INSTABILITY OF SOLUTIONS TO DOUBLE PHASE PROBLEMS INVOLVING THE GRUSHIN OPERATOR AND EXPONENTIAL NONLINEARITY

QUANG THANH KHUAT, DAO TRONG QUYET, AND NGUYEN VAN BIET

ABSTRACT. In this paper, we are concerned with the following problem in the whole space $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$-\mathrm{div}_G(w_1(z)|\nabla_G u|^{p-2}\nabla_G u + w_2(z)|\nabla_G u|^{q-2}\nabla_G u) = f(z)e^u,$$

where ∇_G is the Grushin gradient, $\Delta_G = \mathrm{div}_G \circ \nabla_G$ is the Grushin operator, $q \geq p \geq 2$ and $w_1, w_2, f \in L^1_{loc}(\mathbb{R}^N)$ are three nonnegative functions satisfying some growth conditions at infinity. Using energy methods and nonlinear integral estimates, we obtain the instability of weak solutions of the above problem.

1. Introduction and Main Results

In this paper, we split $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and write $z = (x, y) \in \mathbb{R}^N$, where $x \in \mathbb{R}^{N_1}, y \in \mathbb{R}^{N_2}$. Let α be a nonnegative constant and define the Grushin gradient as follows

$$\nabla_G = (\nabla_x, |x|^\alpha \nabla_y),$$

where ∇_x, ∇_y are standard Euclidean gradients in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$ respectively. The aim of this paper is to study the instability of weak solutions for the following problem in the whole space $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$-\mathrm{div}_G(w_1(z)|\nabla_G u|^{p-2}\nabla_G u + w_2(z)|\nabla_G u|^{q-2}\nabla_G u) = f(z)e^u, \tag{1}$$

where $q \geq p \geq 2$ and $w_1, w_2, f \in L^1_{loc}(\mathbb{R}^N)$ are three nonnegative functions satisfying the following condition: there is $R_0, C_1, C_2, C_3 > 0$ and $\delta_1, \delta_2, \theta \in \mathbb{R}$ such that

$$\begin{aligned} w_1(z) &\leq C_1|z|_G^{\delta_1}, \\ w_2(z) &\leq C_2|z|_G^{\delta_2}, \\ f(z) &\geq C_3|z|_G^{\theta}, \end{aligned} \tag{2}$$

for all $|z|_G > R_0$. Here,

$$|z|_G = \left(|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2\right)^{\frac{1}{2(\alpha+1)}},$$

where $|x|, |y|$ are the Euclidean norms in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$ respectively.

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Recall that in the case $\alpha = 0, w_1 \geq 0, w_2 = 0, p = 2$, (1) becomes the Laplace-type equation. In [13, 17], the authors proved the instability of solutions for the equation

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^N$$

within the condition of $N \leq 9$. For the Hénon equation

$$-\Delta u = |x|^\theta e^u \quad \text{in } \mathbb{R}^N,$$

where $\theta > -2$, Wang and Ye [33] obtained the nonexistence result of stable solutions in dimension $N < 10 + 4\theta$.

Next, in the case $\alpha = 0, w_1 \geq 0, w_2 = 0, p \geq 2$, (1) becomes the p-Laplace equation. The results about the instability of solutions, the reader can be found in [26] for the equation

$$-\Delta_p u = f(z)e^u \quad \text{in } \mathbb{R}^N$$

and in [7, 25, 29] for the more general nonlinear p-Laplace equations.

Considering (1) in the case $\alpha = 0, w_1 \geq 0, w_2 \geq 0, q \geq p \geq 2$, the left-hand side becomes the double phase operator. In particular, when the weight function $w_1 = 1$, following the ideas in the papers [12, 17], Phuong Le [28] proved that the equation

$$-\text{div}(|\nabla u|^{p-2}\nabla u + w_2(z)|\nabla u|^{q-2}\nabla u) = f(z)e^u \text{ in } \mathbb{R}^N$$

has no stable solution under the condition

$$N < \frac{\min\{p, q - \delta_2\}(q+3) + 4\theta}{q-1}.$$

We now consider the general case $\alpha \geq 0, w_1 \geq 0, w_2 \geq 0, q \geq p \geq 2$. Recently, there have been many studies on elliptic equations involving the operator $\Delta_G = \text{div}_G \circ \nabla_G = \Delta_x + |x|^{2\alpha}\Delta_y$, see e.g. [15, 16, 27, 34, 36]. This operator is today usually named Grushin operator. The operators of this kind were first introduced and studied by Franchi and Lanconelli [18]. Recently, they were named by Kogoj and Lanconelli [21] Δ_λ -Laplacians, under the additional assumption that the operators are homogeneous of degree two with respect to a group of dilations, see also [2, 14, 22–24, 31, 34, 35]. The operator considered by Grushin [19] is a very particular case of the Δ_λ-Laplacians, Grushin studied this operator by adding lower terms with complex coefficients, see also [4].

For the equation

$$-\Delta_G u = e^u \text{ in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

Anh et al. [1] proved that this equation does not permit any stable solution if $2 < Q < 10$, where

$$Q = N_1 + (\alpha+1)N_2 \tag{3}$$

is called the homogeneous dimension of $\mathbb{R}^N$ associated to the Grushin operator.

The nonexistence result of stable solutions to the equations involving p-Laplace-type Grushin operator

$$\text{div}_G(w(z)|\nabla_G u|^{p-2}\nabla_G u) = f(z)e^u \text{ in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

was established by Wei et al. in [34].

Notice that double phase differential operators and their respective energy functionals are used for research in nonlinear elasticity theory, strongly anisotropic materials, Lavrentiev's phenomenon, and so on, see e.g. [3, 6, 8, 20, 37–39]. Some results for nonlinear problems controlled by the double-phase operator, the reader can be found in [5, 9–11, 30, 32]. However, to our knowledge, there has not been any

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research on the double phase problem (1) involving the Grushin operator so far. The purpose of this paper is to establish some results about the instability of weak solutions for problem (1).

We now recall some notations which will be used in the sequel. Let $\Omega \subset \mathbb{R}^N$ be an open domain and let $H : \Omega \times [0,\infty) \to [0,\infty)$ be the function $(z,t) \mapsto w_1(z)t^p + w_2(z)t^q$. Put

$$\rho_H(u) = \int_\Omega H(z,|u|) = \int_\Omega (w_1(z)|u|^p + w_2(z)|u|^q)$$

and

$$L^H(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \rho_H(u) < \infty\}.$$

The space $L^H(\Omega)$ is equipped the norm

$$\|u\|_H = \inf\left\{\tau > 0 \mid \rho_H\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Define

$$W^{1,H}(\Omega) = \{u \in L^H(\Omega) \mid |\nabla_G u| \in L^H(\Omega)\},$$

with the norm

$$\|u\|_{1,H} = \||\nabla_G u|\|_H + \|u\|_H.$$

The closure of $C_c^1(\Omega)$ with respect to the $\|\cdot\|_{1,H}$ norm is denoted by $W_0^{1,H}(\Omega)$ and we set

$$W_{loc}^{1,H}(\Omega) = \{u : \Omega \to \mathbb{R} | u\varphi \in W_0^{1,H}(\Omega) \text{ for all } \varphi \in C_c^1(\Omega)\}.$$

In this paper, we understand the solutions of (1) in the sense of weak solutions as follows.

