FURTHER OBSERVATIONS ON CERTAIN $\mathrm{U}_{\mathrm{fin}}$-TYPE SELECTION PRINCIPLES

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ABSTRACT. We study certain weaker variants of $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$, namely $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$. We explore many topological properties of $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$. Certain situations are considered when these weaker variants are equivalent to certain related properties. We also make investigations on these variants using critical cardinalities and Alexandroff duplicate. Few observations on productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ spaces are presented. Besides, we present certain characterizations of $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ using weakly groupable covers. We also obtain many game theoretic observations in this context. Some open problems are given.

Key words and phrases: $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$, productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$, productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$.

1. Introduction

The study of selection principles turns out to be an emerging field in topology and allied areas. A lot of research has been carried out investigating weaker variants of the selection principles. We suggest the readers to consult the papers [7,8,11–16, 18,21] and references therein for recent explorations in this direction. Throughout the paper X stands for a topological space. Let $\mathcal{A}$ and $\mathcal{B}$ be collections consisting of families of subsets of X. Following [10,19], we define

$\mathrm{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(\mathcal{U}_n)$ of elements of $\mathcal{A}$ there exists a sequence (V_n) such that for each n $V_n \in \mathcal{U}_n$ and $\{V_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$\mathrm{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: For each sequence $(\mathcal{U}_n)$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{B}$.

$\mathrm{U}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: For each sequence $(\mathcal{U}_n)$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\cup\mathcal{V}_n : n \in \mathbb{N}\} \in \mathcal{B}$ or there is some n such that $\cup\mathcal{V}_n = X$.

For $\Pi \in \{\mathrm{S}_1, \mathrm{S}_{\mathrm{fin}}, \mathrm{U}_{\mathrm{fin}}\}$, we say that X is a $\Pi(\mathcal{A},\mathcal{B})$ space if X satisfies the selection principle $\Pi(\mathcal{A},\mathcal{B})$. This convention will be used subsequently.

The game $\mathrm{G}_1(\mathcal{A},\mathcal{B})$ on X corresponding to the selection principle $\mathrm{S}_1(\mathcal{A},\mathcal{B})$ is played as follows. Players ONE and TWO play an inning for each positive integer n. In the nth inning ONE chooses a $\mathcal{U}_n \in \mathcal{A}$ and TWO responds by selecting a $U_n \in \mathcal{U}_n$. TWO wins the play $\mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \dots, \mathcal{U}_n, U_n, \dots$ of this game if $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise ONE wins. The game $\mathrm{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ (respectively, $\mathrm{G}_{\mathrm{ufin}}(\mathcal{A},\mathcal{B})$) on X corresponding to the selection principle $\mathrm{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$) can be similarly defined.

It is easy to see that if ONE does not have a winning strategy in the game $\mathrm{G}_1(\mathcal{A},\mathcal{B})$ (respectively, $\mathrm{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, $\mathrm{G}_{\mathrm{ufin}}(\mathcal{A},\mathcal{B})$) on X, then X satisfies $\mathrm{S}_1(\mathcal{A},\mathcal{B})$ (respectively, $\mathrm{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, $\mathrm{U}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$). For $\Sigma \in \{\mathrm{G}_1, \mathrm{G}_{\mathrm{fin}}, \mathrm{G}_{\mathrm{ufin}}\}$, observe that winning strategies for ONE (respectively, TWO) in $\Sigma(\mathcal{A},\mathcal{C})$ (respectively, $\Sigma(\mathcal{A},\mathcal{B})$) implies winning strategies for ONE (respectively, TWO) in $\Sigma(\mathcal{A},\mathcal{B})$ (respectively, $\Sigma(\mathcal{A},\mathcal{C})$) if $\mathcal{B} \subseteq \mathcal{C}$.

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Let $\mathcal{O}$ denote the collection of all open covers of X and Ω denote the collection of all ω-covers of X (recall that an open cover $\mathcal{U}$ of X is said to be an ω-cover [10,19] if X not in $\mathcal{U}$ and for each finite subset F of X there is a set $U \in \mathcal{U}$ such that $F \subseteq U$). Let $\mathcal{O}_D$ denote the family of all sets $\mathcal{U}$ of open subsets of X such that $\cup\mathcal{U}$ is dense in X. Consider the family Ω_D of all sets $\mathcal{U} \in \mathcal{O}_D$ such that for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a $U \in \mathcal{U}$ such that $U \cap V \neq \emptyset$ for all $V \in \mathcal{F}$ and the family Ω^D of all sets $\mathcal{U} \in \Omega_D$ such that for each $\mathcal{U}$ there exists a dense set $Y \subseteq X$ such that each finite set $F \subseteq Y$ is contained in U for some $U \in \mathcal{U}$.

In this article we consider certain weaker variants of the selection principle $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$, namely $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ and their corresponding games $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega_D)$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$. We characterize the selection principles $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ using neighbourhood assignment as well as weakly groupable covers. We observe that if every finite power of a space X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$, then X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. We discuss relation of the selection principles $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ with similar other selection principles. We explore investigations using critical cardinalities, Alexandroff duplicate, mappings and products. Certain investigations on productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ properties are also carried out. We obtain few interesting game theoretic observations in this direction. In particular we observe that the games $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega_D)$, $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$ are equivalent under certain topological assumption. We leave some problems as open.

2. Definitions and terminologies

For undefined notions and terminologies, see [9]. A subset A of a space X is said to be regular-closed (respectively, regular-open) if $\overline{\mathrm{Int}(A)} = A$ (respectively, $\mathrm{Int}(\overline{A}) = A$). An open cover $\mathcal{U}$ of X is said to be a γ-cover [10,19] if $\mathcal{U}$ is infinite and for each $x \in X$, the set $\{U \in \mathcal{U} : x \notin U\}$ is finite. An open cover $\mathcal{U}$ of X is said to be a large cover [10,19] if for each $x \in X$, the set $\{U \in \mathcal{U} : x \in U\}$ is infinite. We use the symbol Γ and Λ to denote the collection of all γ-covers and large covers of X respectively. Note that $\Gamma \subseteq \Omega \subseteq \Lambda \subseteq \mathcal{O}$. An open cover $\mathcal{U}$ of X is said to be weakly groupable [1] if X can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite set $F \subseteq X$ we have $F \subseteq \cup\mathcal{U}_n$ for some n. The collection of all weakly groupable covers of X is denoted by $\mathcal{O}^{wgp}$.

The following families of open sets will be used in our investigation.

