompactness. The measures of noncompactness are functions utilized to quantify the level of noncompactness of a set. A quantitative characteristic $\aleph(A)$ measuring the degree of noncompactness of subset A in metric space, has connection with problems of general topology. The Kuratowski and Hausdorff measure of noncompactness in a metric space are well-known in the literature. Recall the following famous notions of Housdorff measure of noncompactness.

Definition 1.1. [15] Let $(\mathcal{X}, d)$ be a metric space and $\mathbb{B}(\mathcal{X})$ be the family of all nonempty and bounded subsets of $\mathcal{X}$. The function $\alpha : \mathbb{B}(\mathcal{X}) \to [0, \infty)$ defined as

$$\aleph(\mathfrak{C}) = \inf\{r > 0 : \mathfrak{C} \subset \cup_{i=1}^{N} \mathcal{B}(\varkappa_i, r), \varkappa_i \in \mathcal{X}, i = 1, 2, ..., N\},$$

$\forall\ \mathfrak{C} \in \mathbb{B}(\mathcal{X})$, where $\mathcal{B}(\varkappa_i, r)$ is ball with center $\varkappa_i$ and radius r. Then $\aleph$ is said to be Hausdorff measure of noncompactness.

The following are some of the fundamental properties of the Hausdorff measure of noncompactness:

1° $\aleph(\mathcal{U}) = 0$ if and only if $\mathcal{U}$ is relatively compact (i.e., $\overline{\mathcal{U}}$ is compact),
2° $\aleph(\mathcal{U}) = \aleph(\overline{\mathcal{U}})$, $\mathcal{U} \in \mathbb{B}(\mathcal{X})$, where $\overline{\mathcal{U}}$ denotes closure of the set $\mathcal{U}$,
3° $\aleph(\overline{con}(\mathcal{U})) = \aleph(\mathcal{U})$, for all $\mathcal{U} \in \mathbb{B}(\mathcal{X})$,
4° $\mathcal{U} \subset \mathcal{V}$ implies $\aleph(\mathcal{U}) \leq \aleph(\mathcal{V})$, for all $\mathcal{U}, \mathcal{V} \in \mathbb{B}(\mathcal{X})$,
5° $\aleph(\mathcal{U} + \mathcal{V}) \leq \aleph(\mathcal{U}) + \aleph(\mathcal{V})$.

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One of the most significant developments in fixed point theory comes from Darbo, who applied this measure to expand the classical Schauder's theorem [34] to a wide class of operators called condensing operators which are more general then the compact operators. Now, present the combine statement of Darbo [10] and Sadovskii [33] results.

Theorem 1.2. [10,33] *A continuous self-mapping $\mathcal{T}$ on a nonempty, bounded, closed and convex subset $\mathcal{C}$ of a Banach space $\mathcal{X}$, admits a fixed point. If for every $\varnothing \neq \mathcal{U} \subset \mathcal{C}$ satisfying one of the following conditions:*

(D) $\exists\ 0 \leq \Lambda < 1$ *such that* $\aleph(\mathcal{T}(\mathcal{U})) \leq \Lambda \aleph(\mathcal{U})$,

(S) $\aleph(\mathcal{U}) > 0$, $\aleph(\mathcal{T}(\mathcal{U})) < \aleph(\mathcal{U})$.

A mapping satisfying condition (D) is called Λ-set condensing (due to Darbo [10]) whereas satisfying (S) is called as $\aleph$-condensing (due to Sadovskii [33]). Secondly, weakening the compactness of mapping $\mathcal{T}$ is an important factor of improvement. There is a lot of work going on in this direction (cf. [5, 11, 14, 27, 28] and references therein). Generalizing the condition (ii) in Theorem 1.1 is the third significant area for development. Burton [6] successfully replaced the contraction mapping in this direction with large contractions, which are described as follows:

Definition 1.2. A self mapping $\mathcal{T}$ on a metric space $(\mathcal{X}, d)$ is said to be large contraction if $\forall\ \varepsilon > 0$, $\exists\ \delta < 1$ such that $d(\vartheta, \varkappa) \geq \varepsilon$ implies $d(\mathcal{T}\vartheta, \mathcal{T}\varkappa) \leq \delta d(\vartheta, \varkappa)$, $\forall \vartheta, \varkappa \in \mathcal{X}$.

Successively, Przeradki [31] actualized the fixed points for the sum of 'compact operator'and *'generalized contraction'* using the concept of Housdorff measure of noncompactness and $\aleph$-condensing mapping.

Definition 1.3. The self mapping $\mathcal{T}$ on $(\mathcal{X}, d)$ is said to be a generalized contraction if there exists a function $\gamma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that

$$\sup_{(a \leq d(\vartheta, \varkappa) \leq b)} \gamma(\vartheta, \varkappa) < 1 \text{ for all } b \geq a > 0$$

and $\forall\ \varkappa, \vartheta \in \mathcal{X}$

$$d(\mathcal{T}\vartheta, \mathcal{T}\varkappa) \leq \gamma(\vartheta, \varkappa) d(\vartheta, \varkappa).$$

Przeradzki [31] demonstrated that the *generalized contractions* are a proper generalization of *large contractions*. Together with Sadovskii's and condition $(\mathcal{T} + \mathcal{S})(\mathcal{C}) \subset \mathcal{C}$, Przeradzki in [31] also shown that *generalized contractions* are $\aleph$-condensing, which led to a major improvement of Krasnosel'skii's result. This way he succeed in improving Krasnosel'skii's as well as Burton's result [7]. Park [29] made a significant contribution in this direction as well. Recently, Wardowski [41] further extended the Krasnosel'skii's result by applying φ-F-contraction. Patle *et al.* has assured in [13, 30] that the sum of a Z-contraction and a compact operator admits a fixed point.

On the other hand, the Hilfer in [17] presented the idea of the fractional derivative operator ${}^{H}\mathcal{D}^{\breve{\mu},\breve{\nu}}_{p^+}$ with two parameters $\breve{\mu} \in (n-1, n), n \in \mathbb{N}$ and $\breve{\nu}(0 \leq \breve{\nu} \leq 1)$, which includes the theory of fractional differential equations

(FDEs) involving RL fractional derivative ($\breve{\nu} = 0$) and Caputo fractional derivative ($\breve{\nu} = 1$). Later on, the fundamental work on the initial value problems involving Hilfer fractional derivatives were studied by Furati *et al.* [12]. Wang and Zhang [40], discussed the existence of solutions to nonlocal initial value problem for differential equations with Hilfer fractional derivative. In [2], Almeida presented Ψ-Caputo fractional derivative and investigated many fascinating properties of this operator. The discovery of new physical phenomena and the study of chaotic systems have given rise to the proposition of new fractional differential and integral operators that would allow a better description of such systems [9, 16, 32, 38]. In this context, Sousa and Oliveira [37] have recently proposed a fractional derivative operator, which they called Ψ-Hilfer operator that has the spacial property of unifying several different fractional operators, that is, of generalizing those fractional operators. The Ψ-Hilfer derivative allows for a more flexible characterization of memory effects and non-local behaviors and better analytical properties compared to Riemann-Liouville and Caputo derivatives. This flexibility often leads to more accurate modeling of complex systems exhibiting anomalous diffusion or non-local behaviors [3, 18]. The existence and uniqueness of the solution to fractional boundary value problems (BVPs) have acquired a lots of interest due to its qualitative properties. Mali and Kucche [26] considered the nonlocal boundary value problem for implicit Ψ-Hilfer FDEs to discuss the existence and Ulam stability results on the problem. Besides, Shatanawi *et al.* [35] implemented the Krasnoselskii's fixed point approach and Banach contraction principle to obtained a solution for generalized Hilfer operators. The detailed study about existence and uniqueness of the solution to Ψ-Hilfer FDEs and development of Ψ-Hilfer FDEs can be seen in [20, 23, 39].

Motivated by the above study, in the present paper, we consider the following Ψ-Hilfer hybrid FDEs of the form

$$\begin{cases} {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}[\vartheta(\upsilon) - f(\upsilon, \vartheta(\upsilon))] = g(\upsilon, \vartheta(\upsilon)), \text{ a.e. } \upsilon \in (p, T], \\ I_{p^+}^{1-\eta;\Psi}[\vartheta(\upsilon) - f(\upsilon, \vartheta(\upsilon))](p) = \vartheta_p \in \mathbb{R}. \end{cases} \tag{1.2}$$

where ${}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}(\cdot)$ denotes the Ψ-Hilfer fractional derivative of order $0 < \breve{\mu} < 1$ and type $0 \leq \breve{\nu} \leq 1$, $I_{p^+}^{1-\eta;\Psi}$ is the Ψ-RL fractional integral of order $1-\eta, \eta = \breve{\mu} + \breve{\nu}(1-\breve{\mu})$ with respect to function Ψ [21]. For $f(\cdot, \vartheta(\cdot))$ and $g(\cdot, \vartheta(\cdot)) \in C([p, T] \times \mathbb{R}, \mathbb{R})$ with $p < T \leq q$.

The structure of this paper is organized as follows. Section 2 introduces the necessary notations, basic definitions, and preliminary facts that will be utilized in subsequent sections. In Section 3, we investigate $\mathcal{L}$-contractions and obtain Krasnosel'skii type fixed point results by utilizing the Hausdorff measure of noncompactness and condensing mappings. In Section 4, our focus is on examining the existence of solutions for hybrid Hilfer fractional differential equations (1.2). Additionally, in Section 5, we present several illustrative examples to demonstrate the applicability and effectiveness of our results.