Definition 1.1. *A function $u \in W_{loc}^{1,H}(\mathbb{R}^N)$ is said to be a weak solution of (1) if $f(z)e^u \in L_{loc}^1(\mathbb{R}^N)$ and for all $\varphi \in C_c^1(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N} \left(w_1(z)|\nabla_G u|^{p-2}\nabla_G u + w_2(z)|\nabla_G u|^{q-2}\nabla_G u\right) \cdot \nabla_G \varphi = \int_{\mathbb{R}^N} f(z)e^u\varphi. \tag{4}$$

Our results concern with the stability of solutions which are defined as follows.

Definition 1.2. *Let u be a weak solution of (1). Then, u is stable if for all $\varphi \in C_c^1(\mathbb{R}^N)$, we have*

$$\begin{aligned}\int_{\mathbb{R}^N} & w_1(z)\left[|\nabla_G u|^{p-2}|\nabla_G \varphi|^2 + (p-2)|\nabla_G u|^{p-4}(\nabla_G u \cdot \nabla_G \varphi)^2\right] \\ &+ \int_{\mathbb{R}^N} w_2(z)\left[|\nabla_G u|^{q-2}|\nabla_G \varphi|^2 + (q-2)|\nabla_G u|^{q-4}(\nabla_G u \cdot \nabla_G \varphi)^2\right] \\ \ge & \int_{\mathbb{R}^N} f(z)e^u\varphi^2.\end{aligned} \tag{5}$$

Our first result establishes the dimensionality condition that the weak solution of (1) is unstable.

Theorem 1.3. *Let u be a weak solution of (1). Assume that*

$$Q < \frac{\min\{p-\delta_1, q-\delta_2\}(q+3) + 4\theta}{q-1}, \tag{6}$$

where the homogeneous dimension Q is given in (3). Then, u is unstable.

Notice that, if the solutions are bounded from below, we have the following result.

Theorem 1.4. *Let u be a weak solution of (1). In addition, u is bounded from below and*

$$\min\{p-\delta_1+\theta, q-\delta_2+\theta\} > 0, \tag{7}$$

then u is unstable.

The next result is established for the solutions which have a suitable decay of the gradient at infinity.

Theorem 1.5. *Let u be a weak solution of (1). If $|\nabla_G u| = O(|z|_G^\beta)$ as $|z|_G \to \infty$ and*

$$\max\{\delta_1+(p-2)\beta, \delta_2+(q-2)\beta\} < \frac{4\theta - Q(q-1)}{q+3} + 2, \tag{8}$$

then u is unstable. Here the homogeneous dimension Q is given in (3).

Remark that, our results generalize that in [28] from the Laplace operator to the Grushin operator. More precisely, when $w_1 = 1$ and $\alpha = 0$, our results recover that in [28]. Moreover, our results are also extensions of that in [34] to the double phase problems. Inspired by [15, 28, 34], our approach in this paper is also based on the energy method and nonlinear integral estimates.

The rest of this paper is organized as follows. In Sect. 2, we prove some a priori estimates for stable weak solutions of (1), which are used in proving of the main results afterward. In Sect. 3, we prove the main results of this paper.

2. Some a priori estimates for stable weak solutions

Throughout the sequel, we use letter C to denote a generic positive constant which may change from line to line or in the same line. For any $R > 0$, we denote by $\Omega_R = B_1(0,R) \times B_2(0, R^{\alpha+1})$, where $B_1(0,R) \subset \mathbb{R}^{N_1}, B_2(0,R^{\alpha+1}) \subset \mathbb{R}^{N_2}$ are the Euclidean balls.

To prove Theorems 1.3 and 1.4, we need the following a priori estimate.

Proposition 2.1. *Let u be a stable weak solution of (1) and $\gamma \in \left(0, \frac{4}{q-1}\right)$. Then, for all $\eta \in C_c^1(\mathbb{R}^N; [0,1])$ and $\nabla_G \eta = 0$ in Ω_{R_0}, there is a positive constant C depending on p, q, γ such that*

$$\begin{aligned}\int_{\mathbb{R}^N} f(z) e^{(\gamma+1)u} \eta^{q(\gamma+1)} \le & C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_1(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta|^{p(\gamma+1)} \\ & + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_2(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta|^{q(\gamma+1)}.\end{aligned}$$

Proof. Let $k \in \mathbb{N}$, and define

$$g_k(t) = \begin{cases} e^{\frac{\gamma t}{2}}, & t < k, \\ \left[\frac{\gamma}{2}(t-k)+1\right] e^{\frac{\gamma k}{2}}, & t \ge k, \end{cases}$$

and

$$h_k(t) = \begin{cases} e^{\gamma t}, & t < k, \\ [\gamma(t-k)+1] e^{\gamma k}, & t \ge k. \end{cases}$$

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By direct calculations, we have

$$g_k'(t) = \begin{cases} \dfrac{\gamma}{2} e^{\frac{\gamma t}{2}}, & t < k, \\ \dfrac{\gamma}{2} e^{\frac{\gamma k}{2}}, & t \geq k, \end{cases} \quad \text{and} \quad h_k'(t) = \begin{cases} \gamma e^{\gamma t}, & t < k, \\ \gamma e^{\gamma k}, & t \geq k. \end{cases}$$

It follows that

$$\begin{aligned} & g_k(t)^2 \geq h_k(t), \quad g_k'(t)^2 = \frac{\gamma}{4} h_k'(t), \\ & g_k(t)^\sigma g_k'(t)^{2-\sigma} + h_k(t)^\sigma h_k'(t)^{1-\sigma} \leq C_{\gamma,\sigma} e^{\gamma t}, \end{aligned} \tag{9}$$

for all $t \in \mathbb{R}$ and $\sigma \geq 2$.