- $\overline{\mathcal{O}}$: The family of all sets $\mathcal{U}$ of open subsets of X such that $\{\overline{U} : U \in \mathcal{U}\}$ covers X.
- $\overline{\Omega}$: The family of all sets $\mathcal{U} \in \overline{\mathcal{O}}$ such that each finite set $F \subseteq X$ is contained in $\overline{U}$ for some $U \in \mathcal{U}$.
- $\overline{\Gamma}$: The family of all sets $\mathcal{U} \in \overline{\mathcal{O}}$ such that for each $x \in X$, the set $\{U \in \mathcal{U} : x \notin \overline{U}\}$ is finite.
- $\overline{\Lambda}$: The family of all sets $\mathcal{U} \in \overline{\mathcal{O}}$ such that for each $x \in X$, the set $\{U \in \mathcal{U} : x \in \overline{U}\}$ is infinite.
- $\overline{\mathcal{O}^{wgp}}$: The family of all sets $\mathcal{U} \in \overline{\mathcal{O}}$ such that $\mathcal{U}$ can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite set $F \subseteq X$ there exists a n such that $F \subseteq \overline{\cup\mathcal{U}_n}$.
- $\overline{\Lambda^{wgp}}$: The family of all sets $\mathcal{U} \in \overline{\Lambda}$ such that $\mathcal{U} \in \overline{\mathcal{O}^{wgp}}$.
- Γ_D: The family of all sets $\mathcal{U} \in \mathcal{O}_D$ such that for each nonempty open set $U \subseteq X$, the set $\{V \in \mathcal{U} : U \cap V = \emptyset\}$ is finite.
- Λ_D: The family of all sets $\mathcal{U} \in \mathcal{O}_D$ such that for each nonempty open set $U \subseteq X$, the set $\{V \in \mathcal{U} : U \cap V \neq \emptyset\}$ is infinite.

$\mathcal{O}^{wgp}{}_D$: The family of all sets $\mathcal{U} \in \mathcal{O}_D$ such that $\mathcal{U}$ can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a n such that $V \cap (\cup \mathcal{U}_n) \neq \emptyset$ for all $V \in \mathcal{F}$.

$\Lambda^{wgp}{}_D$: The family of all sets $\mathcal{U} \in \Lambda_D$ such that $\mathcal{U} \in \mathcal{O}^{wgp}{}_D$.

Γ^D: The family of all sets $\mathcal{U} \in \Gamma_D$ such that for each $\mathcal{U}$ there exists a dense set $Y \subseteq X$ such that for each $x \in Y$, the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

Λ^D: The family of all sets $\mathcal{U} \in \Lambda_D$ such that for each $\mathcal{U}$ there exists a dense set $Y \subseteq X$ such that for each $x \in Y$, the set $\{U \in \mathcal{U} : x \in U\}$ is infinite.

$\mathcal{O}^{wgpD}$: The family of all sets $\mathcal{U} \in \mathcal{O}^{wgp}{}_D$ such that for each $\mathcal{U}$ there exists a dense set $Y \subseteq X$ and $\mathcal{U}$ can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each finite set $F \subseteq Y$ is contained in $\cup \mathcal{U}_n$ for some n.

Λ^{wgpD}: The family of all sets $\mathcal{U} \in \Lambda^D$ such that $\mathcal{U} \in \mathcal{O}^{wgpD}$.

All the families defined above are assumed to be infinite. Observe that

(1) $\overline{\Gamma} \subseteq \overline{\Omega} \subseteq \overline{\Lambda} \subseteq \overline{\mathcal{O}}$
(2) $\Gamma_D \subseteq \Omega_D \subseteq \Lambda_D \subseteq \mathcal{O}_D$
(3) $\Gamma^D \subseteq \Omega^D \subseteq \Lambda^D \subseteq \mathcal{O}_D$
(4) $\overline{\mathcal{O}} \subseteq \mathcal{O}_D$
(5) $\overline{\Gamma} \subseteq \Gamma_D$ and $\Gamma^D \subseteq \Gamma_D$
(6) $\overline{\Omega} \subseteq \Omega_D$ and $\Omega^D \subseteq \Omega_D$
(7) $\overline{\Lambda} \subseteq \Lambda_D$ and $\Lambda^D \subseteq \Lambda_D$.

Also note that every countable member of $\overline{\Omega}$ (respectively, Ω_D, Ω^D) is a member of $\overline{\mathcal{O}^{wgp}}$ (respectively, $\mathcal{O}^{wgp}{}_D$, $\mathcal{O}^{wgpD}$). A space X is said to be almost Lindelöf (respectively, weakly Lindelöf) if for every open cover $\mathcal{U}$ of X there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V} \in \overline{\mathcal{O}}$ (respectively, $\mathcal{V} \in \mathcal{O}_D$) (see [11, 22]).

The following terminologies will also be used throughout our study.

$\mathcal{G}$: The family of all covers $\mathcal{U}$ of the space X for which each element of $\mathcal{U}$ is a G_δ set.

$\mathcal{G}_K$: The family of all sets $\mathcal{U}$ where X is not in $\mathcal{U}$, each element of $\mathcal{U}$ is a G_δ set, and for each compact set $C \subseteq X$ there is a $U \in \mathcal{U}$ such that $C \subseteq U$.

$\mathcal{G}_\Omega$: The family of all covers $\mathcal{U} \in \mathcal{G}$ such that for each finite set $F \subseteq X$ there is a $U \in \mathcal{U}$ such that $F \subseteq U$.

$\mathcal{G}_\Gamma$: The family of all covers $\mathcal{U} \in \mathcal{G}$ which are infinite and each infinite subset of $\mathcal{U}$ is a cover of X.

$\mathcal{G}_{\overline{\Gamma}}$: The family of all covers $\mathcal{U} \in \mathcal{G}$ which are infinite and for each $x \in X$, the set $\{U \in \mathcal{U} : x \notin \overline{U}\}$ is finite.

$\mathcal{G}_D$: The family of all sets $\mathcal{U}$ where each element of $\mathcal{U}$ is a G_δ set and $\cup \mathcal{U}$ is dense in X.

$\mathcal{G}_{D_\Gamma}$: The family of all sets $\mathcal{U}$ where each element of $\mathcal{U}$ is a G_δ set and for each nonempty open set $U \subseteq X$, the set $\{V \in \mathcal{U} : U \cap V = \emptyset\}$ is finite.

$\mathcal{G}_{\Gamma_D}$: The family of all sets $\mathcal{U}$ where each element of $\mathcal{U}$ is a G_δ set and for each $\mathcal{U}$ there exists a dense set $Y \subseteq X$ such that for each $x \in Y$, the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

$\overline{\mathcal{G}}$: The family of all sets $\mathcal{U}$ such that every $U \in \mathcal{U}$ is a G_δ set and $\{\overline{U} : U \in \mathcal{U}\}$ covers X.

