ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

ARITHMETIC PROGRESSIONS OF INTEGERS THAT ARE RELATIVELY PRIME TO THEIR DIGITAL SUMS

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ABSTRACT. For an integer $b \geq 2$, we call a positive integer b-anti-Niven if it is relatively prime to the sum of the digits in its base-b representation. In this article, we investigate the maximum lengths of arithmetic progressions of b-anti-Niven numbers.

1. Introduction

Throughout this paper, let $b \geq 2$ be an integer. For all positive integers n, let $s_b(n)$ denote the sum of the digits in the base-b expansion of n, i.e., if $n = \sum_{j=0}^{m} a_j b^j$, where m is a nonnegative integer and $0 \leq a_j \leq b-1$ are integers for each $0 \leq j \leq m$, then $s_b(n) = \sum_{j=0}^{m} a_j$.

For positive integers n, d, and t, we call the sequence $\{n + jd : 0 \leq j \leq t-1\}$ a *d-AP of length t* and we call the sequence $\{n + jd : j \geq 0\}$ a *d-AP of infinite length*. A positive integer n is *b-Niven* if $s_b(n) \mid n$. If every term of a d-AP is b-Niven, we call it a *b-Niven d-AP*. We note that a b-Niven 1-AP is a sequence of consecutive Niven numbers.

In 1993, Cooper and Kennedy [1] showed that the maximum length of a 10-Niven 1-AP is 20. Grundman [3] generalized this result in 1994 by showing that the maximum length of a b-Niven 1-AP is $2b$. These maximum lengths were shown to be attainable by Wilson [7]. More recently, Grundman, Harrington, and Wong [4] investigated maximum length b-Niven d-APs for $d > 1$ and Harrington, Litman, and Wong [5] showed that every infinite d-AP contains infinitely many b-Niven numbers.

In 1975, Olivier [6] studied sets $S_b = \{n \in \mathbb{Z} : \gcd(n, s_b(n)) = 1\}$ and showed that the natural density of these sets is $\frac{6}{\pi^2} \prod_{p|(b-1)} \frac{p}{p+1}$. In 1997, Cooper and Kennedy [2] published a weaker result that established Olivier's density as an upper bound for the density of S_{10}.

In this paper, we define a positive integer n to be *b-anti-Niven* if $\gcd(s_b(n), n) = 1$. If every term of a d-AP is b-anti-Niven, then we call it a *b-anti-Niven d-AP*. In Section 2 we give necessary and sufficient conditions on d, b, and n for which the d-AP $\{n + jd : j \geq 0\}$ contains at least one b-anti-Niven number. We also show that there is no b-anti-Niven d-AP of infinite length, but for any b and t, there are infinitely many b-anti-Niven d-APs of length t. In Section 3 we investigate the maximum length of b-anti-Niven d-APs when b and d satisfy various constraints.

2. b-anti-Niven Numbers in d-APs

In this section, we are going to give several general results on b-anti-Niven numbers in d-APs.

2020 *Mathematics Subject Classification.* 11A63, 11B25.
Key words and phrases. Niven, anti-Niven, arithmetic progressions.

Lemma 2.1. *Let δ be a positive integer such that $\delta \mid (b-1)$. Then for all positive integers n, $\delta \mid n$ if and only if $\delta \mid s_b(n)$.*

Proof. Let $n = \sum_{j=0}^{m} a_j b^j$, where m is a nonnegative integer and $0 \le a_j \le b-1$ are integers for each $0 \le j \le m$. The proof follows from the simple observation that $b \equiv 1 \pmod{\delta}$ and thus $\sum_{j=0}^{m} a_j b^j \equiv \sum_{j=0}^{m} a_j \pmod{\delta}$. □

The following lemma was proven by Harrington, Litman, and Wong [5].