Let $r \geq q, \epsilon \in (0,1)$ and $\psi \in C_c^1(\mathbb{R}^N)$ satisfying $0 \leq \psi \leq 1$. By density arguments, we observe that (4) is true for all $\varphi \in W_0^{1,H}(\mathbb{R}^N)$. Moreover, if $u \in W_{loc}^{1,H}(\mathbb{R}^N)$ then $h_k(u) \in W_{loc}^{1,H}(\mathbb{R}^N)$. This implies that $h_k(u)\psi^r \in W_0^{1,H}(\mathbb{R}^N)$. We take $\varphi = h_k(u)\psi^r$ in (4) to get that

$$\begin{aligned} & \int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^p h_k'(u)\psi^r + r\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2} h_k(u)\psi^{r-1}\nabla_G u \cdot \nabla_G \psi \\ & \quad + \int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^q h_k'(u)\psi^r + r\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2} h_k(u)\psi^{r-1}\nabla_G u \cdot \nabla_G \psi \\ & = \int_{\mathbb{R}^N} f(z) e^u h_k(u)\psi^r. \end{aligned} \tag{10}$$

Applying Young's inequality, we obtain

$$\begin{aligned} & -r\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2} h_k(u)\psi^{r-1}\nabla_G u \cdot \nabla_G \psi \\ & \quad \leq r\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-1} h_k(u)\psi^{r-1}|\nabla_G \psi| \\ & \quad \leq \int_{\mathbb{R}^N} \Bigg[\epsilon\Big(w_1(z)^{\frac{p-1}{p}}|\nabla_G u|^{p-1} h_k'(u)^{\frac{p-1}{p}}\psi^{\frac{(p-1)r}{p}}\Big)^{\frac{p}{p-1}} \\ & \qquad\qquad + C_\epsilon\Big(w_1(z)^{\frac{1}{p}} h_k(u) h_k'(u)^{\frac{1-p}{p}}\psi^{\frac{r-p}{p}}|\nabla_G \psi|\Big)^p\Bigg] \\ & \quad = \epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^p h_k'(u)\psi^r + C_\epsilon\int_{\mathbb{R}^N} w_1(z) h_k(u)^p h_k'(u)^{1-p}\psi^{r-p}|\nabla_G \psi|^p. \end{aligned}$$

In the same way, we have

$$\begin{aligned} & -r\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2} h_k(u)\psi^{r-1}\nabla_G u \cdot \nabla_G \psi \\ & \quad \leq \epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^q h_k'(u)\psi^r + C_\epsilon\int_{\mathbb{R}^N} w_2(z) h_k(u)^q h_k'(u)^{1-q}\psi^{r-q}|\nabla_G \psi|^q. \end{aligned}$$

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Substituting these two estimates into (10) and using (9), we get that

$$
\begin{aligned}
(1-\epsilon)&\int_{\mathbb{R}^N}\big(w_1(z)|\nabla_G u|^p+w_2(z)|\nabla_G u|^q\big)h_k'(u)\psi^r\\
&\le \int_{\mathbb{R}^N} f(z)e^u h_k(u)\psi^r + C_\epsilon\int_{\mathbb{R}^N} w_1(z)h_k(u)^p h_k'(u)^{1-p}\psi^{r-p}|\nabla_G\psi|^p\\
&\quad + C_\epsilon\int_{\mathbb{R}^N} w_2(z)h_k(u)^q h_k'(u)^{1-q}\psi^{r-q}|\nabla_G\psi|^q\\
&\le \int_{\mathbb{R}^N} f(z)e^u h_k(u)\psi^r + C_\epsilon\int_{\mathbb{R}^N} w_1(z)e^{\gamma u}\psi^{r-p}|\nabla_G\psi|^p\\
&\quad + C_\epsilon\int_{\mathbb{R}^N} w_2(z)e^{\gamma u}\psi^{r-q}|\nabla_G\psi|^q.
\end{aligned}
\tag{11}
$$

Applying Schwart's inequality for the stability condition (5), we receive

$$
\begin{aligned}
(p-1)&\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}|\nabla_G\varphi|^2+(q-1)\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}|\nabla_G\varphi|^2\\
&\ge\int_{\mathbb{R}^N} f(z)e^u\varphi^2
\end{aligned}
\tag{12}
$$

for all $\varphi\in C_c^1(\mathbb{R}^N)$.

By density arguments, we have (12) hold true for all $\varphi\in W_0^{1,H}(\mathbb{R}^N)$. Moreover, if $u\in W^{1,H}_{loc}(\mathbb{R}^N)$ then $g_k(u)\in W^{1,H}_{loc}(\mathbb{R}^N)$. This implies that $g_k(u)\psi^{\frac{r}{2}}\in W_0^{1,H}(\mathbb{R}^N)$. Taking $\varphi=g_k(u)\psi^{\frac{r}{2}}$ in (12), we have

$$
\begin{aligned}
(p-1)&\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}\Big|g_k'(u)\psi^{\frac{r}{2}}\nabla_G u+\frac{r}{2}g_k(u)\psi^{\frac{r-2}{2}}\nabla_G\psi\Big|^2\\
&+(q-1)\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}\Big|g_k'(u)\psi^{\frac{r}{2}}\nabla_G u+\frac{r}{2}g_k(u)\psi^{\frac{r-2}{2}}\nabla_G\psi\Big|^2\\
&\ge\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r.
\end{aligned}
$$

Using the inequality

$$
|z_1+z_2|^2\le(1+\tau)|z_1|^2+C_\tau|z_2|^2\quad\text{for } z_1,z_2\in\mathbb{R}^N,\tau>0,
$$

we arrive at

$$
\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r \le&\Big(p-1+\frac{\epsilon}{2}\Big)\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^p g_k'(u)^2\psi^r\\
&+A_\epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}g_k(u)^2\psi^{r-2}|\nabla_G\psi|^2\\
&+\Big(q-1+\frac{\epsilon}{2}\Big)\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^q g_k'(u)^2\psi^r\\
&+B_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}g_k(u)^2\psi^{r-2}|\nabla_G\psi|^2.
\end{aligned}
\tag{13}
$$

Employing Young's inequality, we obtain

$$
\begin{aligned}
A_\epsilon&\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}g_k(u)^2\psi^{r-2}|\nabla_G\psi|^2\\
&\le\int_{\mathbb{R}^N}\Big[\frac{\epsilon}{2}\Big(w_1(z)^{\frac{p-2}{p}}|\nabla_G u|^{p-2}g_k'(u)^{\frac{2(p-2)}{p}}\psi^{\frac{(p-2)r}{p}}\Big)^{\frac{p}{p-2}}
\end{aligned}
$$

$$+ C_\epsilon\Big(w_1(z)^{\frac{2}{p}} g_k(u)^2 g_k'(u)^{\frac{2(2-p)}{p}} \psi^{\frac{2(r-p)}{p}} |\nabla_G\psi|^2\Big)^{\frac{p}{2}}\Big]$$

$$= \frac{\epsilon}{2}\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^p g_k'(u)^2\psi^r + C_\epsilon\int_{\mathbb{R}^N} w_1(z)g_k(u)^p g_k'(u)^{2-p}\psi^{r-p}|\nabla_G\psi|^p.$$