$\overline{\mathcal{G}_\Omega}$: The family of all sets $\mathcal{U} \in \overline{\mathcal{G}}$ such that for each finite set $F \subseteq X$ there exists a $U \in \mathcal{U}$ such that $F \subseteq U$.

Consider the Baire space $\mathbb{N}^\mathbb{N}$. A natural pre-order $\leq^*$ on $\mathbb{N}^\mathbb{N}$ is defined by $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many n. A subset D of $\mathbb{N}^\mathbb{N}$ is said to be dominating if for each $g \in \mathbb{N}^\mathbb{N}$ there exists a $f \in D$ such that $g \leq^* f$. A subset A of $\mathbb{N}^\mathbb{N}$ is said to be bounded if there is a $g \in \mathbb{N}^\mathbb{N}$ such that $f \leq^* g$ for all $f \in A$. Let

$\mathfrak{d}$ be the minimum cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$ and $\mathfrak{b}$ be the minimum cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$.

The Alexandroff duplicate [4, 9] $AD(X)$ of a space X is defined as follows. $AD(X) = X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighbourhood of $(x,0) \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{(x,1)\})$, where U is a neighbourhood of x in X. For a space X, $\mathsf{PR}(X)$ denotes the space of all nonempty finite subsets of X with the Pixley-Roy topology [7]. The collection $\{[A,U] : A \in \mathsf{PR}(X),\ U \text{ open in } X\}$ is a base for the Pixley-Roy topology, where $[A,U] = \{B \in \mathsf{PR}(X) : A \subseteq B \subseteq U\}$ for each $A \in \mathsf{PR}(X)$ and each open set U in X. A space X is said to be cosmic if it has a countable network. A space X satisfies the countable chain condition (in short, CCC) if every family of disjoint nonempty open subsets of X is countable. A space satisfying the CCC is called a CCC space. For a space X, $e(X) = \sup\{|Y| : Y \text{ is a closed and discrete subspace of } X\}$ is said to be the extent of X. For any two spaces X and Y, $X \oplus Y$ denote the topological sum of X and Y. For any families $\mathcal{U}$ and $\mathcal{V}$ of subsets of X we denote the set $\{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$ by $\mathcal{U} \wedge \mathcal{V}$.

3. The selection principles $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$

3.1. **Interrelationships**

The proof of the following result is straightforward.

Lemma 3.1.

(1) *A space X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ if and only if for every sequence $(\mathcal{U}_n)$ of open covers of X there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite collection $\mathcal{F}$ of nonempty open subsets of X, the set $\{n \in \mathbb{N} : U \cap (\cup \mathcal{V}_n) \neq \emptyset \text{ for all } U \in \mathcal{F}\}$ is infinite.*

(2) *A space X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ if and only if for every sequence $(\mathcal{U}_n)$ of open covers of X there exist a dense subset Y of X and a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each finite set $F \subseteq Y$ is contained in $\cup \mathcal{V}_n$ for infinitely many n.*

$$\begin{tikzcd}
\text{Lindelöf} \arrow[r] & \text{almost Lindelöf} \arrow[rr] & & \text{weakly Lindelöf} \\
\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \arrow[u] \arrow[r] & \mathrm{S}_{\mathrm{fin}}(\mathcal{O},\overline{\mathcal{O}}) \arrow[u] \arrow[rr] & & \mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D) \arrow[u] \\
 & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D) \arrow[dr] & \\
\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega) \arrow[uu] \arrow[r] \arrow[urr] & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega}) \arrow[uu] \arrow[rr] & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D) \arrow[uu] \\
 & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma^D) \arrow[uu] \arrow[dr] & \\
\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) \arrow[uu] \arrow[r] \arrow[urr] & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma}) \arrow[uu] \arrow[rr] & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma_D) \arrow[uu]
\end{tikzcd}$$

Figure 1. Weaker variants of $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$ and $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$

For a Tychonoff space X let βX denote the Stone-Čech compactification of X.

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Example 3.1. There exists a space which satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ (and hence satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$) but does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$ (i.e. does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$).
Let D be the discrete space of cardinality ω_1. Consider $X=(\beta D\times(\omega+1))\setminus((\beta D\setminus D)\times\{\omega\})$ as a subspace of $\beta D\times(\omega+1)$. Since $\beta D\times\omega$ is a σ-compact dense subset of X, X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ (see Section 3.2 below). Now X is not almost Lindelöf (see [22, Example 2.3]) and so X does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$.

It is well known that every CCC space is weakly Lindelöf [17, 22]. In the next example we show that a CCC space may not be $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$.

Example 3.2. There is a CCC space which does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$.
Let $Z=\mathsf{PR}(\mathbb{P})$, where $\mathbb{P}$ is the space of irrationals. Clearly $\mathbb{P}$ is a cosmic space which does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. It has been observed that $\mathsf{PR}(X)$ is CCC for every regular cosmic space X [17] and also if $\mathsf{PR}(X)$ satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$, then each finite power of X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ [7, Theorem 2A]. Thus Z is a CCC space which does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$. The last shows that Z does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$.

Proposition 3.1.

(1) *A space X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ if and only if for every sequence $(\mathcal{U}_n)$ of covers of X by regular-open sets there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite collection $\mathcal{F}$ of nonempty regular-open sets there is a n such that $U\cap(\cup\mathcal{V}_n)\neq\emptyset$ for all $U\in\mathcal{F}$.*

(2) *Suppose that X is regular. Then X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ if and only if for every sequence $(\mathcal{U}_n)$ of covers of X by regular-open sets there exist a dense subset Y of X and a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite set $F\subseteq Y$ there is a n such that $F\subseteq\cup\mathcal{V}_n$.*

Proof. We only give proof of (1), and (2) can be similarly verified.