2. Preliminaries

2.1. Mathematical Background (from $\mathcal{L}$-contraction)

To establish an extension of Krasnoselskii's fixed point theorem, we utilize the following concepts.

Definition 2.1. (Jleli and Samet [19]) Let Γ be the collection of all functions $\Theta : (0,\infty) \to (1,\infty)$ fulfilling the conditions:

(Θ_1) $\Theta(\upsilon_1) \le \Theta(\upsilon_2)$ for all $\upsilon_1 < \upsilon_2$,
(Θ_2) for each sequence $\{\upsilon_i\} \subset (0,\infty)$, $\lim\limits_{i\to\infty} \Theta(\upsilon_i) = 1 \Leftrightarrow \lim\limits_{i\to\infty} \upsilon_i = 0^+$,
(Θ_3) there exist $k \in (0,1)$ and $\ell \in (0,\infty)$ such that

$$\frac{\Theta(\upsilon) - 1}{\upsilon^k} = \ell.$$

Ahmad *et al.* [1] substituted condition (Θ_3) with the requirement that Θ be continuous on the interval $(0,\infty)$. This idea is utilised in the current paper as well.

For example,

(i) Consider the function $\Theta : (0,\infty) \to (1,\infty)$ defined by $\Theta(\upsilon) := e^{\sqrt{\upsilon}}$ belong to Γ.
(ii) The function $\Theta : (0,\infty) \to (1,\infty)$ defined by $\Theta(\upsilon) := e^{\sqrt{\upsilon e^{\upsilon}}}$ belong to Γ.

A Θ-contraction is a self-mapping $\mathcal{T}$ on a metric space $(\mathcal{X}, d)$ satisfying the following condition: there exist $\Theta \in \Gamma$ and $k \in (0,1)$ such that for any $\varkappa, \vartheta \in \mathcal{X}$

$$d(\mathcal{T}\varkappa, \mathcal{T}\vartheta) \neq 0 \implies \Theta(d(\mathcal{T}\varkappa, \mathcal{T}\vartheta)) \le [\Theta(d(\varkappa, \vartheta))]^k.$$

Further, it has been demonstrated in [19] that every Θ-contraction on complete metric space $(\mathcal{X}, d)$ admits one fixed point.

Definition 2.2. [8] Suppose Σ is the set of all functions $\varXi : [1,\infty)\times[1,\infty) \to \mathbb{R}$ satisfying the following conditions:

(Ξ-1) $\varXi(1,1) = 1$,
(Ξ-2) $\varXi(\upsilon_1, \upsilon_2) < \frac{\upsilon_2}{\upsilon_1}$ for all $\upsilon_1, \upsilon_2 > 1$,
(Ξ-3) if there are two sequences $\{s_j\}$ and $\{\upsilon_j\}$ in $(1,\infty)$ with $s_j \ge \upsilon_j$ and $\lim\limits_{j\to\infty} s_j = \lim\limits_{j\to\infty} \upsilon_j > 1$, then $\limsup\limits_{j\to\infty} \varXi(\upsilon_j, s_j) < 1$.

For example,

(a) $\varXi(\upsilon_1, \upsilon_2) = \frac{\upsilon_2^k}{\upsilon_1}$ for all $\upsilon_1, \upsilon_2 \ge 1$ where $k \in (0,1)$;
(b) $\varXi(\upsilon_1, \upsilon_2) = \frac{\upsilon_2}{\upsilon_1 \varphi(\upsilon_2)}$ for all $\upsilon_1, \upsilon_2 \ge 1$ where $\varphi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semi-continuous such that $\varphi^{-1}(\{1\}) = \{1\}$.

Definition 2.3. Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{T}$ be a self-mapping. We say that $\mathcal{T}$ is an $\mathcal{L}$-contraction if there exist $\Theta \in \Gamma$ and $\varXi \in \Sigma$, for any distinct points $\varkappa$ and ϑ in $\mathcal{X}$ such that $d(\mathcal{T}\varkappa, \mathcal{T}\vartheta) > 0$, we have

$$\varXi\Big(\Theta(d(\mathcal{T}\varkappa, \mathcal{T}\vartheta)), \Theta(d(\varkappa, \vartheta))\Big) \ge 1. \tag{2.1}$$

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On a complete metric space, any $\mathcal{L}$-contraction has a unique fixed point. This result was first established in [8]. The fixed point theorem obtained by Zada *et al.* [42] extends the class of Darbo type fixed point results by introducing the notion of $\mathcal{L}_{\aleph}$ contraction, which involves a simulation function. The statement can be given as:

Theorem 2.1. *Let $\mathcal{A}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{X}$. A continuous mapping $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ which is $\mathcal{L}_{\aleph}$ contraction with respect to Ξ, that is, $\mathcal{T}$ satisfies*

$$\Xi\Big(\Theta(\aleph(\mathcal{T}(\mathcal{C}))), \Theta(\aleph(\mathcal{C}))\Big) \geq 1,$$

where $\Theta \in \Gamma$ and for all nonempty subset $\mathcal{C} \subseteq \mathcal{A}$ with $\aleph(\mathcal{C}) > 0$. Then $\mathcal{T}$ admits fixed point.

2.2. Terminology used in fractional calculus and its outcomes

Let $[p, q]$ $(0 \leq p < q < \infty)$ be an interval with a finite length on the non-negative part of the real number line, denoted by $\mathbb{R}^+$, and $C[p, q]$ be space of function f that are continuous over the interval $[p, q]$ with the norm

$$\|f\|_{C[p,q]} := \max_{\upsilon \in [p,q]} |f(\upsilon)|.$$

Suppose Ψ is a function in $C^1([p, q], \mathbb{R})$ that is increasing and satisfies $\Psi'(\upsilon) \neq 0$ for all $\upsilon \in [p, q]$. The weighted space $C_{1-\eta;\Psi}[p, q]$ consists of continuous functions $f : (p, q] \to \mathbb{R}$ such that $(\Psi(\upsilon) - \Psi(p))^{1-\eta} f(\upsilon)$ belongs to $C[p, q]$, where $0 \leq \eta < 1$, and is defined by [37]. The norm on this space is given by

$$\|f\|_{C_{1-\eta;\Psi}[p,q]} := \|(\Psi(\upsilon)-\Psi(p))^{1-\eta} f(\upsilon)\|_{C[p,q]} = \max_{\upsilon \in [p,q]} |(\Psi(\upsilon)-\Psi(p))^{1-\eta} f(\upsilon)|.$$

The weighted space $C^n_{\eta;\Psi}[p, q]$ of continuous functions f on $(p, q]$ is defined by

$$C^n_{\eta;\Psi}[p, q] = \{f : (p, q] \to \mathbb{R} : f(\upsilon) \in C^{n-1}[p, q];\ f^n(\upsilon) \in C_{\eta;\Psi}[p, q]\},\ 0 \leq \eta < 1,$$

equipped with the norm

$$\|f\|_{C^n_{\eta;\Psi}[p,q]} = \sum_{i=0}^{n-1} \|f^{(i)}\|_{C[p,q]} + \|f^{(n)}\|_{C^n_{\eta;\Psi}[p,q]}.$$

Definition 2.4. [21] Assume f is a function with integrable absolute value defined on the interval $[p, q]$. The Ψ-Riemann-Liouville (Ψ-RL) fractional integral of order $\breve{\mu} > 0$ $(\breve{\mu} \in \mathbb{R})$ of the function f is defined as

$$I^{\breve{\mu};\Psi}_{p^+} f(\upsilon) = \frac{1}{\Gamma(\breve{\mu})} \int_p^{\upsilon} \Psi'(\varrho)(\Psi(\upsilon) - \Psi(\varrho))^{\breve{\mu}-1} f(\varrho) d\varrho. \tag{2.2}$$

Definition 2.5. [37] The expression for the Ψ-Hilfer fractional derivative of a function f with $0 < \breve{\mu} < 1$ and type $0 \leq \breve{\nu} \leq 1$ is provided by

$$^{H}\mathcal{D}^{\breve{\mu},\breve{\nu};\Psi}_{p^+} f(\upsilon) = I^{\breve{\nu}(1-\breve{\mu});\Psi}_{p^+} \left(\frac{1}{\Psi'(\upsilon)} \frac{d}{d\upsilon}\right) I^{(1-\breve{\nu})(1-\breve{\mu});\Psi}_{p^+} f(\upsilon). \tag{2.3}$$

Lemma 2.1. [21,37] *Suppose $\varepsilon, \delta > 0$ and $\rho > n$. Then*

(i) $I_{p^+}^{\breve{\mu};\Psi} I_{p^+}^{\varepsilon;\Psi} h(\upsilon) = I_{p^+}^{\breve{\mu}+\varepsilon;\Psi} h(\upsilon)$.
(ii) $I_{p^+}^{\breve{\mu};\Psi} (\Psi(\upsilon) - \Psi(p))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\breve{\mu}+\delta)} (\Psi(\upsilon) - \Psi(p))^{\breve{\mu}+\delta-1}$.
(iii) ${}^H\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi} (\Psi(\upsilon) - \Psi(p))^{\eta-1} = 0$.
(iv) ${}^H\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi} (\Psi(\upsilon) - \Psi(p))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho-\breve{\mu})} (\Psi(\upsilon) - \Psi(p))^{\rho-\breve{\mu}-1}$.