By the same argument, we have

$$\begin{aligned}
&B_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}g_k(u)^2\psi^{r-2}|\nabla_G\psi|^2\\
&\quad\le \frac{\epsilon}{2}\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^q g_k'(u)^2\psi^r + C_\epsilon\int_{\mathbb{R}^N} w_2(z)g_k(u)^q g_k'(u)^{2-q}\psi^{r-q}|\nabla_G\psi|^q.
\end{aligned}$$

Putting these two estimates back into (13) gives

$$\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r \le&(p-1+\epsilon)\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^p g_k'(u)^2\psi^r\\
&+ C_\epsilon\int_{\mathbb{R}^N} w_1(z)g_k(u)^p g_k'(u)^{2-p}\psi^{r-p}|\nabla_G\psi|^p\\
&+ (q-1+\epsilon)\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^q g_k'(u)^2\psi^r\\
&+ C_\epsilon\int_{\mathbb{R}^N} w_2(z)g_k(u)^q g_k'(u)^{2-q}\psi^{r-q}|\nabla_G\psi|^q.
\end{aligned}$$

Under the condition that $q \ge p$ and (9), we receive

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r\\
&\quad\le (q-1+\epsilon)\int_{\mathbb{R}^N}(w_1(z)|\nabla_G u|^p + w_2(z)|\nabla_G u|^q)g_k'(u)^2\psi^r\\
&\qquad+ C_\epsilon\int_{\mathbb{R}^N} w_1(z)e^{\gamma u}\psi^{r-p}|\nabla_G\psi|^p + C_\epsilon\int_{\mathbb{R}^N} w_2(z)e^{\gamma u}\psi^{r-q}|\nabla_G\psi|^q \qquad (14)\\
&\quad= \frac{(q-1+\epsilon)\gamma}{4}\int_{\mathbb{R}^N}(w_1(z)|\nabla_G u|^p + w_2(z)|\nabla_G u|^q)h_k'(u)\psi^r\\
&\qquad+ C_\epsilon\int_{\mathbb{R}^N} w_1(z)e^{\gamma u}\psi^{r-p}|\nabla_G\psi|^p + C_\epsilon\int_{\mathbb{R}^N} w_2(z)e^{\gamma u}\psi^{r-q}|\nabla_G\psi|^q.
\end{aligned}$$

Combining (11) and (14), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r \le \frac{(q-1+\epsilon)\gamma}{4(1-\epsilon)}&\int_{\mathbb{R}^N} f(z)e^u h_k(u)\psi^r\\
&+ C_\epsilon\int_{\mathbb{R}^N} w_1(z)e^{\gamma u}\psi^{r-p}|\nabla_G\psi|^p\\
&+ C_\epsilon\int_{\mathbb{R}^N} w_2(z)e^{\gamma u}\psi^{r-q}|\nabla_G\psi|^q.
\end{aligned}$$

From (9), we derive

$$\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r \le \frac{(q-1+\epsilon)\gamma}{4(1-\epsilon)}&\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^r\\
&+ C_\epsilon\int_{\mathbb{R}^N} w_1(z)e^{\gamma u}\psi^{r-p}|\nabla_G\psi|^p
\end{aligned}$$

$$+ C_\epsilon \int_{\mathbb{R}^N} w_2(z) e^{\gamma u} \psi^{r-q} |\nabla_G \psi|^q.$$

Therefore,

$$\begin{aligned} &D_\epsilon \int_{\mathbb{R}^N} f(z) e^u g_k(u)^2 \psi^r \\ &\leq C_\epsilon \int_{\mathbb{R}^N} w_1(z) e^{\gamma u} \psi^{r-p} |\nabla_G \psi|^p + C_\epsilon \int_{\mathbb{R}^N} w_2(z) e^{\gamma u} \psi^{r-q} |\nabla_G \psi|^q, \end{aligned}$$

where

$$D_\epsilon := 1 - \frac{(q-1+\epsilon)\gamma}{4(1-\epsilon)}.$$

Since $\lim_{\epsilon \to 0^+} D_\epsilon = 1 - \frac{(q-1)\gamma}{4} > 0$, then we may choose ϵ sufficiently close to zero such that $D_\epsilon > 0$. We also choose $r = q$. It follows that

$$\int_{\mathbb{R}^N} f(z) e^u g_k(u)^2 \psi^q \leq C \int_{\mathbb{R}^N} w_1(z) e^{\gamma u} \psi^{q-p} |\nabla_G \psi|^p + C \int_{\mathbb{R}^N} w_2(z) e^{\gamma u} |\nabla_G \psi|^q.$$

Applying Fatou's lemma when letting $k \to \infty$, we get that

$$\int_{\mathbb{R}^N} f(z) e^{(\gamma+1)u} \psi^q \leq C \int_{\mathbb{R}^N} w_1(z) e^{\gamma u} \psi^{q-p} |\nabla_G \psi|^p + C \int_{\mathbb{R}^N} w_2(z) e^{\gamma u} |\nabla_G \psi|^q. \quad (15)$$

Next, take $\psi = \eta^{\gamma+1}$ in (15) and applying the Young inequality to find that

$$\begin{aligned} &\int_{\mathbb{R}^N} f(z) e^{(\gamma+1)u} \eta^{q(\gamma+1)} \\ &\leq C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_1(z) e^{\gamma u} |\nabla_G \eta|^p \eta^{q\gamma+q-p} + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_2(z) e^{\gamma u} |\nabla_G \eta|^q \eta^{q\gamma} \\ &\leq \int_{\mathbb{R}^N \setminus \Omega_{R_0}} \left\{ \frac{1}{4} \Big(f(z)^{\frac{\gamma}{\gamma+1}} e^{\gamma u} \eta^{q\gamma+q-p} \Big)^{\frac{\gamma+1}{\gamma}} + C \Big(w_1(z) f(z)^{-\frac{\gamma}{\gamma+1}} |\nabla_G \eta|^p \Big)^{\gamma+1} \right\} \\ &\quad + \int_{\mathbb{R}^N \setminus \Omega_{R_0}} \left\{ \frac{1}{4} \Big(f(z)^{\frac{\gamma}{\gamma+1}} e^{\gamma u} \eta^{q\gamma} \Big)^{\frac{\gamma+1}{\gamma}} + C \Big(w_2(z) f(z)^{-\frac{\gamma}{\gamma+1}} |\nabla_G \eta|^q \Big)^{\gamma+1} \right\} \\ &\leq \frac{1}{4} \int_{\mathbb{R}^N \setminus \Omega_{R_0}} f(z) e^{(\gamma+1)u} \eta^{q(\gamma+1)} + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_1(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta|^{p(\gamma+1)} \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^N \setminus \Omega_{R_0}} f(z) e^{(\gamma+1)u} \eta^{q(\gamma+1)} + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_2(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta|^{q(\gamma+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^N} f(z) e^{(\gamma+1)u} \eta^{q(\gamma+1)} \leq & C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_1(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta|^{p(\gamma+1)} \\ & + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_2(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta|^{q(\gamma+1)}. \end{aligned}$$

The proof is finished. $\square$

The following result, which is a variation of Proposition 2.1, will be used to prove Theorem 1.5.