(1). It is enough to prove the reverse implication. From the given sequence $(\mathcal{U}_n)$ of open covers of X, we first construct for each n a cover $\mathcal{W}_n=\{\mathrm{Int}(\overline{U}):U\in\mathcal{U}_n\}$ of X by regular-open sets. Now choose a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$ and for each finite collection $\mathcal{F}$ of nonempty regular-open sets there is a $n\in\mathbb{N}$ such that $U\cap(\cup\mathcal{H}_n)\neq\emptyset$ for all $U\in\mathcal{F}$. For each n and each $V\in\mathcal{H}_n$ we can choose a $U_V\in\mathcal{U}_n$ such that $\mathrm{Int}(\overline{U_V})=V$. Also for each n $\mathcal{V}_n=\{U_V:V\in\mathcal{H}_n\}$ is a finite subset of $\mathcal{U}_n$. Let $\mathcal{F}$ be a finite collection of nonempty open sets. Now first choose $\mathcal{F}'=\{\mathrm{Int}(\overline{U}):U\in\mathcal{F}\}$ and then choose a $m\in\mathbb{N}$ such that $V\cap(\cup\mathcal{H}_m)\neq\emptyset$ for all $V\in\mathcal{F}'$. Consequently $U\cap(\cup\mathcal{V}_m)\neq\emptyset$ for all $U\in\mathcal{F}$. Thus X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$. □

Recall that a neighbourhood assignment for a space (X,τ) is a function $\mathcal{N}:X\to\tau$ such that $x\in\mathcal{N}(x)$ for each $x\in X$.

Theorem 3.1.

(1) *A space X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ if and only if for each sequence $(\mathcal{N}_n)$ of neighbourhood assignments of X there exists a sequence (F_n) of finite subsets of X such that $\{\mathcal{N}(F_n):n\in\mathbb{N}\}\in\Omega_D$.*

(2) *A space X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ if and only if for each sequence $(\mathcal{N}_n)$ of neighbourhood assignments of X there exists a sequence (F_n) of finite subsets of X such that $\{\mathcal{N}(F_n):n\in\mathbb{N}\}\in\Omega^D$.*

Proof. Since the proof of (1) and (2) are similar, we only present the proof of (1). Let $(\mathcal{N}_n)$ be a sequence of neighbourhood assignments of X. For each n we define $\mathcal{U}_n=\{\mathcal{N}_n(x):x\in X\}$. Then $(\mathcal{U}_n)$ is a sequence of open covers of X and since X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$, there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\cup\mathcal{V}_n:n\in\mathbb{N}\}\in\Omega_D$. This gives us a sequence (F_n) of finite subsets of X such that $\mathcal{V}_n = \{\mathcal{N}_n(x) : x \in F_n\}$ for each n. It follows that $\{\mathcal{N}_n(F_n) : n \in \mathbb{N}\} \in \Omega_D$.

Conversely let $(\mathcal{U}_n)$ be a sequence of open covers of X. For each $x \in X$ there exists a $U_x^{(n)} \in \mathcal{U}_n$ such that $x \in U_x^{(n)}$. Let $(\mathcal{N}_n)$ be a sequence of neighbourhood assignments of X defined by $\mathcal{N}_n(x) = U_x^{(n)}$ for each n and each $x \in X$. By the given hypothesis, we get a sequence (F_n) of finite subsets of X such that $\{\mathcal{N}_n(F_n) : n \in \mathbb{N}\} \in \Omega_D$. For each n choose $\mathcal{V}_n = \{\mathcal{N}_n(x) : x \in F_n\}$. Then the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. □

The following result is from [5], we sketch the proof for convenience of the reader.

Theorem 3.2 ([5]). *Every paracompact* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$ *space is* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$.

Proof. Let X be a paracompact $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$ space. Let $(\mathcal{U}_n)$ be a sequence of open covers of X. Choose a sequence $(\mathcal{W}_n)$ such that for each n $\mathcal{W}_n$ is a locally finite open refinement of $\mathcal{U}_n$. Now fix n. For each $x \in X$ there is an open set U_x containing x such that $U_x \subseteq V$ for some $V \in \mathcal{W}_n$ and $\{W \in \mathcal{W}_n : W \cap U_x \neq \emptyset\}$ is finite. Observe that $\mathcal{H}_n = \{U_x : x \in X\}$ is an open refinement of $\mathcal{W}_n$. Applying the hypothesis to the sequence $(\mathcal{H}_n)$ of open covers we obtain a sequence $(\mathcal{K}_n)$ such that for each n $\mathcal{K}_n$ is a finite subset of $\mathcal{H}_n$ and for each finite set $F \subseteq X$ there is a n such that $F \subseteq \overline{\cup \mathcal{K}_n}$. Clearly for each n the collection $\mathcal{F}_n = \{V \in \mathcal{W}_n : V \cap U \neq \emptyset \text{ for some } U \in \mathcal{K}_n\}$ is finite. For each $V \in \mathcal{F}_n$, choose a $U_V \in \mathcal{U}_n$ such that $V \subseteq U_V$ and define $\mathcal{V}_n = \{U_V : V \in \mathcal{F}_n\}$. Now the property $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$ is witnessed by the sequence $(\mathcal{V}_n)$. □

An open cover $\mathcal{U}$ of a space X is called star finite if every $U \in \mathcal{U}$ intersects only finitely many $V \in \mathcal{U}$. A space X is called hypocompact if every open cover $\mathcal{U}$ of X has a star finite open refinement. Since every hypocompact space is paracompact, we obtain the following.

Corollary 3.1. *Every hypocompact* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$ *space is* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$.

But the above results (Theorem 3.2 and Corollary 3.1) do not hold in the context of $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. The Baire space is a typical counter-example to it (see Remark 3.1).

Theorem 3.3. *If every finite power of a space X satisfies* $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$, *then X satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

Proof. Let $(\mathcal{U}_n)$ be a sequence of open covers of X. Let $\{N_k : k \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite subsets. For each k and each $m \in N_k$ let $\mathcal{W}_m = \{U_1 \times U_2 \times \cdots \times U_k : U_1, U_2, \ldots, U_k \in \mathcal{U}_m\}$. Clearly $(\mathcal{W}_m : m \in N_k)$ is a sequence of open covers of X^k. Apply the $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ property of X^k to $(\mathcal{W}_m : m \in N_k)$ to obtain a sequence $(\mathcal{H}_m : m \in N_k)$ of finite sets such that $\mathcal{H}_m \subseteq \mathcal{W}_m$ for each $m \in N_k$ and $\cup\{\cup \mathcal{H}_m : m \in N_k\}$ is dense in X^k. For each $m \in N_k$ we can express every $H \in \mathcal{H}_m$ as $H = U_1(H) \times U_2(H) \times \cdots \times U_k(H)$, where $U_1(H), U_2(H), \ldots, U_k(H) \in \mathcal{U}_m$. Choose $\mathcal{V}_m = \{U_i(H) : 1 \leq i \leq k, H \in \mathcal{H}_m\}$. Then $\mathcal{V}_m$ is a finite subset of $\mathcal{U}_m$ for each $m \in N_k$. Thus we obtain a sequence $(\mathcal{V}_n)$ of finite sets with $\mathcal{V}_n \subseteq \mathcal{U}_n$. To complete the proof, choose a finite collection $\mathcal{F} = \{U_1, U_2, \ldots, U_p\}$ of nonempty open subsets of X. Since $U_1 \times U_2 \times \cdots \times U_p$ is a nonempty open set in X^p, there is a $m_0 \in N_p$ such that $(U_1 \times U_2 \times \cdots \times U_p) \cap (\cup \mathcal{H}_{m_0}) \neq \emptyset$, which in turn implies that $U_i \cap (\cup \mathcal{V}_{m_0}) \neq \emptyset$ for each $1 \leq i \leq p$. Clearly $(\mathcal{V}_n)$ witnesses that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. □

Note that $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G})$ implies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$. A similar observation for $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ is presented in the next result.