Lemma 2.2. [37] *If $\gamma \in C^n[p,q], n-1 < \breve{\mu} < n$ and $0 \le \breve{\nu} \le 1$, then*

(i) $I_{p^+}^{\breve{\mu};\Psi} {}^H\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi} \gamma(\upsilon) = \gamma(\upsilon) - \sum_{k=1}^{n} \frac{(\Psi(\upsilon)-\Psi(p))^{\eta-k}}{\Gamma(\eta-k+1)} \gamma_\Psi^{[n-k]} I_{p^+}^{(1-\breve{\nu})(n-\breve{\mu});\Psi} \gamma(p)$, *where* $\gamma_\Psi^{[n-k]}\gamma(\upsilon) = \left(\frac{1}{\Psi'(\upsilon)} \frac{d}{dt}\right)^{n-k} \gamma(\upsilon)$.
(ii) ${}^H\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi} I_{p^+}^{\breve{\mu};\Psi} \gamma(\upsilon) = \gamma(\upsilon)$.
(iii) $I_{p^+}^{\breve{\alpha};\Psi} \gamma(p) = \lim_{\upsilon \to p^+} I_{p^+}^{\breve{\alpha};\Psi} \gamma(\upsilon) = 0,\ n-1 \le \breve{\mu} < \alpha$.

3. Krasnosel'skii type fixed point results

We enunciate this section by showing $\mathcal{L}$-contraction to be a $\aleph$-condensing map and, then using Sadovskii's theorem we derive fixed point theorem for sum of a compact operator with $\mathcal{L}$-contraction. Before stating the results we wish to show that $\mathcal{L}$-contraction is not a particular instance of a *generalized contraction*.

Example 1. Consider $\mathcal{X}$ as the set $\mathcal{X} = \left\{\varkappa_k = \frac{k}{\sqrt{2}} - \sqrt{2} + 2^{-k+\frac{1}{2}} : k \in \mathbb{N}\right\}$, equipped with the metric $d(\varkappa, \vartheta) = |\varkappa - \vartheta|$, for all $\varkappa, \vartheta \in \mathcal{X}$. The space $(\mathcal{X}, d)$ is complete. We then define the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{T}\varkappa_k = \begin{cases} \varkappa_1 & \text{if } k = 1, \\ \\ \varkappa_{k-1} & \text{if } k \ge 2. \end{cases}$$

Firstly, note that $\mathcal{T}$ does not satisfy the condition of being a generalized contraction. Specifically, for any $k \in \mathbb{N}$, we can see that

$$|\varkappa_{k+1} - \varkappa_k| = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2^{k+1}}.$$

Hence

$$\frac{1}{2\sqrt{2}} \le |\varkappa_{k+1} - \varkappa_k| < \frac{1}{\sqrt{2}},\ \forall\ k \in \mathbb{N}.$$

Assuming to the contrary, that $\mathcal{T}$ is generalized contraction, there exists a function γ satisfying second condition in Definition 1.3. Then for $k \ge 2$, we get

$$\gamma(\varkappa_{k+1}, \varkappa_k) \ge \frac{d(\mathcal{T}\varkappa_{k+1}, \mathcal{T}\varkappa_k)}{d(\varkappa_{k+1}, \varkappa_k)} = \frac{|\varkappa_k - \varkappa_{k-1}|}{|\varkappa_{k+1} - \varkappa_k|} = \frac{2^{k+1} - 4}{2^{k+1} - 2}.$$

So by first condition of Definition 1.3, we get

$$1 > \sup_{\left(\frac{1}{2\sqrt{2}} < d(\varkappa,\vartheta) < \frac{1}{\sqrt{2}}\right)} \gamma(\varkappa, \vartheta) \ge \gamma(\varkappa_{k+1}, \varkappa_k) \ge \frac{2^{k+1} - 4}{2^{k+1} - 2}.$$

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Taking the limit as n approaches infinity leads to a contradiction. Now to show $\mathcal{T}$ to be a $\mathcal{L}$-contraction, we need to consider a mapping $\varphi : (0,\infty) \to (0,\infty)$ defined as

$$\varphi(\tau) = \begin{cases} 1 + \dfrac{\tau - 1}{\tau + 1} & \text{if } 0 < \tau < 1, \\ \\ 1 + \dfrac{\tau - k}{\tau + 1} & \text{if } (k-1) \leq \tau < k, \ k \geq 2. \end{cases}$$

Then one can observe that $\limsup\limits_{\tau \to r^+} \varphi(\tau) < 1$ for any $r \geq 0$. Now for any $m, n \in \mathbb{N}$, $k > l \geq 2$, we have

$$|\varkappa_l - \varkappa_k| = \frac{(k-l)}{\sqrt{2}} + 2^{-k+\frac{1}{2}} - 2^{-l+\frac{1}{2}}$$

and

$$\frac{(k-l)}{\sqrt{2}} - 1 < |\varkappa_l - \varkappa_k| < \frac{(k-l)}{\sqrt{2}}.$$

Hence, we obtain

$$\begin{aligned} |\mathcal{T}\varkappa_l - \mathcal{T}\varkappa_k| &= \left(1 + \frac{2^{-k+\frac{1}{2}} - 2^{-l+\frac{1}{2}}}{|\varkappa_l - \varkappa_k|}\right)|\varkappa_l - \varkappa_k| \\ &< \left(1 + \frac{|\varkappa_l - \varkappa_k| - \frac{(k-l)}{\sqrt{2}}}{|\varkappa_l - \varkappa_k| + 1}\right)|\varkappa_l - \varkappa_k| \\ &= \varphi(|\varkappa_l - \varkappa_k|)|\varkappa_l - \varkappa_k|. \end{aligned}$$

Also, for any $k \geq 2$ we get

$$|\varkappa_k - \varkappa_1| = \frac{k}{\sqrt{2}} - \sqrt{2} + 2^{-k+\frac{1}{2}}$$

and

$$\frac{k-1}{\sqrt{2}} - 1 < \frac{k-2}{\sqrt{2}} < |\varkappa_k - \varkappa_1| < \frac{k-1}{\sqrt{2}},$$

which returns us with

$$\begin{aligned} |\mathcal{T}\varkappa_k - \mathcal{T}\varkappa_1| &= \left(1 + \frac{2^{-k+\frac{1}{2}} - \frac{1}{\sqrt{2}}}{|\varkappa_k - \varkappa_1|}\right)|\varkappa_k - \varkappa_1| \\ &< \left(1 + \frac{|\varkappa_k - \varkappa_1| - \frac{(k-1)}{\sqrt{2}}}{|\varkappa_k - \varkappa_1| + 1}\right)|\varkappa_k - \varkappa_1| \\ &< \varphi(|\varkappa_k - \varkappa_1|)|\varkappa_k - \varkappa_1|. \end{aligned}$$

Define $\Theta : [0,\infty) \to [1,\infty)$ as $\Theta(\upsilon) = e^{\upsilon}$ and $L = \limsup\limits_{\tau \to r^+} \varphi(\tau) < 1$ for any $r \geq 0$. Then

$$\frac{[\Theta(|\varkappa - \vartheta|)]^L}{\Theta(|\mathcal{T}\varkappa - \mathcal{T}\vartheta|)} \geq 1.$$

Select $\Xi(\upsilon,\varrho)=\dfrac{\varrho^L}{\upsilon}$, we obtain

$$\Xi(\Theta(|\mathcal{T}\varkappa-\mathcal{T}\vartheta|),\Theta(|\varkappa-\vartheta|))\geq 1.$$

Hence, $\mathcal{T}$ is an $\mathcal{L}$-contraction but not later one.

Theorem 3.1. *Every $\mathcal{L}$-contraction $\mathcal{T}$ on metric space $(\mathcal{X},d)$ is $\aleph$-condensing.*

Proof. Take any nonempty subset $\mathcal{C}$ of $\mathcal{X}$ with positive Housdorff measure of noncompactness. Let two sequences $\{p_n\}$ and $\{q_n\}$ defined by $p_n=\aleph(\mathcal{C})-\varepsilon_n>0$ and $q_n=\aleph(\mathcal{C})+\varepsilon_n>0$ where $\varepsilon_n\to 0$ as $n\to\infty$. Then, take $\upsilon_n=\Theta(p_n)=\Theta(\aleph(\mathcal{C})-\varepsilon_n)>1$ and $\varrho_n=\Theta(q_n)=\Theta(\aleph(\mathcal{C})+\varepsilon_n)>1$. Therefore, $\lim\limits_{n\to\infty}\upsilon_n=\lim\limits_{n\to\infty}\varrho_n=\Theta(\aleph(\mathcal{C}))>1$ and $\upsilon_n\leq\varrho_n$. From (Ξ-3) we obtain

$$\limsup_{n\to\infty}\Xi(\upsilon_n,\varrho_n)<1 \tag{3.1}$$

Choosing $\varepsilon=\sup\{\varepsilon_n\}$ sufficiently small, form (3.1), there exists $\delta<1$ such that

$$\Xi(\upsilon,\varrho)<\delta, \tag{3.2}$$

where $\varrho\in[\Theta(\aleph(\mathcal{C})-\varepsilon),\Theta(\aleph(\mathcal{C})))$ and $\upsilon\in(\Theta(\aleph(\mathcal{C})),\Theta(\aleph(\mathcal{C})+\varepsilon)]$. Assume that $\mathcal{R}$ is equal to $\aleph(\mathcal{C})+\varepsilon$, and select a finite $\mathcal{R}$-net of $\mathcal{C}$, i.e.