Proposition 2.2. *Let u be a stable weak solution of (1) and $\gamma \in \left(0, \frac{4}{q-1}\right)$. Then, for all $\eta \in C_c^1(\mathbb{R}^N;[0,1])$ and $\nabla_G \eta = 0$ in Ω_{R_0}, there is a positive constant C depending on p, q, γ such that*

$$\begin{aligned}\int_{\mathbb{R}^N} f(z) e^{(\gamma+1)u} \eta^{2(\gamma+1)} \leq C &\int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_1(z)^{\gamma+1} f(z)^{-\gamma} (|\nabla_G u|^{p-2} |\nabla_G \eta|^2)^{\gamma+1} \\ &+ C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w_2(z)^{\gamma+1} f(z)^{-\gamma} (|\nabla_G u|^{q-2} |\nabla_G \eta|^2)^{\gamma+1}.\end{aligned}$$

Proof. The following proof is similar to that of Proposition 2.1. We use the functions g_k, h_k, ψ as in the proof of Proposition 2.1.

Let $\epsilon \in (0,1)$. Using $\varphi = h_k(u)\psi^2 \in W_0^{1,H}(\mathbb{R}^N)$ as a test function in (4) gives us

$$\begin{aligned}&\int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^p h_k'(u) \psi^2 + 2 \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^{p-2} h_k(u) \psi \nabla_G u \cdot \nabla_G \psi \\ &\quad + \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^q h_k'(u) \psi^2 + 2 \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^{q-2} h_k(u) \psi \nabla_G u \cdot \nabla_G \psi \\ &= \int_{\mathbb{R}^N} f(z) e^u h_k(u) \psi^2.\end{aligned} \tag{16}$$

Applying Young's inequality, we obtain

$$\begin{aligned}&- 2 \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^{p-2} h_k(u) \psi \nabla_G u \cdot \nabla_G \psi \\ &\leq 2 \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^{p-1} h_k(u) \psi |\nabla_G \psi| \\ &\leq \int_{\mathbb{R}^N} \epsilon \Big(w_1(z)^{\frac{1}{2}} |\nabla_G u|^{\frac{p}{2}} h_k'(u)^{\frac{1}{2}} \psi \Big)^2 + C_\epsilon \Big(w_1(z)^{\frac{1}{2}} |\nabla_G u|^{\frac{p-2}{2}} h_k(u) h_k'(u)^{-\frac{1}{2}} |\nabla_G \psi| \Big)^2 \\ &= \epsilon \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^p h_k'(u) \psi^2 + C_\epsilon \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^{p-2} h_k(u)^2 h_k'(u)^{-1} |\nabla_G \psi|^2.\end{aligned}$$

In the same way, we have

$$\begin{aligned}&-2 \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^{q-2} h_k(u) \psi \nabla_G u \cdot \nabla_G \psi \\ &\quad \leq \epsilon \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^q h_k'(u) \psi^2 + C_\epsilon \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^{q-2} h_k(u)^2 h_k'(u)^{-1} |\nabla_G \psi|^2.\end{aligned}$$

Substituting these two estimates into (16) and using (9), we get that

$$\begin{aligned}&(1-\epsilon) \int_{\mathbb{R}^N} \big(w_1(z) |\nabla_G u|^p + w_2(z) |\nabla_G u|^q \big) h_k'(u) \psi^2 \\ &\quad \leq \int_{\mathbb{R}^N} f(z) e^u h_k(u) \psi^2 + C_\epsilon \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^{p-2} h_k(u)^2 h_k'(u)^{-1} |\nabla_G \psi|^2 \\ &\quad\quad + C_\epsilon \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^{q-2} h_k(u)^2 h_k'(u)^{-1} |\nabla_G \psi|^2 \\ &\quad \leq \int_{\mathbb{R}^N} f(z) e^u h_k(u) \psi^2 + C_\epsilon \int_{\mathbb{R}^N} w_1(z) |\nabla_G u|^{p-2} e^{\gamma u} |\nabla_G \psi|^2 \\ &\quad\quad + C_\epsilon \int_{\mathbb{R}^N} w_2(z) |\nabla_G u|^{q-2} e^{\gamma u} |\nabla_G \psi|^2.\end{aligned} \tag{17}$$

Next, taking $\varphi = g_k(u)\psi \in W_0^{1,H}(\mathbb{R}^N)$ in (12), we obtain

$$
\begin{aligned}
&(p-1)\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}\left|g_k'(u)\psi\nabla_G u + g_k(u)\nabla_G\psi\right|^2 \\
&\quad + (q-1)\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}\left|g_k'(u)\psi\nabla_G u + g_k(u)\nabla_G\psi\right|^2 \\
&\geq \int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2.
\end{aligned}
$$

Using the inequality

$$
|z_1+z_2|^2 \leq (1+\tau)|z_1|^2 + C_\tau|z_2|^2 \quad \text{for } z_1, z_2 \in \mathbb{R}^N, \tau > 0,
$$

we arrive at

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2 \\
&\leq (p-1+\epsilon)\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^p g_k'(u)^2\psi^2 + C_\epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}g_k(u)^2|\nabla_G\psi|^2 \\
&\quad + (q-1+\epsilon)\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^q g_k'(u)^2\psi^2 + C_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}g_k(u)^2|\nabla_G\psi|^2.
\end{aligned}
$$

Under the condition that $q \geq p$ and (9), we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2 \\
&\leq (q-1+\epsilon)\int_{\mathbb{R}^N} (w_1(z)|\nabla_G u|^p + w_2(z)|\nabla_G u|^q)g_k'(u)^2\psi^2 \\
&\quad + C_\epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G\psi|^2 + C_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G\psi|^2 \qquad (18)\\
&= \frac{(q-1+\epsilon)\gamma}{4}\int_{\mathbb{R}^N} (w_1(z)|\nabla_G u|^p + w_2(z)|\nabla_G u|^q)h_k'(u)\psi^2 \\
&\quad + C_\epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G\psi|^2 + C_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G\psi|^2.
\end{aligned}
$$