Theorem 3.4. $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_D)$ *implies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

Proof. Let X satisfy $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_D)$. To show that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ we pick a sequence $(\mathcal{U}_n)$ of open covers of X. We may assume that for each n $\mathcal{U}_n$ is closed for finite unions. Let $\{N_k : k \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets. For each k and each $n \in N_k$ choose $\mathcal{W}_n = \{U^k : U \in \mathcal{U}_n\}$. Then for each k $(\mathcal{W}_n : n \in N_k)$ is a sequence of open covers of X^k. Fix k. Let $\mathcal{U} = \{\cap_{n \in N_k} W_n : W_n \in \mathcal{W}_n\}$. Obviously $\mathcal{U} \in \mathcal{G}_K$ for X^k. Without loss of generality we suppose that $\mathcal{U} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_k$, where each $\mathcal{H}_i \in \mathcal{G}_K$ for X. Applying the $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_D)$ property of X, for each $1 \leq i \leq k$, we get a countable set $\mathcal{C}_i \subseteq \mathcal{H}_i$ such that $\cup \mathcal{C}_i$ is dense in X and subsequently we have a countable set $\mathcal{V} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_k \subseteq \mathcal{U}$ such that $\cup \mathcal{V}$ is dense in X^k. Put $\mathcal{V} = \{\cap_{n \in N_k} W_n^m : W_n^m \in \mathcal{W}_n, m \in N_k\}$. Later we choose $\mathcal{V}' = \{W_n^n \in \mathcal{W}_n : n \in N_k\}$. Then $\cup \mathcal{V} \subseteq \cup \mathcal{V}'$ and so $\cup \mathcal{V}'$ is also dense in X^k. For each $n \in N_k$ let $\mathcal{V}_n = \{U \in \mathcal{U}_n : U^k \in \mathcal{V}'\}$. The sequence $(\mathcal{V}_n)$ now witnesses for $(\mathcal{U}_n)$ that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. □

Lemma 3.2. *Every weakly Lindelöf P-space satisfies* $\mathrm{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$.

Proof. Let X be a weakly Lindelöf P-space. Pick $\mathcal{U} \in \mathcal{G}_K$. Since X is a P-space, $\mathcal{U}$ is an open cover of X. Also since X is weakly Lindelöf, there is a countable set $\mathcal{V} \subseteq \mathcal{U}$ such that $\cup \mathcal{V}$ is dense in X. Again by the property of a P-space we can say that the set $\cup \{\overline{V} : V \in \mathcal{V}\}$ is closed in X. Since $\overline{\cup \mathcal{V}}$ is the smallest closed set in X containing $\cup \mathcal{V}$, $\overline{\cup \mathcal{V}} \subseteq \cup \{\overline{V} : V \in \mathcal{V}\}$ and hence $\cup \{\overline{V} : V \in \mathcal{V}\} = X$. Thus X satisfies $\mathrm{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$. □

By Lemma 3.2 and using the implications of Figures 1 and 2, we obtain the following.

Theorem 3.5. *For a P-space X the following properties are equivalent.*

(1) *almost Lindelöf*
(2) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \overline{\mathcal{O}})$
(3) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$
(4) $\mathrm{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$
(5) *weakly Lindelöf*
(6) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$
(7) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$
(8) $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_D)$.

Corollary 3.2. *For a P-space* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ *implies* $\mathrm{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$ *(so implies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$*).*

Note that for a regular space the Lindelöf and almost Lindelöf properties are equivalent. Again using Theorem 3.5 and implications of Figures 1 and 2 the following result can be easily verified.

Theorem 3.6. *For a regular P-space X the following properties are equivalent.*

(1) *Lindelöf*
(2) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$
(3) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$
(4) *almost Lindelöf*
(5) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \overline{\mathcal{O}})$
(6) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$
(7) $\mathrm{S}_1(\mathcal{G}_K, \overline{\mathcal{G}})$
(8) *weakly Lindelöf*
(9) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$
(10) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$
(11) $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_D)$.
(12) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

3.2. **Preservation like properties**

For a set $Y \subseteq \mathbb{N}^{\mathbb{N}}$, $\mathsf{maxfin}(Y)$ is defined as

$$\mathsf{maxfin}(Y) = \{\max\{f_1, f_2, \dots, f_k\} : f_1, f_2, \dots, f_k \in Y \text{ and } k \in \mathbb{N}\},$$

where $\max\{f_1, f_2, \dots, f_k\} \in \mathbb{N}^{\mathbb{N}}$ is given by

$$\max\{f_1, f_2, \dots, f_k\}(n) = \max\{f_1(n), f_2(n), \dots, f_k(n)\} \text{ for all } n \in \mathbb{N}.$$

Theorem 3.7. *Let X be Lindelöf and $\kappa < \mathfrak{d}$. If X is a union of κ many* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma_D)$ *spaces, then X satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