$$\mathcal{C}\subset\bigcup_{i=1}^{j}\mathcal{B}(\varkappa_i,\mathcal{R}),\ \ \varkappa_1,\ldots,\varkappa_j\in\mathcal{X}. \tag{3.3}$$

Denote $\mathcal{R}'=\aleph(\mathcal{C})-\varepsilon$. It is our aim to prove that an $\mathcal{R}'$-net exists for $\mathcal{T}(\mathcal{C})$. Indeed, let $\vartheta\in\mathcal{T}(\mathcal{C})$ and there exist $\varkappa\in\mathcal{C}$ such that $\mathcal{T}\varkappa=\vartheta$. Then from (3.3) there exists $i\in\{1,2,\ldots,j\}$ such that $d(\varkappa,\varkappa_i)<\mathcal{R}$, thus $\Theta(d(\varkappa,\varkappa_i))<\Theta(\aleph(\mathcal{C})+\varepsilon)$. If $\mathcal{T}\varkappa=\mathcal{T}\varkappa_i$, then $d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)<\mathcal{R}'$ (trivially). Suppose that $\mathcal{T}\varkappa\neq\mathcal{T}\varkappa_i$, then we have two cases:

(i) If $0<d(\varkappa,\varkappa_i)<\mathcal{R}'$, then as $\mathcal{T}$ is $\mathcal{L}$-contraction and (Ξ-2), we obtain

$$1\leq\Xi(\Theta(d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)),\Theta(d(\varkappa_i,\varkappa)))<\frac{\Theta(d(\varkappa_i,\varkappa))}{\Theta(d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa))},$$

which yields, $\Theta(d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa))<\Theta(d(\varkappa_i,\varkappa))$. Since Θ is nondecreasing, thus $d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)<d(\varkappa_i,\varkappa)<\mathcal{R}'$.

(ii) If $\mathcal{R}'\leq d(\varkappa_i,\varkappa)<\mathcal{R}$, then either $d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)<\mathcal{R}'$ or $d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)\geq\mathcal{R}'$. Assume that $d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)\geq\mathcal{R}'$, then due to (3.2), we have

$$\Xi(\Theta(d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)),\Theta(d(\varkappa_i,\varkappa)))<\delta,$$

which yields a contradiction of $\mathcal{T}$ is a $\mathcal{L}$-contraction.

So $d(\mathcal{T}\varkappa_i,\mathcal{T}\varkappa)<\mathcal{R}'$ holds in both cases. Thus, $\mathcal{T}(\mathcal{C})$ has $\mathcal{R}'$-net, which gives

$$\aleph(\mathcal{T}(\mathcal{C}))\leq\mathcal{R}'<\aleph(\mathcal{C}).$$

□

Now, we can prove the desired result for sum of two operators as our main goal.

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Theorem 3.2. *Let $\mathcal{C}$ be a nonempty closed convex and bounded subset of a Banach space $\mathcal{X}$. If $\mathcal{T}:\mathcal{C}\to\mathcal{X}$ is a $\mathcal{L}$-contraction and $\mathcal{S}:\mathcal{C}\to\mathcal{X}$ is compact operator such that $(\mathcal{T}+\mathcal{S})(\mathcal{C})\subset\mathcal{C}$, then $\mathcal{T}+\mathcal{S}$ admits fixed point.*

Proof. Let $\mathcal{K}\subset\mathcal{C}$ which is not relatively compact (i.e. $\aleph(\mathcal{K})\neq 0$). Then,

$$\aleph((\mathcal{T}+\mathcal{S})(\mathcal{K}))\leq\aleph(\mathcal{T}(\mathcal{K})+\mathcal{S}(\mathcal{K}))\leq\aleph(\mathcal{T}(\mathcal{K}))+\aleph(\mathcal{S}(\mathcal{K})). \tag{3.4}$$

By utilizing the properties of measure of noncompactness, along with Theorem 3.1 and the observation that $\mathcal{S}$ is a compact operator (i.e., $\aleph(\mathcal{S}(\mathcal{K}))=0$), we can derive the following:

$$\aleph((\mathcal{T}+\mathcal{S})(\mathcal{K}))\leq\aleph(\mathcal{T}(\mathcal{K}))<\aleph(\mathcal{K}), \tag{3.5}$$

which conclude that $\mathcal{T}+\mathcal{S}$ is $\aleph$-condensing mapping; thus by Sadovskii's theorem, $\mathcal{T}+\mathcal{S}$ admits fixed point. □

The above theorem implies the following corollaries.

Corollary 3.1. *Let $\mathcal{C}$ be a nonempty closed convex and bounded subset of a Banach space $\mathcal{X}$. If $\mathcal{T}:\mathcal{C}\to\mathcal{X}$ is Θ-contraction and $\mathcal{S}:\mathcal{C}\to\mathcal{X}$ is compact operator satisfying $(\mathcal{T}+\mathcal{S})(\mathcal{C})\subset\mathcal{C}$, then $\mathcal{T}+\mathcal{S}$ admits a fixed point.*

Proof. Define $\Xi_B:[1,\infty)\times[1,\infty)\to\mathbb{R}$ by $\Xi_B(\upsilon,\varrho)=\frac{\varrho^k}{\upsilon}$ for all $\varrho,\upsilon\geq 1$, where $k\in(0,1)$. Note that, the mapping $\mathcal{T}$ is $\mathcal{L}$-contraction with respect to $\Xi_B\in\Sigma$. Thus the result follows by taking $\Xi=\Xi_B$ in Theorem 3.2. □

Corollary 3.2. *Let $\mathcal{C}$ be a nonempty closed convex and bounded subset of a Banach space $\mathcal{X}$. If $\mathcal{T}:\mathcal{C}\to\mathcal{X}$ is contraction and $\mathcal{S}:\mathcal{C}\to\mathcal{X}$ is compact operator satisfying $(\mathcal{T}+\mathcal{S})(\mathcal{C})\subset\mathcal{C}$, then $\mathcal{T}+\mathcal{S}$ admits a fixed point.*

Proof. Consider the function $\Theta:(0,\infty)\to(1,\infty)$ defined by $\Theta(r):=e^{\sqrt{r}}$ clearly $\Theta\in\Gamma$ and Ξ_B as defined in proof of Corollary 3.1. Note that, the mapping $\mathcal{T}$ is $\mathcal{L}$-contraction with respect to $\Xi_B\in\Sigma$. Thus the result follows by Theorem 3.2. □

4. Existence of solutions to Ψ-Hilfer hybrid fractional differential equations

It is crucial and inspiring for researchers studying fractional calculus to propose a fractional differentiation or integration operator. However, unifying numerous definitions with a single fractional operator is a challenging and complex task. There are several classes (definitions) of fractional derivatives that are special instances of the Ψ-Hilfer fractional derivative stated in Eq (2.3). Hence, the nonlinear Cauchy problem proposed encompasses the Cauchy problems for those classes of fractional derivatives as specific instances. The Ψ-Hilfer hybrid FDEs:

$${}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}[\vartheta(\upsilon)-f(\upsilon,\vartheta(\upsilon))]=g(\upsilon,\vartheta(\upsilon)),\ \text{a.e. } \upsilon\in(p,T], \tag{4.1}$$

$$I_{p^+}^{1-\eta;\Psi}[\vartheta(\upsilon)-f(\upsilon,\vartheta(\upsilon))](p)=\vartheta_p\in\mathbb{R}, \tag{4.2}$$

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where ${}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}(\cdot)$ denotes the Ψ-Hilfer fractional derivative of order $0 < \breve{\mu} < 1$ and type $0 \le \breve{\nu} \le 1$, $I_{p^+}^{1-\eta;\Psi}$ is the Ψ-RL fractional integral of order $1-\eta, \eta = \breve{\mu} + \breve{\nu}(1-\breve{\mu})$ with respect to function Ψ [21]. For $f(\cdot,\vartheta(\cdot))$ and $g(\cdot,\vartheta(\cdot)) \in C([p,T]\times\mathbb{R},\mathbb{R})$ with $p < T \le q$.

The fractional derivatives in these categories are achieved by selecting an appropriate function $\Psi(\cdot)$ and taking the limits of Ψ-Hilfer hybrid FDEs (4.1)-(4.2) as $\breve{\nu}$ approaches 0 or 1.