Combining (17) and (18), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2 &\leq \frac{(q-1+\epsilon)\gamma}{4(1-\epsilon)}\int_{\mathbb{R}^N} f(z)e^u h_k(u)\psi^2 \\
&\quad + C_\epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G\psi|^2 + C_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G\psi|^2.
\end{aligned}
$$

From (9), we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2 &\leq \frac{(q-1+\epsilon)\gamma}{4(1-\epsilon)}\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2 \\
&\quad + C_\epsilon\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G\psi|^2 + C_\epsilon\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G\psi|^2.
\end{aligned}
$$

Therefore,

$$
D_\epsilon\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2
$$

$$\leq C_\epsilon \int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G \psi|^2 + C_\epsilon \int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G \psi|^2,$$

where

$$D_\epsilon := 1 - \frac{(q-1+\epsilon)\gamma}{4(1-\epsilon)}.$$

Since $\lim_{\epsilon\to 0^+} D_\epsilon = 1 - \frac{(q-1)\gamma}{4} > 0$, then we can fix some ϵ sufficiently close to zero such that $D_\epsilon > 0$. Hence

$$\begin{aligned}&\int_{\mathbb{R}^N} f(z)e^u g_k(u)^2\psi^2 \\ &\quad\leq C\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G \psi|^2 + C\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G \psi|^2.\end{aligned}$$

Applying Fatou's lemma when letting $k\to\infty$, we get that

$$\begin{aligned}&\int_{\mathbb{R}^N} f(z)e^{(\gamma+1)u}\psi^2 \\ &\quad\leq C\int_{\mathbb{R}^N} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G \psi|^2 + C\int_{\mathbb{R}^N} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G \psi|^2.\end{aligned} \tag{19}$$

Next, take $\psi = \eta^{\gamma+1}$ in (19), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f(z)e^{(\gamma+1)u}\eta^{2(\gamma+1)} \leq\, & C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_1(z)|\nabla_G u|^{p-2}e^{\gamma u}|\nabla_G \eta|^2\eta^{2\gamma} \\ &+ C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_2(z)|\nabla_G u|^{q-2}e^{\gamma u}|\nabla_G \eta|^2\eta^{2\gamma}.\end{aligned}$$

Applying the Young inequality to find that

$$\begin{aligned}&\int_{\mathbb{R}^N} f(z)e^{(\gamma+1)u}\eta^{2(\gamma+1)} \\ &\leq \int_{\mathbb{R}^N\setminus\Omega_{R_0}} \frac{1}{4}\Big(f(z)^{\frac{\gamma}{\gamma+1}}e^{\gamma u}\eta^{2\gamma}\Big)^{\frac{\gamma+1}{\gamma}} + C\Big(w_1(z)f(z)^{-\frac{\gamma}{\gamma+1}}|\nabla_G u|^{p-2}|\nabla_G\eta|^2\Big)^{\gamma+1} \\ &\quad+ \int_{\mathbb{R}^N\setminus\Omega_{R_0}} \frac{1}{4}\Big(f(z)^{\frac{\gamma}{\gamma+1}}e^{\gamma u}\eta^{2\gamma}\Big)^{\frac{\gamma+1}{\gamma}} + C\Big(w_2(z)f(z)^{-\frac{\gamma}{\gamma+1}}|\nabla_G u|^{q-2}|\nabla_G\eta|^2\Big)^{\gamma+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} f(z)e^{(\gamma+1)u}\eta^{2(\gamma+1)} \leq & C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_1(z)^{\gamma+1}f(z)^{-\gamma}(|\nabla_G u|^{p-2}|\nabla_G\eta|^2)^{\gamma+1} \\ &+ C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_2(z)^{\gamma+1}f(z)^{-\gamma}(|\nabla_G u|^{q-2}|\nabla_G\eta|^2)^{\gamma+1}.\end{aligned}$$

The proof is finished. □

3. Proof of main results

Let $R > 0$. In the sequel, we denote by $\Omega_{2R} = B_1(0,2R)\times B_2(0,2R^{\alpha+1})$, where $B_1(0,2R)\subset\mathbb{R}^{N_1}$, $B_2(0,2R^{\alpha+1})\subset\mathbb{R}^{N_2}$ are the Euclidean balls, and consider functions

$$\eta_{1,R}(x) = \eta_1\left(\frac{|x|}{R}\right), \eta_{2,R}(y) = \eta_2\left(\frac{|y|}{R^{\alpha+1}}\right),$$

with $\eta_1, \eta_2 \in C_c^\infty([0,+\infty)), 0 \le \eta_1, \eta_2 \le 1$,

$$\eta_i(t) = \begin{cases} 1 & \text{in} \quad [0,1], \\ 0 & \text{in} \quad [2,+\infty), \end{cases}$$

and

$$|\nabla_x \eta_{1,R}| \le \frac{C}{R}, |\nabla_y \eta_{2,R}| \le \frac{C}{R^{\alpha+1}},$$

for some constant $C > 0$.

The following lemma is necessary to prove main results.

Lemma 3.1. *The following assertions are true.*

(i) *There is a constant $C > 0$ independent of R such that*

$$|\nabla_G \eta_R| \le \frac{C}{R}, \forall z \in \Omega_{2R},$$

where $\eta_R = \eta_{1,R}\eta_{2,R}$.

(ii) *There is a constant $C > 0$ independent of R such that if $z \in \Omega_{2R}$ then*

$$|z|_G \le CR.$$

(iii) *If $z \notin \Omega_R$ then $|z|_G > R$.*

Proof. Proof of (i). We have

$$\nabla_G \eta_R = (\nabla_x \eta_R, |x|^\alpha \nabla_y \eta_R) = (\eta_{2,R}\nabla_x \eta_{1,R}, |x|^\alpha \eta_{1,R}\nabla_y \eta_{2,R}).$$

For any $z = (x,y) \in \Omega_{2R}$, we have $x \in B_1(0,2R)$. Hence

$$|x| \le 2R.$$

Combining this with the hypothesis about functions $\eta_{i,R}, i = 1,2$, there is a constant $C > 0$ independent of R such that

$$|\nabla_G \eta_R|^2 = \eta_{2,R}^2 |\nabla_x \eta_{1,R}|^2 + |x|^{2\alpha}\eta_{1,R}^2 |\nabla_y \eta_{2,R}|^2 \le \frac{C}{R^2}, \forall z \in \Omega_{2R}.$$

This implies

$$|\nabla_G \eta_R| \le \frac{C}{R}, \forall z \in \Omega_{2R}.$$

Proof of (ii). For any $z = (x,y) \in \Omega_{2R}$, we have

$$|x| \le 2R \text{ and } |y| \le 2R^{\alpha+1}.$$

Hence

$$\begin{aligned} |z|_G &= \left(|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2\right)^{\frac{1}{2(\alpha+1)}} \\ &\le \left[(2R)^{2(\alpha+1)} + (\alpha+1)^2(2R^{\alpha+1})^2\right]^{\frac{1}{2(\alpha+1)}}. \end{aligned}$$

From direct calculation we obtain

$$|z|_G \le CR,$$

where $C > 0$ is independent of R.