Lemma 2.2 ([5, Proposition 2.6])**.** *Let $\xi = \gcd(s_b(n), s_b(d), b-1)$. Then there exists a positive multiple $\overline{d}$ of d such that $\gcd(s_b(n), s_b(\overline{d})) = \xi$.*

Theorem 2.3. *The d-AP of infinite length $\{n + jd : j \ge 0\}$ contains a b-anti-Niven number if and only if $\gcd(n,d,b-1) = 1$.*

Proof. Assuming that $\gcd(n,d,b-1) = 1$, we have $\gcd(s_b(n), s_b(d), b-1) = 1$ by Lemma 2.1. By Lemma 2.2, there exists a positive multiple $\overline{d}$ of d such that $\gcd(s_b(n), s_b(\overline{d})) = 1$. Let k be a positive integer such that $s_b(n) + k \cdot s_b(\overline{d}) = p$ is a prime with $p > \max(b, \overline{d})$, and we further let $m_0 = \lfloor \log_b(n) \rfloor + 1$ and $m_i = m_{i-1} + \lfloor \log_b(\overline{d}) \rfloor + 1$ for all $1 \le i \le k$. Consider $n + j\overline{d}$ and $n + j'\overline{d}$, where $j = \sum_{i=0}^{k} b^{m_i}$ and $j' = j - b^{m_k} + b^{m_k+1}$. Note that both $s_b(n + j\overline{d})$ and $s_b(n + j'\overline{d})$ are equal to $s_b(n) + k \cdot s_b(\overline{d}) = p$, and $(n + j'\overline{d}) - (n + j\overline{d}) = b^{m_k}(b-1)\overline{d}$ is not divisible by p since $p > \max(b, \overline{d})$. Hence, at least one of $n + j\overline{d}$ and $n + j'\overline{d}$ is our desired b-anti-Niven number in the given d-AP.

Conversely, if $\gcd(n,d,b-1) = \delta > 1$, then $\delta \mid \gcd(n + jd, s_b(n + jd))$ for all integers $j \ge 0$ by Lemma 2.1. Therefore, $\{n + jd : j \ge 0\}$ does not contain any b-anti-Niven numbers. □

The following theorem is a consequence of a result of Harrington, Litman, and Wong [5] who showed that every arithmetic progression of infinite length contains at least one b-Niven number n with $s_b(n) \ne 1$.

Theorem 2.4. *For any positive integer d, there is no b-anti-Niven d-AP of infinite length.*

Our next theorem shows that there exist arithmetic progressions of arbitrary length containing only b-anti-Niven numbers.

Theorem 2.5. *For every positive integer t, there exist positive integers n and d such that $\{n + jd : 0 \le j \le t-1\}$ is a b-anti-Niven d-AP of length t.*

Proof. Let m be a positive integer such that $b^m/(m(b-1)+1) \ge t$, and let $d = b(b^m-1)(m(b-1)+1)$. Consider the d-AP $\{(d+1) + jd : 0 \le j \le t-1\}$. For all $0 \le j \le t-1$, note that $\widetilde{j} = (j+1)(m(b-

$1)+1) \leq b^m$. Hence,

$$\begin{aligned} s_b((d+1)+jd) &= s_b(\widetilde{j}(b^{m+1}-b)+1) \\ &= s_b(\widetilde{j}(b^{m+1}-b))+1 \\ &= s_b\big((\widetilde{j}-1)b^{m+1}+b(b^m-1-(\widetilde{j}-1))\big)+1 \\ &= s_b(\widetilde{j}-1)+s_b\big(b^m-1-(\widetilde{j}-1)\big)+1 \\ &= s_b(\widetilde{j}-1)+s_b\left(\sum_{j=0}^{m-1}(b-1)b^j-(\widetilde{j}-1)\right)+1 \\ &= s_b(\widetilde{j}-1)+m(b-1)-s_b(\widetilde{j}-1)+1 \\ &= m(b-1)+1. \end{aligned}$$

Since $(d+1)+jd \equiv 1 \pmod{m(b-1)+1}$, we conclude that $(d+1)+jd$ is b-anti-Niven for all $0 \leq j \leq t-1$. □

Although Theorem 2.5 shows that there are b-anti-Niven d-APs of arbitrary length, the maximum length of a b-anti-Niven d-AP is bounded above by $b-2$ for many values of b and d, as shown in the following theorem.