- For $f(\upsilon,\vartheta(\upsilon)) = 0$, the obtained initial value problem is from the paper of Sousa and Oliveira [36] of the form
$$ {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}\vartheta(\upsilon) = g(\upsilon,\vartheta(\upsilon)),\ \text{a.e. } \upsilon \in (p,T], $$
$$ I_{p^+}^{1-\eta;\Psi}\vartheta(\upsilon)(p^+) = \vartheta_p \in \mathbb{R},\ \eta = \breve{\mu} + \breve{\nu}(1-\breve{\mu}), $$
where ${}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}(\cdot)$ is the Ψ-Hilfer fractional operator.
- For $\Psi(\upsilon) = \upsilon$, the obtain results in the paper of Gabeleh *et al.* [13] to nonlinear Hilfer FDEs of the form
$$ {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu}}[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))] = g(\upsilon,\vartheta(\upsilon)),\ \text{a.e. } \upsilon \in (p,T], $$
$$ I_{p^+}^{1-\eta}[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))](p^+) = \vartheta_p \in \mathbb{R},\ \eta = \breve{\mu} + \breve{\nu}(1-\breve{\mu}), $$
where ${}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu}}(\cdot)$ is the Hilfer fractional operator.
- For $\Psi(\upsilon) = \upsilon$ and $f(\upsilon,\vartheta(\upsilon)) = 0$, the obtained initial value problem in the paper of Furati *et al.* [12] of the form
$$ {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu}}\vartheta(\upsilon) = g(\upsilon,\vartheta(\upsilon)),\ \text{a.e. } \upsilon \in (p,T], $$
$$ I_{p^+}^{1-\eta}\vartheta(\upsilon)(p^+) = \vartheta_p \in \mathbb{R},\ \eta = \breve{\mu} + \breve{\nu}(1-\breve{\mu}). $$
- The study conducted by Lakshmikantham *et al.* [24] on nonlinear FDEs that contain Riemann-Liouville fractional derivative is included in the obtained outcomes for $f(\upsilon,\vartheta(\upsilon)) = 0, \breve{\nu} = 0$, and $\Psi(\upsilon) = \upsilon$.
$$ {}^{RL}\mathcal{D}_{p^+}^{\breve{\mu}}\vartheta(\upsilon) = g(\upsilon,\vartheta(\upsilon)),\ \text{a.e. } \upsilon \in (p,T], $$
$$ I_{p^+}^{1-\breve{\mu}}[\vartheta(\upsilon)](p^+) = \vartheta_p \in \mathbb{R}, $$
- Lakshmikantham and Vatsala [25] conducted a study of FDEs that are nonlinear and involve the Caputo derivative, where $f(\upsilon,\vartheta(\upsilon)) = 0$, $\breve{\nu} = 1$ (or $\eta = 1$), and $\Psi(\upsilon) = \upsilon$.
$$ {}^{C}\mathcal{D}_{p^+}^{\breve{\mu}}\vartheta(\upsilon) = g(\upsilon,\vartheta(\upsilon)),\ \text{a.e. } \upsilon \in (p,T], $$
$$ \vartheta(p) = \vartheta_p \in \mathbb{R}, $$
- The precise cases of the operators are also determined by the choice of parameter p. For instance, since $\ln\upsilon$ is not defined for $p = 0$, select $\Psi(\upsilon) = \ln\upsilon$ and $f(\upsilon,\vartheta(\upsilon)) = 0$, take the limit $\breve{\nu} \to 0$, and put $p = 1$ to recover results for the Hadamard fractional derivative.

An equivalence between integral equation and the problem (4.1)-(4.2) is provided by the lemma below.

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Lemma 4.1. *The Cauchy problem for hybrid FDEs (4.1)-(4.2) has a solution $\vartheta \in C_{1-\eta;\Psi}([p,T],\mathbb{R})$ such that $\vartheta - f(\cdot,\vartheta(\cdot)) \in C_{1-\eta;\Psi}([p,T],\mathbb{R})$ if and only if it satisfies the integral equation expressed by*

$$\vartheta(\upsilon) = f(\upsilon,\vartheta(\upsilon)) + \frac{\vartheta_p}{\Gamma(\eta)}(\Psi(\upsilon)-\Psi(p))^{\eta-1} + I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\vartheta(\upsilon)),\ \upsilon \in (0,T]. \quad (4.3)$$

Proof. Assume that $\vartheta \in C_{1-\eta;\Psi}([p,T],\mathbb{R})$ be the solution of hybrid FDEs (4.1)-(4.2). Applying the Ψ-RL operator $I_{p^+}^{\breve{\mu};\Psi}$ to the both side of the Eq.(4.1) and property of Lemma 2.2 (i), we have

$$[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))] - \frac{(\Psi(\upsilon)-\Psi(p))^{\eta-1}}{\Gamma(\eta)} C = I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\vartheta(\upsilon)) \quad (4.4)$$

By applying Eq.(4.2) we get $C = \vartheta_p$ and the above equation can be expressed as

$$\vartheta(\upsilon) = f(\upsilon,\vartheta(\upsilon)) + \frac{\vartheta_p}{\Gamma(\eta)}(\Psi(\upsilon)-\Psi(p))^{\eta-1} + I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\vartheta(\upsilon)),\ \upsilon \in (0,T].$$

Now, we prove sufficient part. Let ϑ satisfy Eq. (4.3). Then, it also satisfies Eq.(4.1)-(4.2). Nevertheless, applying the Ψ-Hilfer fractional derivative ${}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}$ to both sides of Eq.(4.3) and using Lemma 2.1(iii) and Lemma 2.2(ii), we obtain

$$\begin{aligned} {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))] &= \frac{\vartheta_p}{\Gamma(\eta)}\ {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi}(\Psi(\upsilon)-\Psi(p))^{\eta-1} + \\ &\quad {}^{H}\mathcal{D}_{p^+}^{\breve{\mu},\breve{\nu};\Psi} I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\vartheta(\upsilon)) \\ &= g(\upsilon,\vartheta(\upsilon)). \end{aligned}$$

It is necessary to confirm whether the initial condition (4.2) is fulfilled. Applying $I_{p^+}^{1-\eta;\Psi}$ on Eq.(4.3), we have

$$I_{p^+}^{1-\eta;\Psi}[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))] = I_{p^+}^{1-\eta;\Psi}\frac{\vartheta_p}{\Gamma(\eta)}(\Psi(\upsilon)-\Psi(p))^{\eta-1} + I_{p^+}^{1-\eta;\Psi} I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\vartheta(\upsilon))$$

Using the Lemma 2.1(i) and Lemma 2.1(ii) we have

$$I_{p^+}^{1-\eta;\Psi}[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))] = \vartheta_p + I_{p^+}^{1-\eta+\breve{\mu};\Psi} g(\upsilon,\vartheta(\upsilon)) = \vartheta_p + I_{p^+}^{1-\breve{\nu}(1-\breve{\mu});\Psi} g(\upsilon,\vartheta(\upsilon))$$

Evaluating the limit as υ approaches p and Lemma 2.2(iii), the above equation simplifies to

$$\left[I_{p^+}^{1-\eta;\Psi}[\vartheta(\upsilon) - f(\upsilon,\vartheta(\upsilon))]\right]_{\upsilon=p} = \vartheta_p.$$

This complete the proof. □

The following outcome deals with the existence of a solution for the hybrid FDEs (4.1)-(4.2).

Theorem 4.1. *Suppose that the following assumptions:*

(H1) *A bounded function $f \in C([p,T]\times\mathbb{R},\mathbb{R})$ fulfills the following requirements:*

1. *The mapping* $\vartheta \mapsto \vartheta - f(\upsilon, \vartheta(\upsilon))$ *is monotonic increasing in* $\mathbb{R}$ *almost everywhere with respect to* $\upsilon \in (0, T]$;
2. *There exists a positive constant* $L < 1$ *such that*

$$|f(\upsilon, \varkappa(\upsilon)) - f(\upsilon, \vartheta(\upsilon))| \leq L|\varkappa(\upsilon) - \vartheta(\upsilon)|.$$

(H2) *Let* $g \in C([p, T] \times \mathbb{R}, \mathbb{R})$ *and there exists continuous function* $K : [p, T] \to \mathbb{R}$ *such that*

$$|g(\upsilon, \vartheta)| \leq K(\upsilon), \text{ a.e. } \upsilon \in [0, 1] \text{ and } \vartheta \in \mathbb{R}.$$

Then there exists a solution $\vartheta \in C_{1-\eta;\Psi}([p, T], \mathbb{R})$ *to the hybrid FDEs* (4.1)-(4.2).

Proof. Consider $\mathbb{B}_R = \{\varkappa \in C_{1-\eta;\Psi}([p, T], \mathbb{R}) : \|\varkappa\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})} \leq R\}$, where,

$$R := M + \left\{ \left| \frac{\vartheta_p}{\Gamma(\eta)} \right| + \frac{(\Psi(T) - \Psi(p))^{\breve{\mu}}}{\Gamma(\breve{\mu} + 1)} \|K\|_{C_{1-\eta;\Psi}[p,T]} \right\}$$

and M is bound on f that is, $\|f\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})} \leq M$. One can observe that $\mathbb{B}_R$ belongs to the class of closed, bounded, and convex subsets of $C_{1-\eta;\Psi}([p, T], \mathbb{R})$. Define two operators $\mathcal{T} : \mathbb{B}_R \to C_{1-\eta;\Psi}([p, T], \mathbb{R})$ and $\mathcal{S} : \mathbb{B}_R \to C_{1-\eta;\Psi}([p, T], \mathbb{R})$ as

$$\mathcal{T}\vartheta(\upsilon) = f(\upsilon, \vartheta(\upsilon))$$

and

$$\mathcal{S}\vartheta(\upsilon) = \frac{\vartheta_p}{\Gamma(\eta)}(\Psi(\upsilon) - \Psi(p))^{\eta-1} + I_{p^+}^{\breve{\mu};\Psi} g(\upsilon, \vartheta(\upsilon)), \text{ for each } \upsilon \in (0, \mathrm{T}].$$