Proof of (iii). For any $z = (x,y) \notin \Omega_R$, we have

$$|x| > R \text{ and } |y| > R^{\alpha+1}.$$

Therefore, we get

$$|z|_G = \left(|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2\right)^{\frac{1}{2(\alpha+1)}}$$

$$> \left[R^{2(\alpha+1)} + (\alpha+1)^2 (R^{\alpha+1})^2\right]^{\frac{1}{2(\alpha+1)}}.$$

Simple calculation we have $|z|_G > R$. □

Note that the function η_R is defined as in Lemma 3.1 satisfying $\eta_R \in C_c^1(\mathbb{R}^N), 0 \leq \eta_R \leq 1$ in $\mathbb{R}^N$ and

$$\begin{cases} \eta_R = 1 & \text{in } \Omega_R, \\ \eta_R = 0 & \text{in } \mathbb{R}^N \setminus \Omega_{2R}, \\ |\nabla_G \eta_R| \leq \dfrac{C}{R} & \text{in } \Omega_{2R} \setminus \Omega_R. \end{cases} \tag{20}$$

Now, we use the contrary method to prove our main results. We assume that the weak solution u of (1) is stable. Then, applying the a priori estimates for stable weak solutions in Section 2 to derive a contradiction. Hence, we have the conclusion of theorems.

3.1. **Proof Theorem 1.3.**

Proof. Suppose conversely that u is stable. For all $R > R_0$, applying Proposition 2.1 with $\gamma \in \left(0, \frac{4}{q-1}\right)$ and $\eta = \eta_R$, where η_R is chosen as in Lemma 3.1, there is a positive constant C independent of R such that

$$\int_{\Omega_R} f(z) e^{(\gamma+1)u} \leq C \int_{\Omega_{2R} \setminus \Omega_R} w_1(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta_R|^{p(\gamma+1)} + C \int_{\Omega_{2R} \setminus \Omega_R} w_2(z)^{\gamma+1} f(z)^{-\gamma} |\nabla_G \eta_R|^{q(\gamma+1)}.$$

Since $|z|_G > R > R_0, \forall z \notin \Omega_R$ and using (2), (20) we obtain

$$\int_{\Omega_R} f(z) e^{(\gamma+1)u} \leq C \int_{\Omega_{2R} \setminus \Omega_R} |z|_G^{\delta_1(\gamma+1)} |z|_G^{-\theta\gamma} R^{-p(\gamma+1)} + C \int_{\Omega_{2R} \setminus \Omega_R} |z|_G^{\delta_2(\gamma+1)} |z|_G^{-\theta\gamma} R^{-q(\gamma+1)}.$$

Combining this with $|z|_G > R, \forall z \notin \Omega_R$ and $|z|_G \leq CR, \forall z \in \Omega_{2R}$, we have

$$\int_{\Omega_R} f(z) e^{(\gamma+1)u} \leq C R^{\lambda}, \tag{21}$$

where

$$\lambda = Q - \min\{\theta\gamma + (p - \delta_1)(\gamma+1), \theta\gamma + (q - \delta_2)(\gamma+1)\}.$$

Using (6) yields

$$\lim_{\gamma \to \left(\frac{4}{q-1}\right)^-} \lambda = Q - \frac{\min\{p - \delta_1, q - \delta_2\}(q+3) + 4\theta}{q-1} < 0.$$

So we can fix some γ sufficiently close to $\frac{4}{q-1}$ such that $\lambda < 0$. Letting $R \to \infty$ in (21), we have a contradiction. Therefore, we get the conclusion of the theorem. □

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3.2. **Proof Theorem 1.4.**

Proof. Suppose conversely that u is stable. Since u is bounded from below then there is $M < 0$ such that $u(z) > M$ for a.e. $z \in \mathbb{R}^N$. It follows that

$$M < (1-\epsilon)M < (1-\epsilon)u,$$

for all $\epsilon \in (0,1)$. Hence

$$e^M \int_{\mathbb{R}^N} f(z)e^{(\gamma+\epsilon)u}\psi^q \le \int_{\mathbb{R}^N} e^{(1-\epsilon)u} f(z)e^{(\gamma+\epsilon)u}\psi^q = \int_{\mathbb{R}^N} f(z)e^{(\gamma+1)u}\psi^q.$$

Combine this with the inequality (15) in the proof of Proposition 2.1, we obtain

$$\int_{\mathbb{R}^N} f(z)e^{(\gamma+\epsilon)u}\psi^q \le C\int_{\mathbb{R}^N} w_1(z)e^{\gamma u}\psi^{q-p}|\nabla_G\psi|^p + C\int_{\mathbb{R}^N} w_2(z)e^{\gamma u}|\nabla_G\psi|^q, \tag{22}$$

where $C > 0$ independent of ϵ. Taking $\psi = \eta^{\frac{\gamma+\epsilon}{\epsilon}}$ in (22) and employing Young's inequality, we receive