Theorem 2.6. *For $b > 2$ and a positive integer d, let p be the smallest prime such that $p \mid (b-1)$ and $p \nmid d$. Then every b-anti-Niven d-AP has length at most $p-1$.*

Proof. Since $p \nmid d$, every d-AP of length p contains a multiple of p. By Lemma 2.1, this multiple of p is not b-anti-Niven. Hence, the maximum length of a d-AP that contains only b-anti-Niven numbers is at most $p-1$. □

3. Maximum Length b-anti-Niven d-APs

Theorem 2.6 in the previous section gives a bound on the maximum length of certain b-anti-Niven d-APs. We begin this section by demonstrating that there are instances when this bound is achieved. Theorems 3.2 and 3.3 investigate 1-APs, i.e. sequences of consecutive b-anti-Niven numbers, and 2-APs, respectively. The following lemma will be a common tool in establishing these two theorems.

Lemma 3.1. *For all finite collections of distinct primes $q_1, q_2, \ldots, q_t$, there exist infinitely many positive integers m such that $b^m \equiv b \pmod{q_1 q_2 \cdots q_t}$.*

Proof. Without loss of generality, assume that there exists $0 \leq t' \leq t$ such that $q_i \nmid b$ for all $1 \leq i \leq t'$ and $q_i \mid b$ for all $t'+1 \leq i \leq t$. By Euler's theorem, $b^{k\varphi(q_1 q_2 \cdots q_{t'})} \equiv 1 \pmod{q_1 q_2 \cdots q_{t'}}$ for every positive integer k. Hence, $m = k\varphi(q_1 q_2 \cdots q_{t'})+1$ is our desired choice of integer. □

Theorem 3.2. *For $b > 2$, let p be the smallest prime such that $p \mid (b-1)$. Then the maximum length of a sequence of consecutive b-anti-Niven numbers is $p-1$. Furthermore, there exist infinitely many such sequences of length $p-1$.*

Proof. By Theorem 2.6, the maximum length of a b-anti-Niven 1-AP is at most $p-1$. It remains to show that such sequences occur infinitely often. Let $q_1, q_2, \ldots, q_t$ be all primes less than p. By

Lemma 3.1, there exist infinitely many positive integers m such that $b^m \equiv b \pmod{q_1 q_2 \cdots q_t}$. Now, for all $0 \le j \le p-2$, we have $s_b(b^m + j) = j+1$. Since $j+1 < p$, for any prime divisor q of $j+1$, we have $b^m + j \equiv b + j \equiv b - 1 \not\equiv 0 \pmod{q}$. Therefore, $\gcd(b^m + j, s_b(b^m + j)) = 1$, implying that $\{b^m + j : 0 \le j \le p-2\}$ forms a sequence of $p-1$ consecutive b-anti-Niven numbers. □

Theorem 3.3. *Let $b > 2$ be such that $b \neq 2^r + 1$ for any integer r, and let p be the smallest odd prime such that $p \mid (b-1)$. Then the maximum length of a b-anti-Niven 2-AP is $p-1$. Furthermore, there exist infinitely many such sequences of length $p-1$.*

Proof. By Theorem 2.6, the maximum length of a b-anti-Niven 2-AP is at most $p-1$. It remains to show that such sequences occur infinitely often. Let $q_1, q_2, \ldots, q_t$ be all primes less than or equal to b. By Lemma 3.1, there exist infinitely many positive integers m such that $b^m \equiv b \pmod{q_1 q_2 \cdots q_t}$.

Consider the case when b is even. For all $0 \le j \le p-2$, we have $s_b(b^m + 2j + 1) = 2(j+1)$. Since $j+1 < p$, for any prime divisor q of $2(j+1)$, we have $b^m + 2j + 1 \equiv b + 2j + 1 \equiv b - 1 \not\equiv 0 \pmod{q}$. Therefore, $\gcd(b^m + 2j + 1, s_b(b^m + 2j + 1)) = 1$, implying that $\{b^m + 2j + 1 : 0 \le j \le p-2\}$ forms a b-anti-Niven 2-AP of length $p-1$.