Then the hybrid Eq.(4.3) can be equivalently expressed as

$$\mathcal{T}\vartheta(\upsilon) + \mathcal{S}\vartheta(\upsilon) = \vartheta(\upsilon), \text{ for each } \upsilon \in (0, \mathrm{T}].$$

To verify that the operators $\mathcal{T}$ and $\mathcal{S}$ satisfy all the conditions of Theorem 3.2, we proceed as follows:

Step 1:

Operator $\mathcal{T}$ is an $\mathcal{L}$-contraction. Indeed using the hypothesis (H1)(2), we have

$$\begin{aligned}
|(\Psi(\upsilon) - \Psi(p))^{1-\eta}(\mathcal{T}\varkappa(\upsilon) - \mathcal{T}\vartheta(\upsilon))| &= |(\Psi(\upsilon) - \Psi(p))^{1-\eta}(f(\upsilon, \varkappa(\upsilon)) - f(\upsilon, \vartheta(\upsilon)))| \\
&\leq L|(\Psi(\upsilon) - \Psi(p))^{1-\eta}(\varkappa(\upsilon) - \vartheta(\upsilon))| \\
&\leq L\|\varkappa - \vartheta\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})}.
\end{aligned}$$

Consequently,

$$\|\mathcal{T}\varkappa - \mathcal{T}\vartheta\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})} \leq L\|\varkappa - \vartheta\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})}.$$

Define $\Theta : [0, \infty) \to [1, \infty)$ as $\Theta(\upsilon) = e^{\upsilon}$, then

$$\frac{[\Theta(\|\varkappa - \vartheta\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})})]^L}{\Theta(\|\mathcal{T}\varkappa - \mathcal{T}\vartheta\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})})} \geq 1.$$

Select $\Xi(\upsilon,\varrho)=\dfrac{\varrho^L}{\upsilon}$, we obtain

$$\Xi(\Theta(\|\mathcal{T}\varkappa(\upsilon)-\mathcal{T}\vartheta(\upsilon)\|_{C_{1-\eta;\Psi}[p,T]}),\Theta(\|\varkappa(\upsilon)-\vartheta(\upsilon)\|_{C_{1-\eta;\Psi}[p,T]}))\geq 1.$$

Thus, $\mathcal{T}$ is an $\mathcal{L}$-contraction.

Step 2:

Operator $\mathcal{S}$ is compact. Indeed, first we show that $\mathcal{S}$ is continuous. Let $\vartheta_n\to\vartheta$ in $\mathbb{B}_R$. So,

$$\|\mathcal{S}\vartheta_n(\upsilon)-\mathcal{S}\vartheta(\upsilon)\|_{C_{1-\eta;\Psi}[p,T]}=\max_{\upsilon\in[p,T]}\{(\Psi(\upsilon)-\Psi(p))^{1-\eta}(\mathcal{S}\vartheta_n(\upsilon)-\mathcal{S}\vartheta(\upsilon))\}$$
$$\leq\max_{\upsilon\in[p,T]}\frac{(\Psi(\upsilon)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})}\int_p^{\upsilon}\Psi'(\varrho)(\Psi(\upsilon)-\Psi(p))^{\breve{\mu}-1}|g(\varrho,\vartheta_n(\varrho))-g(\varrho,\vartheta(\varrho))|d\varrho.$$

Since g is continuous and Lebesgue dominated convergence theorem, we have

$$\|\mathcal{S}\vartheta_n(\upsilon)-\mathcal{S}\vartheta(\upsilon)\|_{C_{1-\eta;\Psi}[p,T]}\to 0 \text{ as n}\to\infty.$$

Hence, it follows that $\mathcal{S}$ is a continuous operator.

We will now prove that the set $\mathcal{S}(\mathbb{B}_R)=\{\mathcal{S}\vartheta:\vartheta\in\mathbb{B}_R\}$ is uniformly bounded and equicontinuous. Let $\vartheta\in\mathbb{B}_R$ and $\upsilon\in[p,T]$. Then

$$\begin{aligned}|(\Psi(\upsilon)-\Psi(p))^{1-\eta}\mathcal{S}\vartheta(\upsilon)|&=\left|\frac{\vartheta_p}{\Gamma(\eta)}+(\Psi(\upsilon)-\Psi(p))^{1-\eta}I_{p^+}^{\breve{\mu};\Psi}g(\varrho,\vartheta(\varrho))\right|\\&\leq\left|\frac{\vartheta_p}{\Gamma(\eta)}\right|+(\Psi(\upsilon)-\Psi(p))^{1-\eta}I_{p^+}^{\breve{\mu};\Psi}|g(\varrho,\vartheta(\varrho))|\\&\leq\left|\frac{\vartheta_p}{\Gamma(\eta)}\right|+\frac{(\Psi(\upsilon)-\Psi(p))^{1-\eta}|K(\upsilon)|}{\Gamma(\breve{\mu})}\int_p^{\upsilon}\Psi'(\varrho)(\Psi(\upsilon)-\Psi(\varrho))^{\breve{\mu}-1}d\varrho\\&\leq\left|\frac{\vartheta_p}{\Gamma(\eta)}\right|+\frac{(\Psi(T)-\Psi(p))^{\breve{\mu}}}{\Gamma(\breve{\mu}+1)}\|K\|_{C_{1-\eta;\Psi}[p,T]}.\end{aligned}$$

Hence,

$$\|\mathcal{S}\vartheta\|_{C_{1-\eta;\Psi}[p,T]}\leq\left|\frac{\vartheta_p}{\Gamma(\eta)}\right|+\frac{(\Psi(T)-\Psi(p))^{\breve{\mu}}}{\Gamma(\breve{\mu}+1)}\|K\|_{C_{1-\eta;\Psi}[p,T]}. \tag{4.5}$$

Consequently, $\mathcal{S}(\mathbb{B}_R)$ is uniformly bounded. To demonstrate that $\mathcal{S}(\mathbb{B}_R)$ has equicontinuity. Let $\vartheta\in\mathbb{B}_R$ and $\upsilon_1,\upsilon_2\in[p,T]$ with $\upsilon_1<\upsilon_2$. Then

$$\begin{aligned}&|(\Psi(\upsilon_2)-\Psi(p))^{1-\eta}\mathcal{S}\vartheta(\upsilon_2)-(\Psi(\upsilon_1)-\Psi(p))^{1-\eta}\mathcal{S}\vartheta(\upsilon_1)|\\&=\Bigg|\left\{\frac{\vartheta_p}{\Gamma(\eta)}+\frac{(\Psi(\upsilon_2)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})}\int_p^{\upsilon_2}\Psi'(\varrho)(\Psi(\upsilon_2)-\Psi(\varrho))^{\breve{\mu}-1}g(\varrho,\vartheta(\varrho))d\varrho\right\}\\&\quad-\left\{\frac{\vartheta_p}{\Gamma(\eta)}+\frac{(\Psi(\upsilon_1)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})}\int_p^{\upsilon_1}\Psi'(\varrho)(\Psi(\upsilon_1)-\Psi(\varrho))^{\breve{\mu}-1}g(\varrho,\vartheta(\varrho))d\varrho\right\}\Bigg|\\&\leq\Bigg|\frac{(\Psi(\upsilon_2)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})}\int_p^{\upsilon_2}\Psi'(\varrho)(\Psi(\upsilon_2)-\Psi(\varrho))^{\breve{\mu}-1}g(\varrho,\vartheta(\varrho))d\varrho\\&\quad-\frac{(\Psi(\upsilon_1)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})}\int_p^{\upsilon_1}\Psi'(\varrho)(\Psi(\upsilon_1)-\Psi(\varrho))^{\breve{\mu}-1}g(\varrho,\vartheta(\varrho))d\varrho\Bigg|\end{aligned}$$

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$$
\begin{aligned}
&\leq \Bigg| \frac{(\Psi(\upsilon_2)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})} \int_p^{\upsilon_2} \Psi'(\varrho)(\Psi(\upsilon_2)-\Psi(\varrho))^{\breve{\mu}-1} K(\varrho) d\varrho \\
&\quad - \frac{(\Psi(\upsilon_1)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})} \int_p^{\upsilon_1} \Psi'(\varrho)(\Psi(\upsilon_1)-\Psi(\varrho))^{\breve{\mu}-1} K(\varrho) d\varrho \Bigg| \\
&\leq \Bigg| \frac{(\Psi(\upsilon_2)-\Psi(p))^{1-\eta}|K(\upsilon_2)|}{\Gamma(\breve{\mu})} \int_p^{\upsilon_2} \Psi'(\varrho)(\Psi(\upsilon_2)-\Psi(\varrho))^{\breve{\mu}-1} d\varrho \\
&\quad - \frac{(\Psi(\upsilon_1)-\Psi(p))^{1-\eta}|K(\upsilon_1)|}{\Gamma(\breve{\mu})} \int_p^{\upsilon_1} \Psi'(\varrho)(\Psi(\upsilon_1)-\Psi(\varrho))^{\breve{\mu}-1} d\varrho \Bigg| \\
&\leq \frac{\|K\|_{C_{1-\eta;\Psi}[p,T]}}{\Gamma(\breve{\mu}+1)} \{\Psi(\upsilon_2)-\Psi(p))^{\breve{\mu}} - \Psi(\upsilon_1)-\Psi(p))^{\breve{\mu}}\}.
\end{aligned}
$$

Since the function Ψ is continuous. Therefore, $|(\Psi(\upsilon_2)-\Psi(p))^{1-\eta}\mathcal{S}\vartheta(\upsilon_2) - (\Psi(\upsilon_1)-\Psi(p))^{1-\eta}\mathcal{S}\vartheta(\upsilon_1)| \to 0$ as $\upsilon_2 \to \upsilon_1$. This follows that $\mathcal{S}(\mathbb{B}_{\mathbb{R}})$ is equicontinuous. Thus by applying Arzelà-Ascoli theorem, we can conclude that $\mathcal{S}(\mathbb{B}_R)$ is relatively compact. Hence the operator $\mathcal{S}: \mathbb{B}_R \to C_{1-\eta;\Psi}([p,T],\mathbb{R})$ is compact.