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Proof. Let κ be a cardinal smaller than $\mathfrak{d}$ and $X = \cup_{\alpha<\kappa} X_\alpha$ with each X_α satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma_D)$. To show that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ we pick a sequence $(\mathcal{U}_n)$ of open covers of X, say for each n $\mathcal{U}_n = \{U_m^{(n)} : m \in \mathbb{N}\}$. For each $\alpha < \kappa$ we get a sequence $(\mathcal{V}_n^{(\alpha)})$ such that for each n $\mathcal{V}_n^{(\alpha)}$ is a finite subset of $\mathcal{U}_n$ and for every nonempty open set $U \subseteq X_\alpha$, $U \cap (\cup \mathcal{V}_n^{(\alpha)}) \neq \emptyset$ for all but finitely many n. Next for each $\alpha < \kappa$ we define $f_\alpha : \mathbb{N} \to \mathbb{N}$ by $f_\alpha(n) = \min\{m \in \mathbb{N} : \mathcal{V}_n^{(\alpha)} \subseteq \{U_i^{(n)} : i \leq m\}\}$. Choose $Y = \{f_\alpha : \alpha < \kappa\}$. Obviously the cardinality of $\mathsf{maxfin}(Y)$ is less than $\mathfrak{d}$. Then there exists a $g \in \mathbb{N}^{\mathbb{N}}$ such that for each finite subset A of κ we get $g \not\leq^* f_A$ with $f_A \in \mathsf{maxfin}(Y)$. For each n $\mathcal{V}_n = \{U_i^{(n)} : i \leq g(n)\}$ is a finite subset of $\mathcal{U}_n$. We now show that the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. Then we can find a finite subset A of κ such that $\mathcal{F} = \cup_{\alpha \in A} \mathcal{F}_\alpha$ with for each $\alpha \in A$, $\{U \cap X_\alpha : U \in \mathcal{F}_\alpha\}$ is a finite collection of nonempty of open sets of X_α. For each $\alpha \in A$ select a $n_\alpha \in \mathbb{N}$ such that for each $U \in \mathcal{F}_\alpha$, $U \cap (\cup \mathcal{V}_n^{(\alpha)}) \neq \emptyset$ for all $n \geq n_\alpha$. Choose $n_0 = \max\{n_\alpha : \alpha \in A\}$. Then we get a $n_1 \in \mathbb{N}$ such that $n_1 > n_0$ and $f_A(n_1) < g(n_1)$. Observe that for each $\alpha \in A$ and each $U \in \mathcal{F}_\alpha$, $U \cap (\cup_{i \leq f_\alpha(n_1)} U_i^{(n_1)}) \neq \emptyset$ i.e. $U \cap (\cup_{i \leq f_A(n_1)} U_i^{(n_1)}) \neq \emptyset$. Thus for each $\alpha \in A$ and each $U \in \mathcal{F}_\alpha$, $U \cap (\cup_{i \leq g(n_1)} U_i^{(n_1)}) \neq \emptyset$ i.e. $U \cap (\cup \mathcal{V}_{n_1}) \neq \emptyset$. Hence X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. $\square$

The proof of next two results (Theorems 3.8 and 3.9) are similar to the proof of Theorem 3.7 with necessary modifications and so are omitted.

Theorem 3.8. *Let X be Lindelöf and $\kappa < \mathfrak{d}$. If X is a union of κ many $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma^D)$ spaces, then X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.*

Let κ be any cardinal. We say that the collection $\{X_\alpha : \alpha < \kappa\}$ is a Ω-wrapping if for each finite set $F \subseteq \cup_{\alpha<\kappa} X_\alpha$ there exists a $\beta < \kappa$ such that $F \subseteq X_\beta$.

Theorem 3.9. *Let X be Lindelöf and $\kappa < \mathfrak{b}$. Suppose that $X = \cup_{\alpha<\kappa} X_\alpha$ and the collection $\{X_\alpha : \alpha < \kappa\}$ is a Ω-wrapping. If each X_α satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$, then X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.*

We do not know whether the above result holds for $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ and thus, we present an open question.

Problem 3.1. *Let X be Lindelöf and $\kappa < \mathfrak{b}$. Suppose that $X = \cup_{\alpha<\kappa} X_\alpha$ and the collection $\{X_\alpha : \alpha < \kappa\}$ is a Ω-wrapping. If each X_α satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$, does then X satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$?*

A collection $\{X_\alpha : \alpha < \kappa\}$ in a space X is said to be a strongly Ω-wrapping if for each $\alpha < \kappa$ and any dense set $Y_\alpha \subseteq X_\alpha$ the collection $\{Y_\alpha : \alpha < \kappa\}$ is a Ω-wrapping.

Theorem 3.10. *Let X be Lindelöf and $\kappa < \mathfrak{b}$. Suppose that $X = \cup_{\alpha<\kappa} X_\alpha$ and $\{X_\alpha : \alpha < \kappa\}$ is a strongly Ω-wrapping. If each X_α satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$, then X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.*

Proof. Let $(\mathcal{U}_n)$ be a sequence of open covers of X. We can assume that for each n $\mathcal{U}_n = \{U_m^{(n)} : m \in \mathbb{N}\}$. For each $\alpha < \kappa$ we get a dense subset Y_α and a sequence $(\mathcal{V}_n^{(\alpha)})$ such that for each n $\mathcal{V}_n^{(\alpha)}$ is a finite subset of $\mathcal{U}_n$ and each finite set $F \subseteq Y_\alpha$ is contained in $\cup \mathcal{V}_n^{(\alpha)}$ for infinitely many n. By the given condition, $\{Y_\alpha : \alpha < \kappa\}$ is a Ω-wrapping. For each $\alpha < \kappa$ we define $f_\alpha : \mathbb{N} \to \mathbb{N}$ by $f_\alpha(n) = \min\{m \in \mathbb{N} : \mathcal{V}_n^{(\alpha)} \subseteq \{U_i^{(n)} : i \leq m\}\}$. Since $\kappa < \mathfrak{b}$, there exists a $g \in \mathbb{N}^{\mathbb{N}}$ such that for each $\alpha < \kappa$ we get $f_\alpha \leq^* g$. Clearly $Y = \cup_{\alpha<\kappa} Y_\alpha$ is a dense subset of X and for each n $\mathcal{V}_n = \{U_i^{(n)} : i \leq g(n)\}$ is a finite subset of $\mathcal{U}_n$. It is easy to observe that Y and the sequence $(\mathcal{V}_n)$ witness for $(\mathcal{U}_n)$ that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. $\square$

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Note that all the variants described in Figure 1 are preserved under clopen subsets. Observe that every regular-closed subset of a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ space is $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$. But this result does not hold for $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$.

Example 3.3.