Next, consider the case when b is odd. For all $0 \le j \le (p-1)/2$, we have $s_b(b^m + b - p + 2j) = 1 + b - p + 2j$. Since $1 + b - p + 2j \le b$, for any prime divisor q of $1 + b - p + 2j$, we have $b^m + b - p + 2j \equiv 2b - p + 2j \equiv 2b - p + 2j - 2(1 + b - p + 2j) \equiv p - 2j - 2 \pmod{q}$. Note that $-1 \le p - 2j - 2 \le p - 2$, so none of these odd numbers share a common prime factor with $b-1$. Hence, $\gcd(p - 2j - 2, 1 + b - p + 2j) = \gcd(p - 2j - 2, b - 1) = 1$, implying that $p - 2j - 2 \not\equiv 0 \pmod{q}$. Thus, $\gcd(b^m + b - p + 2j, s_b(b^m + b - p + 2j)) = 1$ when $0 \le j \le (p-1)/2$.

Furthermore, for all $0 \le j \le (p-5)/2$, we have $s_b(b^m + b + 1 + 2j) = 3 + 2j$. Since $3 + 2j \le p - 2 < b$, for any prime divisor q of $3 + 2j$, we have $b^m + b + 1 + 2j \equiv 2b + 1 + 2j \equiv 2b + 1 + 2j - (3 + 2j) \equiv 2(b-1) \pmod{q}$. Note that $2(b-1) \not\equiv 0 \pmod{q}$ since q is an odd prime less than p. Thus, $\gcd(b^m + b + 1 + 2j, s_b(b^m + b + 1 + 2j)) = 1$ when $0 \le j \le (p-5)/2$. Therefore, $\{b^m + b - p + 2j : 0 \le j \le (p-1)/2\} \cup \{b^m + b + 1 + 2j : 0 \le j \le (p-5)/2\}$ forms a b-anti-Niven 2-AP of length $p-1$. □

So far, the theorems in this section have shown that the bound provided in Theorem 2.6 is attainable. However, there are infinitely many instances when the maximum length of b-anti-Niven d-APs does not attain this bound. The following theorem illustrates one such instance.

Theorem 3.4. *Let $b \ge 6$ be even, and let $3 \le d \le b/2$ be an odd integer. Then the maximum length of a b-anti-Niven d-AP is at most $\lceil 2b/d \rceil + 2$.*

Note that when $b-1$ is an odd prime, the bound given by Theorem 2.6 is $b-2$, while the bound given by Theorem 3.4 is strictly smaller when $b > 15$.

Proof of Theorem 3.4. Suppose there are two consecutive terms from a d-AP in the interval $[ab, ab + b - 1]$ that are b-anti-Niven. Let these two numbers be $ab + a_0$ and $ab + a_0 + d$ for some nonnegative integers a and $a_0 \le b - 1 - d$. Recalling that d is odd, there exists $\chi \in \{0, 1\}$ so that $a_0 + \chi d$ is even. As a result, $ab + a_0 + \chi d$ is also even since b is even. Since $ab + a_0 + \chi d$ is b-anti-Niven, $s_b(ab + a_0 + \chi d) = s_b(a) + a_0 + \chi d$ is odd, implying that $s_b(a)$ is odd.

Note that there are at most $\lceil 2b/d \rceil$ terms from a d-AP in the interval $[ab,(a+1)b+(b-1)]$. Hence, if there is a b-anti-Niven d-AP of length $\lceil 2b/d \rceil + 3$, then there exists a nonnegative integer a such that each of the intervals $[ab, ab+b-1]$, $[(a+1)b,(a+1)b+b-1]$, and $[(a+2)b,(a+2)b+b-1]$ contains at least two terms from this d-AP. From the above observation, we conclude that $s_b(a)$, $s_b(a+1)$, and $s_b(a+2)$ are all odd, which is a contradiction. This establishes an upper bound for the maximum length of a b-anti-Niven d-AP as stated in the theorem. □

We now turn our attention to d-APs for which Theorem 2.6 does not apply.