Step 3:

For $\vartheta \in C_{1-\eta;\Psi}([p,T],\mathbb{R})$, $\vartheta = \mathcal{T}\vartheta + \mathcal{S}\varkappa \implies \vartheta \in \mathbb{B}_R$, for all $\varkappa \in \mathbb{B}_R$.

Let any $\vartheta \in C_{1-\eta;\Psi}([p,T],\mathbb{R})$ and $\varkappa \in \mathbb{B}_R$ such that $\vartheta = \mathcal{T}\vartheta + \mathcal{S}\varkappa$.

$$
\begin{aligned}
&|(\Psi(\upsilon)-\Psi(p))^{1-\eta}\vartheta(\upsilon)| = |(\Psi(\upsilon)-\Psi(p))^{1-\eta}(\mathcal{T}\vartheta(\upsilon)+\mathcal{S}\varkappa(\upsilon))| \\
&= \left| (\Psi(\upsilon)-\Psi(p))^{1-\eta} \left(f(\upsilon,\vartheta(\upsilon)) + \frac{\vartheta_p}{\Gamma(\eta)}(\Psi(\upsilon)-\Psi(p))^{\eta-1} + I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\varkappa(\upsilon)) \right) \right| \\
&\leq \|f(\upsilon,\vartheta(\upsilon))\|_{C_{1-\eta;\Psi}([p,T],\mathbb{R})} + \left| \left(\frac{\vartheta_p}{\Gamma(\eta)} + (\Psi(\upsilon)-\Psi(p))^{1-\eta} I_{p^+}^{\breve{\mu};\Psi} g(\upsilon,\varkappa(\upsilon)) \right) \right| \\
&\leq M + \left\{ \left| \frac{\vartheta_p}{\Gamma(\eta)} \right| + \frac{(\Psi(\upsilon)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})} \int_p^{\upsilon} \Psi'(\varrho)(\Psi(\upsilon)-\Psi(\varrho))^{\breve{\mu}-1} |g(\varrho,\varkappa(\varrho))| d\varrho \right\} \\
&\leq M + \left\{ \left| \frac{\vartheta_p}{\Gamma(\eta)} \right| + \frac{(\Psi(\upsilon)-\Psi(p))^{1-\eta}}{\Gamma(\breve{\mu})} \int_p^{\upsilon} \Psi'(\varrho)(\Psi(\upsilon)-\Psi(\varrho))^{\breve{\mu}-1} K(\varrho)) d\varrho \right\} \\
&\leq M + \left\{ \left| \frac{\vartheta_p}{\Gamma(\eta)} \right| + \frac{(\Psi(\upsilon)-\Psi(p))^{1-\eta}|K(\upsilon)|}{\Gamma(\breve{\mu})} \int_p^{\upsilon} \Psi'(\varrho)(\Psi(\upsilon)-\Psi(\varrho))^{\breve{\mu}-1} d\varrho \right\} \\
&\leq M + \left\{ \left| \frac{\vartheta_p}{\Gamma(\eta)} \right| + \frac{\|K\|_{C_{1-\eta;\Psi}[p,T]}}{\Gamma(\breve{\mu}+1)} (\Psi(\upsilon)-\Psi(p))^{\breve{\mu}} \right\}.
\end{aligned}
$$

This gives

$$
\|\vartheta\|_{C_{1-\eta;\Psi}[p,T]} \leq M + \left\{ \left| \frac{\vartheta_p}{\Gamma(\eta)} \right| + \frac{(\Psi(\upsilon)-\Psi(p))^{\breve{\mu}}}{\Gamma(\breve{\mu}+1)} \|K\|_{C_{1-\eta;\Psi}[p,T]} \right\} := R.
$$

Consequently $\vartheta \in \mathbb{B}_R$.

From the above steps, we have verified that all the requirements of Theorem 3.2 are fulfilled. Therefore the operator equation $\vartheta = \mathcal{T}\vartheta + \mathcal{S}\vartheta$ has a solution in $\mathbb{B}_R$ which acts as a solution of hybrid FDEs (4.1)-(4.2). □

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5. Illustrative examples

To demonstrate the numerical evidence of the existence result, the following specific instance of Ψ-Hilfer fractional derivatives are considered.

Example 2. Consider the particular case of problem (4.1)-(4.2) for the Ψ-RL FDEs:

$$ {}^{H}\mathcal{D}_{1^+}^{\frac{1}{2},0;\Psi}\vartheta(\upsilon) = \left(\frac{(\Psi(\upsilon)-\Psi(1))^{0.5}+5}{50}\right)\frac{\sqrt{\pi}}{(1+|\vartheta(\upsilon)|)}, \tag{5.1}$$

$$ I_{1^+}^{\frac{1}{2};\Psi}\vartheta(1^+) = 0 \in \mathbb{R}. \tag{5.2}$$

Through a comparison of problem (5.1)-(5.2) with the fundamental Ψ-Hilfer hybrid FDEs (4.1)-(4.2), we obtain $\breve{\mu} = \frac{1}{2}, \breve{\nu} = 0$, in this case $\eta = \frac{1}{2}$. Further $p = 1, T = e, J = [1, e]$ and take four cases for Ψ as

$$\Psi_1(\upsilon) = \ln(\upsilon),\ \Psi_2(\upsilon) = \upsilon,\ \Psi_3(\upsilon) = 2^{\upsilon} \text{ and } \Psi_4(\upsilon) = \pi^{\upsilon}.$$

$$f(\upsilon,\vartheta(\upsilon)) = 0 \text{ and } g(\upsilon,\vartheta(\upsilon)) = \left(\frac{(\Psi(\upsilon)-\Psi(1))^{0.5}+5}{50}\right)\frac{\sqrt{\pi}}{(1+|\vartheta(\upsilon)|)}.$$

Hence condition (H1) is satisfied for any $0 < L < 1$. Also for $\vartheta \in \mathbb{R}$ and $\upsilon \in J$,

$$|g(\upsilon,\vartheta)| \leq \frac{\sqrt{\pi}(\Psi(\upsilon)-\Psi(1))^{0.5}+5\sqrt{\pi}}{50} := K(\upsilon).$$

	$\Psi(\upsilon)=\ln(\upsilon)$	$\Psi(\upsilon)=\upsilon$	$\Psi(\upsilon)=2^{\upsilon}$	$\Psi(\upsilon)=\pi^{\upsilon}$
υ	$K(\upsilon)$	$K(\upsilon)$	$K(\upsilon)$	$K(\upsilon)$
1.0000	0.1772	0.1772	0.1772	0.1772
1.3436	0.1965	0.1980	0.2032	0.2208
1.6873	0.2028	0.2066	0.2164	0.2459
2.0309	0.2070	0.2132	0.2284	0.2715
2.3746	0.2102	0.2188	0.2405	0.3001
2.7182	0.2126, $\|K\| \approx$ 0.2126	0.2237, $\|K\| \approx$ 0.2933	0.2531, $\|K\| \approx$ 0.4788	0.3330, $\|K\| \approx$ 1.4638

TABLE 1. Numerical values of $K(\upsilon)$ for four different cases of Ψ in Example (2) and note that here $\|K\| = \|K\|_{C_{1-\eta;\Psi}[1,e]:\Psi}$

By fixing $M = 0.01$, one may verify that

$$M + \left\{\left|\frac{\vartheta_1}{\Gamma(\eta)}\right| + \frac{(\Psi(e)-\Psi(1))^{\breve{\mu}}}{\Gamma(\breve{\mu}+1)}\|K\|_{C_{1-\eta;\Psi}[1,e]}\right\} := R \approx \begin{cases} 0.2499 & \text{if } \Psi(\upsilon) = \Psi_1 \\ 0.4438 & \text{if } \Psi(\upsilon) = \Psi_2 \\ 1.1663 & \text{if } \Psi(\upsilon) = \Psi_3 \\ 7.2696 & \text{if } \Psi(\upsilon) = \Psi_4. \end{cases}$$

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Thus, all the hypothesis of Theorem 4.1 are fulfilled, the hybrid FDEs (5.1)-(5.2) has solution which is $\vartheta(\upsilon) = \frac{1}{5}(\Psi(\upsilon) - \Psi(1))^{0.5}$.