(1) A regular-closed subset of a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$ space need not be $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$.
Let $A=\{(a_\alpha,-1):\alpha<\omega_1\}\subseteq\{(x,-1):x\geq 0\}\subseteq\mathbb{R}^2$, $Y=\{(a_\alpha,n):\alpha<\omega_1,n\in\mathbb{N}\}$ and $p=(-1,-1)$. Choose $X_1=Y\cup A\cup\{p\}$. We topologize X_1 as follows. (i) Every point of Y is isolated, (ii) for each $\alpha<\omega_1$ a basic neighbourhood of $(a_\alpha,-1)$ is of the form $U_n(a_\alpha,-1)=\{(a_\alpha,-1)\}\cup\{(a_\alpha,m):m\geq n\}$, where $n\in\mathbb{N}$ and (iii) a basic neighbourhood of p is of the form $U_\alpha(p)=\{p\}\cup\{(a_\beta,n):\beta>\alpha,n\in\mathbb{N}\}$, where $\alpha<\omega_1$. We now show that X_1 satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$. Let $(\mathcal{U}_n)$ be a sequence of open covers of X_1. For each n choose a $U_n\in\mathcal{U}_n$ such that $p\in U_n$ and also choose a basic neighbourhood $U_{\beta_n}(p)\subseteq U_n$. Now for each n $X_1\setminus\overline{U_n}$ is at most countable as $\overline{U_{\beta_n}(p)}=U_{\beta_n}(p)\cup\{(a_\beta,-1):\beta>\beta_n\}$. Clearly $K=\cup_{n\in\mathbb{N}}(X_1\setminus\overline{U_n})$ is σ-compact and hence satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$. Apply the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$ property of K to $(\mathcal{U}_n)$ to obtain a sequence $(\mathcal{V}_n')$ such that for each n $\mathcal{V}_n'$ is a finite subset of $\mathcal{U}_n$ and each $x\in K$ belongs to $\cup\mathcal{V}_n'$ for all but finitely many n. Clearly the sequence $(\mathcal{V}_n)$, where $\mathcal{V}_n=\mathcal{V}_n'\cup\{U_n\}$ for each n witnesses that X_1 satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$.
Let X_2 be the space as in Example 3.1. Then X_2 does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$. Assume that $X_1\cap X_2=\emptyset$. Since the cardinality of D is ω_1, we write $D=\{d_\alpha:\alpha<\omega_1\}$. Define a bijection $\varphi:D\times\{\omega\}\to A$ by $\varphi(d_\alpha,\omega)=(a_\alpha,-1)$ for each $\alpha<\omega_1$. Also define Z to be the quotient image of the topological sum $X_1\oplus X_2$ by identifying (d_α,ω) of X_2 with $\varphi(d_\alpha,\omega)$ of X_1 for each $\alpha<\omega_1$. Let $q:X_1\oplus X_2\to Z$ be the quotient map. Now $q(X_2)$ is a regular-closed subset of Z which does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$ as it is homeomorphic to X_2.
We now claim that Z satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$. The claim will follow if we show that Z satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$. Choose a sequence $(\mathcal{U}_n)$ of open covers of Z. Now $q(X_1)$ being the homoeomorphic image of a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$ space, is also $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$. Apply the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$ property of $q(X_1)$ to $(\mathcal{U}_n)$ to obtain a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and each $x\in q(X_1)$ belongs to $\overline{\cup\mathcal{H}_n}$ for all but finitely many n. Again since $q(\beta D\times\omega)$ is homeomorphic to $\beta D\times\omega$, $q(\beta D\times\omega)$ is σ-compact and so satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$. Thus there is a sequence $(\mathcal{K}_n)$ such that for each n $\mathcal{K}_n$ is a finite subset of $\mathcal{U}_n$ and each $x\in q(\beta D\times\omega)$ belongs to $\cup\mathcal{K}_n$ for all but finitely many n. For each n let $\mathcal{V}_n=\mathcal{H}_n\cup\mathcal{K}_n$. The sequence $(\mathcal{V}_n)$ witnesses that Z satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma})$.

(2) A closed subset of a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$) space need not be $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$).
Consider X as in Example 3.1. Now X is a Tychonoff $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ space and hence a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ space. Since $D\times\{\omega\}$ is a discrete closed subset of X with cardinality ω_1, it follows that $D\times\{\omega\}$ fails to satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ as well.

Observe that if a subset Y of a space X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$), then $\overline{Y}$ also satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$). Thus for a dense subset Y of X, if Y satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$), then X also satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$), and also every separable space satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ (so satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$). Surprisingly a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$ space may not satisfy this preservation property. The Baire space X does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$ because X is paracompact and does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$. Since X is separable, there exists a countable dense subset Y of X. Thus Y satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$ but $\overline{Y}=X$ does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$.

Remark 3.1. *The Baire space X is hypocompact and separable. Clearly X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ (so satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$). Thus there is a hypocompact $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ space which does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$ (as X does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega})$).*

Theorem 3.11. *For a space X the following assertions are equivalent.*
(1) *X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$.*
(2) *$AD(X)$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$.*
(3) *$AD(X)$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n)$ be a sequence of open covers of $AD(X)$. For each n and each $x\in X$ let $W_x^{(n)}=(V_x^{(n)}\times\{0,1\})\setminus\{(x,1)\}$ be an open set in $AD(X)$ containing $(x,0)$ such that there is a $U_x^{(n)}\in\mathcal{U}_n$ with $W_x^{(n)}\subseteq U_x^{(n)}$, where $V_x^{(n)}$ is an open set in X containing x. For each n $\mathcal{W}_n=\{V_x^{(n)}:x\in X\}$ is an open cover of X. Apply (1) to $(\mathcal{W}_n)$ to obtain a sequence (F_n) of finite subsets of X such that $(\{V_x^{(n)}:x\in F_n\})$ witnesses the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$ property of X. For each n and each $x\in F_n$ choose a $O_x^{(n)}\in\mathcal{U}_n$ with $(x,1)\in O_x^{(n)}$. Observe that $\mathcal{V}_n=\{U_x^{(n)}:x\in F_n\}\cup\{O_x^{(n)}:x\in F_n\}$ is a finite subset of $\mathcal{U}_n$ for each n. The sequence $(\mathcal{V}_n)$ witnesses that $AD(X)$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$.

(3) $\Rightarrow$ (1). Let $(\mathcal{U}_n)$ be a sequence of open covers of X. Say, $\mathcal{U}_n=\{U_x^{(n)}:x\in X\}$, where $U_x^{(n)}$ is an open set in X containing x for each n. Choose $\mathcal{W}_n=\{(U_x^{(n)}\times\{0,1\})\setminus\{(x,1)\}:x\in X\}\cup\{\{(x,1)\}:x\in X\}$ for each n. Since $(\mathcal{W}_n)$ is a sequence of open covers of $AD(X)$, there are a dense subset Z of $AD(X)$ and a sequence (F_n) of finite subsets of X such that $\{(U_x^{(n)}\times\{0,1\})\setminus\{(x,1)\}:x\in F_n\}\cup\{\{(x,1)\}:x\in F_n\}$ witnesses the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ property of $AD(X)$. For each n $\mathcal{V}_n=\{U_x^{(n)}:x\in F_n\}$ is a finite subset of $\mathcal{U}_n$. It now follows that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$. □

There is a space X satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$) such that $AD(X)$ does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$). Let X be the space as in Example 3.1. Then X is a Tychonoff $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (and also a $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$) space and $A=D\times\{\omega\}$ is an uncountable discrete closed subset of X. Observe that $A\times\{1\}$ is an uncountable discrete clopen subset of $AD(X)$. Thus $AD(X)$ does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ and so does not satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$.