Theorem 3.5. *Let b be even. Then the maximum length of a b-anti-Niven $(b-1)$-AP is $2b+1$. Furthermore, there exist infinitely many such sequences of length $2b+1$.*

Proof. Suppose there is a b-anti-Niven $(b-1)$-AP $\mathscr{S}$ of length at least $2b+1$. Since $\gcd(b-1,b)=1$, there exist two terms in $\mathscr{S}$ that are multiples of b. Let these two terms be ab and $ab+(b-1)b$ for some positive integer a. Note that ab is even, so $s_b(ab) = s_b(a)$ is odd. Since $ab+2(b-1) = (a+1)b+b-2$ is an even term in $\mathscr{S}$, the digit sum $s_b((a+1)b+b-2) = s_b(a+1)+b-2$ must be odd, implying that $s_b(a+1)$ is also odd. Hence, $a = cb+b-1$ for some nonnegative integer c, where $s_b(c)$ is even. In other words, $ab = cb^2+(b-1)b$ and $ab+(b-1)b = (c+1)b^2+(b-2)b$. Also, $s_b(c+1)$ is odd since $s_b((c+1)b^2+(b-2)b) = s_b(c+1)+b-2$ is odd.

Now, note that cb^2 and $s_b(cb^2) = s_b(c)$ are even, so cb^2 is not in $\mathscr{S}$. Similarly, $(c+1)b^2+(b-1)b+b-2$ and $s_b((c+1)b^2+(b-1)b+b-2) = s_b(c+1)+b-1+b-2$ are even, so $(c+1)b^2+(b-1)b+b-2 = cb^2+(2b+2)(b-1)$ is also not in $\mathscr{S}$. Therefore, $\mathscr{S}$ is a subsequence of $\{cb^2+j(b-1) : 1 \le j \le 2b+1\}$, thus the maximum length of a b-anti-Niven $(b-1)$-AP is at most is $2b+1$.

It remains to show that such sequences occur infinitely often. Let c be a nonnegative integer such that $s_b(c+1) = s_b(c)+1$. Then it is not difficult to observe that $s_b(cb^2+j(b-1)) = s_b(c)+b-1$ for $1 \le j \le b$ and $b+2 \le j \le 2b$, and $s_b(cb^2+j(b-1)) = s_b(c)+2(b-1)$ for $j \in \{b+1, 2b+1\}$. Hence, it suffices to show that there exist infinitely many positive integers c such that

- $b \nmid (c+1)$,
- $\gcd(cb^2+j(b-1), s_b(c)+b-1) = 1$ for $1 \le j \le b$ and $b+2 \le j \le 2b$, and
- $\gcd(cb^2+j(b-1), s_b(c)+2(b-1)) = 1$ for $j \in \{b+1, 2b+1\}$.

Let $p_1, p_2, \ldots, p_t$ be all primes less than or equal to $2b$. By Lemma 3.1, there exist infinitely many positive integers m such that $b^{m+1} \equiv b \pmod{p_1 p_2 \cdots p_t}$. In other words, $p_1 p_2 \cdots p_t \mid b(b^m - 1)$. Let $P = b^m + 1$. Since $\gcd(b^m+1, b) = 1$ and $\gcd(b^m+1, b^m-1) = 1$, we have $\gcd(P, p_1 p_2 \cdots p_t) = 1$. Next, consider $q_1, q_2, \ldots, q_v$ be all prime factors of $b^{m-1}+1$. Hence, $b^{m-1} \equiv -1 \pmod{q_1 q_2 \cdots q_v}$. Let $r_1, r_2, \ldots, r_{(P-b+1)/2}$ be positive integers, where $r_{i+1} - r_i \ge m+1$ for all $1 \le i \le (P-b-1)/2$, be defined as follows.

- If $(P-b+1)/2$ is odd, then

$$b^{r_i+2} \equiv \begin{cases} -1 \pmod{q_1 q_2 \cdots q_v} & \text{if } 1 \le i \le (P-b-1)/4; \\ 1 \pmod{q_1 q_2 \cdots q_v} & \text{otherwise.} \end{cases}$$

- If $(P-b+1)/2$ is even, then

$$b^{r_i+2} \equiv \begin{cases} -1 \pmod{q_1q_2\cdots q_\tau} & \text{if } 1 \le i \le (P-b-3)/4; \\ 1 \pmod{q_1q_2\cdots q_\tau} & \text{if } (P-b+1)/4 \le i \le (P-b-3)/2; \\ b \pmod{q_1q_2\cdots q_\tau} & \text{otherwise.} \end{cases}$$