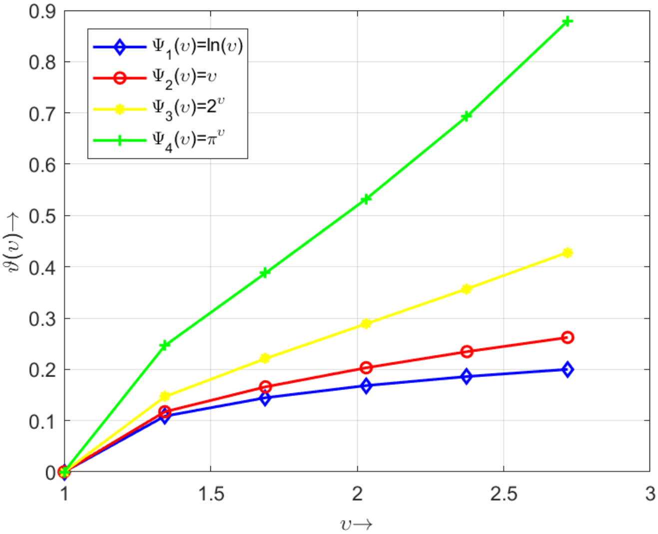

FIGURE 1. 2D-graph for the solutions of Example (2)

Example 3. Taking $\breve{\nu} \to 1, \breve{\mu} = \frac{1}{2}$, in this case $\eta = 1$ and let $p = 0, T = \pi$. Further we obtain a particular instance of problem (4.1)-(4.2) involving the Caputo fractional derivative with function Ψ:

$$ {}^{H}\mathcal{D}_{0^+}^{\breve{\mu},\breve{\nu};\Psi}\left[\vartheta(\upsilon) - \left(\frac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right)\left(\frac{\frac{|\vartheta(\upsilon)|}{3}}{\frac{|\vartheta(\upsilon)|}{3}+1}-1\right)\right] = \frac{\Gamma(\frac{3}{4})(\Psi(\upsilon)-\Psi(0))^{\frac{1}{4}}\left((\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+12\right)}{48\Gamma(\frac{1}{4})}\left(1-\frac{\vartheta(\upsilon)}{\vartheta(\upsilon)+1}\right), \tag{5.3}$$

$$ I_{0^+}^{1-\eta;\Psi}\left[\vartheta(\upsilon) - \left(\frac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right)\left(\frac{\frac{|\vartheta(\upsilon)|}{3}}{\frac{|\vartheta(\upsilon)|}{3}+1}-1\right)\right](0^+) = \frac{1}{60}. \tag{5.4}$$

Consider three cases for Ψ as

$$\Psi_1(\upsilon) = e^{\upsilon}, \Psi_2(\upsilon) = 5^{\upsilon} \text{ and } \Psi_3(\upsilon) = \ln(\upsilon + 0.01).$$

Comparing the problem (5.3)-(5.4) with the basic Ψ-Hilfer hybrid FDEs (4.1)-(4.2) we get

$$ f(\upsilon,\vartheta(\upsilon)) = \left(\frac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right)\left(\frac{\frac{|\vartheta(\upsilon)|}{3}}{\frac{|\vartheta(\upsilon)|}{3}+1}-1\right)$$

	$\Psi(\upsilon)=e^{\upsilon}$		$\Psi(\upsilon)=5^{\upsilon}$		$\Psi(\upsilon)=\ln(\upsilon+0.01)$	
υ	L_Ψ	$K(\upsilon)$	L_Ψ	$K(\upsilon)$	L_Ψ	$K(\upsilon)$
0.0000	1.667e-02	0.0000	1.667e-02	0.0000	1.667e-02	0.0000
0.3490	1.691e-02	0.7087e-01	1.704e-02	0.8404e-01	1.787e-02	0.1414
0.6981	1.713e-02	0.9181e-01	1.747e-02	1.1604e-01	1.804e-02	0.1513
1.0471	1.740e-02	1.1156e-01	1.807e-02	1.5328e-01	1.814e-02	0.1569
1.3962	1.773e-02	1.3298e-01	1.896e-02	2.0369e-01	1.820e-02	0.1608
1.7453	1.815e-02	1.5788e-01	2.030e-02	2.7770e-01	1.825e-02	0.1637
2.0943	1.868e-02	1.8816e-01	2.232e-02	3.9242e-01	1.829e-02	0.1661
2.4434	1.937e-02	2.2617e-01	2.538e-02	5.7708e-01	1.833e-02	0.1681
2.7925	2.025e-02	2.7506e-01	3.003e-02	8.8244e-01	1.836e-02	0.1698
3.1415	2.139e-02 ≈2.140e-02 $\approx L=M$	3.3919e-01 ≈3.392e-01 $\approx \|K\|$	3.710e-02 ≈3.711e-02 $\approx L=M$	13.970e-01 ≈13.971e-01 $\approx \|K\|$	1.838e-02 ≈1.839e-02 $\approx L=M$	0.1713 ≈0.1714 $\approx \|K\|$

TABLE 2. Numerical values of L, M and $K(\upsilon)$ for three different cases of Ψ in Example (3) and note that here $\|K\| = \|K\|_{C_{1-\eta;\Psi}[0,\pi]:\Psi}$

and

$$g(\upsilon,\vartheta(\upsilon)) = \frac{\Gamma(\frac{3}{4})(\Psi(\upsilon)-\Psi(0))^{\frac{1}{4}}\left((\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+12\right)}{48\Gamma(\frac{1}{4})}\left(1-\frac{\vartheta(\upsilon)}{\vartheta(\upsilon)+1}\right).$$

For each $\vartheta,\zeta\in\mathbb{R}$ and $\upsilon\in J$, we have

$$\begin{aligned}|f(\upsilon,\vartheta)-f(\upsilon,\zeta)| &= \left(\frac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right)\left|\frac{\frac{|\vartheta|}{3}}{\frac{|\vartheta|}{3}+1}-\frac{\frac{|\zeta|}{3}}{\frac{|\zeta|}{3}+1}\right| \\ &\le \left(\frac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right)\left|\frac{|\vartheta|}{3}-\frac{|\zeta|}{3}\right| \\ &\le \left(\frac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right)|\vartheta-\zeta| := L_\Psi\,|\vartheta-\zeta|\end{aligned}$$

and the function $\vartheta(\upsilon)-f(\upsilon,\vartheta(\upsilon))$ is increasing. Hence condition (H1) is satisfied with $L\approx 2.140\text{e-}02$, 3.711e-02 and 1.839e-02 for $\Psi(\upsilon)=e^{\upsilon}$, 5^{υ} and $\ln(\upsilon+0.01)$ respectively. Also for $\vartheta\in\mathbb{R}$ and $\upsilon\in J$,

$$|g(\upsilon,\vartheta)| \le \left|\frac{\Gamma(\frac{3}{4})(\Psi(\upsilon)-\Psi(0))^{\frac{1}{4}}\left((\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}+12\right)}{48\Gamma(\frac{1}{4})}\right| := K(\upsilon),$$

and

$$|f(\upsilon,\vartheta)| \le \left(\frac{(\Psi(\pi)-\Psi(0))^{\frac{3}{4}}+36}{2160}\right) := M.$$

Next, one can check that

$$M+\left\{\left|\frac{\vartheta_0}{\Gamma(\eta)}\right|+\frac{(\Psi(\pi)-\Psi(0))^{\breve{\mu}}}{\Gamma(\breve{\mu}+1)}\|K\|_{C_{1-\eta;\Psi}[0,\pi]}\right\} := R \approx \begin{cases} 1.3081 & \text{if } \Psi(\upsilon)=e^{\upsilon} \\ 19.7433 & \text{if } \Psi(\upsilon)=5^{\upsilon} \\ 0.4989 & \text{if } \Psi(\upsilon)=\Psi_3. \end{cases}$$

The hybrid FDEs (5.3)-(5.4) have a solution as a result, satisfying all of Theorem 4.1 hypotheses. Hence, one can check that the solution is the function $\vartheta(\upsilon) = \dfrac{(\Psi(\upsilon)-\Psi(0))^{\frac{3}{4}}}{12}$.

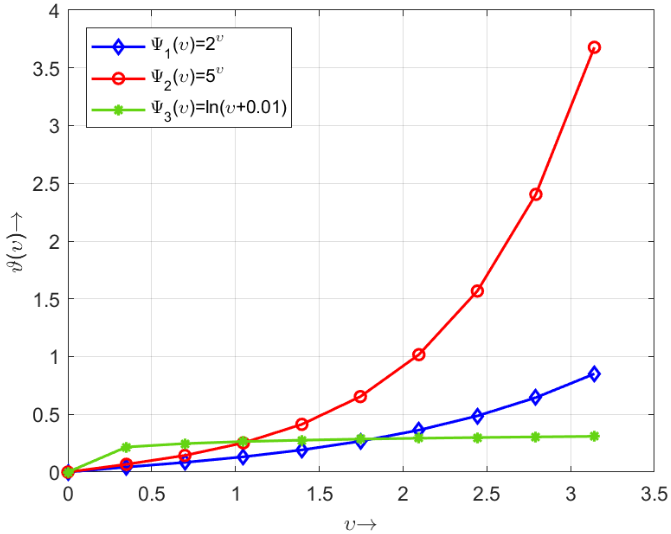

FIGURE 2. 2D-graph for the solutions of Example (3)

6. Conclusion

The work contributes to the growth of fractional calculus, especially in the case of Ψ-Hilfer FDEs. There are some articles that carried out a brief study on existence of solutions of FDEs. So the main purpose of this paper is to study the existence of solutions for the Ψ-Hilfer hybrid FDEs (4.1)-(4.2) by means of Krasnoselskii type fixed point theorem (Theorem 3.2). The proof of Theorem 3.2 is based on Housdorff measure of noncompactness.

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