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^{(\gamma+\epsilon)u}\eta^{\frac{q(\gamma+\epsilon)}{\epsilon}} \\
&\le C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_1(z)e^{\gamma u}|\nabla_G\eta|^p\eta^{\frac{q\gamma}{\epsilon}+q-p} + C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_2(z)e^{\gamma u}|\nabla_G\eta|^q\eta^{\frac{q\gamma}{\epsilon}} \\
&\le \int_{\mathbb{R}^N\setminus\Omega_{R_0}} \left\{\frac{1}{4}\Big(f(z)^{\frac{\gamma}{\gamma+\epsilon}}e^{\gamma u}\eta^{\frac{q\gamma}{\epsilon}+q-p}\Big)^{\frac{\gamma+\epsilon}{\gamma}} + C\Big(w_1(z)f(z)^{-\frac{\gamma}{\gamma+\epsilon}}|\nabla_G\eta|^p\Big)^{\frac{\gamma+\epsilon}{\epsilon}}\right\} \\
&\quad + \int_{\mathbb{R}^N\setminus\Omega_{R_0}} \left\{\frac{1}{4}\Big(f(z)^{\frac{\gamma}{\gamma+\epsilon}}e^{\gamma u}\eta^{\frac{q\gamma}{\epsilon}}\Big)^{\frac{\gamma+\epsilon}{\gamma}} + C\Big(w_2(z)f(z)^{-\frac{\gamma}{\gamma+\epsilon}}|\nabla_G\eta|^q\Big)^{\frac{\gamma+\epsilon}{\epsilon}}\right\} \\
&\le \frac{1}{4}\int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)e^{(\gamma+\epsilon)u}\eta^{\frac{q(\gamma+\epsilon)}{\epsilon}} + C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_1(z)^{\frac{\gamma+\epsilon}{\epsilon}}f(z)^{-\frac{\gamma}{\epsilon}}|\nabla_G\eta|^{\frac{p(\gamma+\epsilon)}{\epsilon}} \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)e^{(\gamma+\epsilon)u}\eta^{\frac{q(\gamma+\epsilon)}{\epsilon}} + C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_2(z)^{\frac{\gamma+\epsilon}{\epsilon}}f(z)^{-\frac{\gamma}{\epsilon}}|\nabla_G\eta|^{\frac{q(\gamma+\epsilon)}{\epsilon}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} f(z)e^{(\gamma+\epsilon)u}\eta^{\frac{q(\gamma+\epsilon)}{\epsilon}} \le C&\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_1(z)^{\frac{\gamma+\epsilon}{\epsilon}}f(z)^{-\frac{\gamma}{\epsilon}}|\nabla_G\eta|^{\frac{p(\gamma+\epsilon)}{\epsilon}} \\
&+ C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w_2(z)^{\frac{\gamma+\epsilon}{\epsilon}}f(z)^{-\frac{\gamma}{\epsilon}}|\nabla_G\eta|^{\frac{q(\gamma+\epsilon)}{\epsilon}}.
\end{aligned} \tag{23}$$

For all $R > R_0$, applying (23) with $\eta = \eta_R$, where η_R is chosen as in Lemma 3.1, we have

$$\begin{aligned}
\int_{\Omega_R} f(z)e^{(\gamma+\epsilon)u} \le C&\int_{\Omega_{2R}\setminus\Omega_R} w_1(z)^{\frac{\gamma+\epsilon}{\epsilon}}f(z)^{-\frac{\gamma}{\epsilon}}|\nabla_G\eta_R|^{\frac{p(\gamma+\epsilon)}{\epsilon}} \\
&+ C\int_{\Omega_{2R}\setminus\Omega_R} w_2(z)^{\frac{\gamma+\epsilon}{\epsilon}}f(z)^{-\frac{\gamma}{\epsilon}}|\nabla_G\eta_R|^{\frac{q(\gamma+\epsilon)}{\epsilon}}.
\end{aligned}$$

By the same argument in the proof of Theorem 1.3, we obtain

$$\int_{\Omega_R} f(z)e^{(\gamma+\epsilon)u} \le CR^\lambda, \tag{24}$$

where

$$\lambda = Q - \min\left\{\frac{\theta\gamma + (p-\delta_1)(\gamma+\epsilon)}{\epsilon}, \frac{\theta\gamma + (q-\delta_2)(\gamma+\epsilon)}{\epsilon}\right\}$$
$$= Q - \frac{\min\{p-\delta_1+\theta, q-\delta_2+\theta\}\gamma}{\epsilon} - \min\{p-\delta_1, q-\delta_2\}.$$

By the assumption (7), we can fix some ϵ sufficiently close to 0 such that $\lambda < 0$. Letting $R \to \infty$ in (24), we have a contradiction. Therefore, we get the conclusion of the theorem. $\square$

3.3. **Proof of Theorem 1.5.**

Proof. Suppose conversely that u is stable. Applying Proposition 2.2 with $\gamma \in \left(0, \frac{4}{q-1}\right)$ and $\eta = \eta_R$, where η_R is chosen as in Lemma 3.1. Then, for all $R > R_0$, there is a positive constant C independent of R such that

$$\int_{\Omega_R} f(z)e^{(\gamma+1)u} \leq C\int_{\Omega_{2R}\setminus\Omega_R} w_1(z)^{\gamma+1}f(z)^{-\gamma}(|\nabla_G u|^{p-2}|\nabla_G\eta_R|^2)^{\gamma+1}$$
$$+ C\int_{\Omega_{2R}\setminus\Omega_R} w_2(z)^{\gamma+1}f(z)^{-\gamma}(|\nabla_G u|^{q-2}|\nabla_G\eta_R|^2)^{\gamma+1}.$$

Since $|z|_G > R > R_0, \forall z \notin \Omega_R$, using (2),(20) and the hypothesis about $|\nabla_G u|$ we obtain

$$\int_{\Omega_R} f(z)e^{(\gamma+1)u} \leq C\int_{\Omega_{2R}\setminus\Omega_R} |z|_G^{\delta_1(\gamma+1)}|z|_G^{-\theta\gamma}|z|_G^{(p-2)\beta(\gamma+1)}R^{-2(\gamma+1)}$$
$$+ C\int_{\Omega_{2R}\setminus\Omega_R} |z|_G^{\delta_2(\gamma+1)}|z|_G^{-\theta\gamma}|z|_G^{(q-2)\beta(\gamma+1)}R^{-2(\gamma+1)}.$$

Combining this with $|z|_G > R, \forall z \notin \Omega_R$ and $|z|_G \leq CR, \forall z \in \Omega_{2R}$, we have

$$\int_{\Omega_R} f(z)e^{(\gamma+1)u} \leq CR^\lambda, \tag{25}$$

where

$$\lambda = Q + \max\{\delta_1 + (p-2)\beta, \delta_2 + (q-2)\beta\}(\gamma+1) - 2(\gamma+1) - \theta\gamma.$$

Using the assumption (8), we get that

$$\lim_{\gamma\to\left(\frac{4}{q-1}\right)^-} \lambda = Q + \max\{\delta_1 + (p-2)\beta, \delta_2 + (q-2)\beta\} \cdot \tfrac{q+3}{q-1} - \tfrac{2(q+3)}{q-1} - \tfrac{4\theta}{q-1} < 0.$$

So we can fix some γ sufficiently close to $\frac{4}{q-1}$ such that $\lambda < 0$. Letting $R \to \infty$ in (25), we have a contradiction. Therefore, we get the conclusion of the theorem. $\square$

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Quang Thanh Khuat, Academy of Finance, Ha Noi, Viet Nam
Email address: khuatquangthanh@hvtc.edu.vn

Dao Trong Quyet, Academy of Finance, Ha Noi, Viet Nam
Email address: daotrongquyet@hvtc.edu.vn

Nguyen Van BIET, Department of Education and Training of Phu Tho Province, Viet Tri, Phu Tho, Viet Nam
Email address: nguyenvanbiet@moet.edu.vn

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