Now, let $c = \sum_{i=1}^{(P-b+1)/2} b^{r_i}(b^m+1)$. Since $b \mid c$, we have $b \nmid (c+1)$. Next, $s_b(c) = P-b+1$ from our construction, thus $s_b(c)+b-1 = P = b^m+1$, which is a factor of c. Recalling that P is relatively prime to all positive integers up to $2b$, we have $\gcd(cb^2+j(b-1),P) = 1$ for all $1 \le j \le 2b$. It remains to prove that $\gcd(cb^2+j(b-1),s_b(c)+2(b-1)) = 1$ for $j \in \{b+1,2b+1\}$. Note that $s_b(c)+2(b-1) = P-b+1+2(b-1) = P+b-1 = b(b^{m-1}+1)$. For any prime factor q of $b^{m-1}+1$, we clearly have $q \nmid b$. Moreover, $q \nmid (b-1)$, or otherwise, $q \mid b(b^{m-1}+1)$ and $q \mid (b-1)$ imply $q \mid P$, contradicting that P is relatively prime to all positive integers up to $2b$. If $(P-b+1)/2$ is odd, then

$$\begin{aligned} cb^2+(b+1)(b-1) &= \left(\sum_{i=1}^{(P-b+1)/2} b^{r_i+2}\right)P+b^2-1 \\ &\equiv P+b^2-1 \\ &\equiv P+b^2-1-(P+b-1) \\ &\equiv b(b-1) \\ &\not\equiv 0 \pmod{q} \end{aligned}$$

and $cb^2+(2b+1)(b-1) = cb^2+(b+1)(b-1)+b(b-1) \equiv 2b(b-1) \not\equiv 0 \pmod{q}$. If $(P-b+1)/2$ is even, then

$$\begin{aligned} cb^2+(b+1)(b-1) &= \left(\sum_{i=1}^{(P-b+1)/2} b^{r_i+2}\right)P+b^2-1 \\ &\equiv 2bP+b^2-1 \\ &\equiv 2bP+b^2-1-2b(P+b-1) \\ &\equiv -(b-1)^2 \\ &\not\equiv 0 \pmod{q} \end{aligned}$$

and $cb^2+(2b+1)(b-1) = cb^2+(b+1)(b-1)+b(b-1) \equiv -(b-1)^2+b(b-1) \equiv b-1 \not\equiv 0 \pmod{q}$. Finally, our proof is completed by noticing that $\gcd(cb^2+j(b-1),b) = 1$ for $j \in \{b+1,2b+1\}$. □

To complete the investigation on 1-APs, we provide the following corollary by choosing $b=2$ in Theorem 3.5.

Corollary 3.6. *For $b=2$, the maximum length of a sequence of consecutive 2-anti-Niven numbers is 5. Furthermore, there exist infinitely many such sequences of length 5.*

4. Concluding Remarks

Theorem 3.3 establishes that the maximum length of a b-anti-Niven 2-AP is at most $b-2$ when $b-1$ has an odd prime divisor. Although we have not established an upper bound for the maximum length of a b-anti-Niven 2-AP when $b = 2^r + 1$ for some nonnegative integer r, the following theorem establishes a lower bound.

Theorem 4.1. *Let $b = 2^r + 1$ for some nonnegative integer r. Then the maximum length of a b-anti-Niven 2-AP is at least b.*

Proof. If $r = 0$, then $b = 2$, and $\{2,4\}$ forms a 2-anti-Niven 2-AP of length 2. If $r > 0$, then b is odd. For all $0 \le j \le (b-1)/2$, we have $\gcd(b+2j, s_b(b+2j)) = \gcd(b+2j, 1+2j) = \gcd(b+2j, b-1) = \gcd(2^r+1+2j, 2^r) = 1$. Furthermore, for all $0 \le j \le (b-3)/2$, we have $\gcd(2b+1+2j, s_b(2b+1+2j)) = \gcd(2b+1+2j, 3+2j) = \gcd(2b+1+2j, 2b-2) = \gcd(2^{r+1}+3+2j, 2^{r+1}) = 1$. Therefore, $\{b+2j : 0 \le j \le (b-1)/2\} \cup \{2b+1+2j : 0 \le j \le (b-3)/2\}$ forms a b-anti-Niven 2-AP of length b. □

Theorem 3.5 establishes that the maximum length of a b-anti-Niven $(b-1)$-AP is at most $2b+1$ when b is even. The following theorem establishes a lower bound when b is an odd prime.

Theorem 4.2. *Let b be an odd prime. Then the maximum length of a b-anti-Niven $(b-1)$-AP is at least $2b+1$.*

Proof. Clearly, 1, b, and b^2 are b-anti-Niven. For all $1 \le j \le b-1$, we have

$$\begin{aligned}\gcd\left(b+j(b-1), s_b(b+j(b-1))\right) &= \gcd\left((j+1)b-j, s_b(jb+b-j)\right)\\ &= \gcd((j+1)b-j, j+b-j)\\ &= \gcd((j+1)b-j, b) = 1\end{aligned}$$

since b is a prime. Furthermore, for all $1 \le j \le b-1$, we have

$$\begin{aligned}\gcd\left(b^2+j(b-1), s_b(b^2+j(b-1))\right) &= \gcd\left(b^2+jb-j, s_b(b^2+(j-1)b+b-j)\right)\\ &= \gcd(b^2+jb-j, 1+j-1+b-j)\\ &= \gcd(b^2+jb-j, b) = 1.\end{aligned}$$

Therefore, $\{1,b\} \cup \{b+j(b-1) : 1 \le j \le b-1\} \cup \{b^2\} \cup \{b^2+j(b-1) : 1 \le j \le b-1\}$ forms a b-anti-Niven $(b-1)$-AP of length $2b+1$. □

Recall that Theorem 3.4 shows that the upper bound on the maximum length of a b-anti-Niven d-AP given by Theorem 2.6 may not be achievable for even b. However, when b is odd, computational data suggests otherwise. Of course, if d is odd, then Theorem 2.6 implies that the maximum length of a b-anti-Niven d-AP is at most 1, which is clearly attainable. It is more interesting to investigate if d is even. The next conjecture addresses this more interesting case and generalizes Theorem 3.3 to other even values of d.

Conjecture 4.3. *Let b be odd such that $b \ne 2^r + 1$ for any positive integer r, let d be even, and let p be the smallest prime such that $p \mid (b-1)$ and $p \nmid d$. Then there exist infinitely many b-anti-Niven d-APs of length $p-1$.*

To partially support Conjecture 4.3, we have verified computationally that for $b \in \{7, 11, 13, 15, 19, 21, 23, 25, 27, 29\}$ and even integers $d \le 100$ such that d is not a multiple of the square-free kernel of $b-1$, there exists at least one b-anti-Niven d-AP of length $p-1$, where p is the smallest prime p satisfying $p \mid (b-1)$ and $p \nmid d$.

Theorem 3.4 established an upper bound for the maximum length of a b-anti-Niven d-AP when $b \ge 6$ is even and $3 \le d \le b/2$ is an odd integer. We conjecture that this bound is attainable for infinitely many pairs (b, d).

Conjecture 4.4. *There exist infinitely many pairs (b, d), where $b \ge 6$ is even and $3 \le d \le b/2$ is an odd integer, for which there is a b-anti-Niven d-AP of length $\lceil 2b/d \rceil + 2$.*

b	d	First term of a b-anti-Niven d-AP of length $\lceil 2b/d \rceil + 2$	b	d	First term of a b-anti-Niven d-AP of length $\lceil 2b/d \rceil + 2$
10	3	1190	56	15	3073
12	3	2005	58	15	3293
14	3	3513	60	5	3537
18	3	6463	62	21	3763
20	5	8779	66	15	4283
22	9	457	68	9	4549
24	5	549	70	9	4825
28	9	3892	72	9	5107
30	5	867	74	15	5389
32	9	3031	78	21	5993
34	15	1126	80	9	6313
36	15	1247	82	15	6631
38	15	1393	84	15	6959
40	15	1549	88	15	7643
42	7	1717	90	15	8017
44	15	1879	92	21	8380
48	7	2251	94	21	8746
50	21	2435	96	25	9099
52	9	2653	98	15	9499
54	15	2849	100	27	9883

TABLE 1. Computational data supporting Conjecture 4.4

5. Acknowledgements

These results are based on work supported by the National Science Foundation under grant numbered MPS-2150299.

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