The proof of the following result is in line of (1) $\Rightarrow$ (2) of Theorem 3.11.

Theorem 3.12. *For a space X if $AD(X)$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$, then X also satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$.*

By Example 3.1, there exists a Tychonoff $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ and hence $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ space X such that $e(X)\geq\omega_1$. However for a T_1 space X it can be shown that if $AD(X)$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$, then $e(X)<\omega_1$. The conclusion also holds if $AD(X)$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$.

Theorem 3.13. *If $f:X\to Y$ is a mapping from a space X satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$) onto a space Y such that for each $x\in X$ and each open set V in Y containing $f(x)$ there is an open set U in X containing x such that $f(\overline{U})\subseteq V$, then Y also satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (respectively, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$).*

Proof. We only sketch the proof for the case $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ and other case can be observed similarly. Let $(\mathcal{U}_n)$ be a sequence of open covers of Y. Let $x\in X$. For each n choose a $V_x^{(n)}\in\mathcal{U}_n$ containing $f(x)$. Subsequently for each n we obtain an open set $U_x^{(n)}$ in X containing x such that $f\left(\overline{U_x^{(n)}}\right)\subseteq V_x^{(n)}$. We now apply the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$ property of X to the sequence $(\mathcal{W}_n)$, where for each n $\mathcal{W}_n=\{U_x^{(n)}:x\in X\}$. Choose a dense set $Z\subseteq X$ and a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n = \{U_{x_i}^{(n)} : 1 \leq i \leq k_n\}$ is a finite subset of $\mathcal{W}_n$ and for each finite set $F \subseteq Z$ there is a n such that $F \subseteq \cup\mathcal{H}_n$. Clearly $f(Z)$ is dense in Y. Let $\mathcal{V}_n = \{V_{x_i}^{(n)} : 1 \leq i \leq k_n\}$ for each n. It is now easy to verify that $f(Z)$ and $(\mathcal{V}_n)$ witness that Y satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. □

Corollary 3.3. *The* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ *and* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ *properties are preserved under continuous mappings.*

Theorem 3.14. *If $f : X \to Y$ is a closed mapping from a space X onto a space Y satisfying* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ *such that for each $x \in X$ and each open set U in X containing x $\overline{f(U)}$ is a neighbourhood of $f(x)$ and $f^{-1}(y)$ is compact in X for each $y \in Y$, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

Proof. Let $(\mathcal{U}_n)$ be a sequence of open covers of X and $y \in Y$. Since $f^{-1}(y)$ is compact, for each n there exists a finite subset $\mathcal{V}_n^y$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \cup\mathcal{V}_n^y$. Again since f is closed, there exists an open set $U_y^{(n)}$ in Y containing y such that $f^{-1}(U_y^{(n)}) \subseteq \cup\mathcal{V}_n^y$. Thus for each n $\mathcal{W}_n = \{U_y^{(n)} : y \in Y\}$ is an open cover of Y. Apply the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ property of Y to $(\mathcal{W}_n)$ to obtain a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n = \{U_{y_i}^{(n)} : 1 \leq i \leq k_n\}$ is a finite subset of $\mathcal{W}_n$ and $\{\cup\mathcal{H}_n : n \in \mathbb{N}\} \in \Omega_D$ for Y. For each n $\mathcal{V}_n = \cup_{1\leq i\leq k_n}\mathcal{V}_n^{y_i}$ is a finite subset of $\mathcal{U}_n$. Let $\mathcal{F} = \{U_i : 1 \leq i \leq k\}$ be a finite collection of nonempty open sets of X. Then for each $1 \leq i \leq k$ there exists a nonempty open set V_i in Y such that $V_i \subseteq \overline{f(U_i)}$. Choose $\mathcal{F}' = \{V_i : 1 \leq i \leq k\}$. Later we get a $n_0 \in \mathbb{N}$ such that $V_i \cap (\cup\mathcal{H}_{n_0}) \neq \emptyset$ for all $1 \leq i \leq k$. It follows that for each $1 \leq i \leq k$, $\overline{f(U_i)} \cap (\cup\mathcal{H}_{n_0}) \neq \emptyset$ and consequently $f(U_i) \cap (\cup\mathcal{H}_{n_0}) \neq \emptyset$. Thus for each $1 \leq i \leq k$, $U_i \cap f^{-1}(\cup\mathcal{H}_{n_0}) \neq \emptyset$ and then it turns into $U_i \cap (\cup\mathcal{V}_{n_0}) \neq \emptyset$. This completes the proof. □

Corollary 3.4.

(1) *If $f : X \to Y$ is an open perfect mapping from a space X onto a space Y satisfying* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$*, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) *If $f : X \to Y$ is an open closed mapping from a space X onto a space Y satisfying* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ *such that $f^{-1}(y)$ is compact in X for each $y \in Y$, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

We also obtain a similar observation for $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ as follows. The proof of it is similar to the proof of Theorem 3.14 with necessary modifications and so is omitted.

Theorem 3.15. *If $f : X \to Y$ is an open closed mapping from a space X onto a space Y satisfying* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ *such that $f^{-1}(y)$ is compact in X for each $y \in Y$, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

Corollary 3.5. *If $f : X \to Y$ is an open perfect mapping from a space X onto a space Y satisfying* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$*, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

The proof of the next result is immediate.

Proposition 3.2. *Let $X = \cup_{m\in\mathbb{N}}X_m$, where $X_m \subseteq X_{m+1}$ for each m.*

(1) *If each X_m satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$*, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) *If each X_m satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$*, then X also satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

In combination with Proposition 3.2, Theorem 3.14 and Theorem 3.15 we obtain the following.

Proposition 3.3. *Let X be a σ-compact space. Then X is*

(1) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

3.3. **The productively properties**

If P is a property of a space, we call a space X productively P if $X \times Y$ has the property P whenever Y has the property P. We start with the following basic observation for dense subsets. If $Y \subseteq X$ is dense in X and $D \subseteq Y$ is dense in Y, then D is dense in X. Also if $D \subseteq X$ is dense in X and $E \subseteq Y$ is dense in Y, then $D \times E$ is dense in $X \times Y$.

Proposition 3.4. *Let D be a dense subset of a space X.*
(1) *If D is productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$*, then X is also productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) *If D is productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$*, then X is also productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

Proof. (1). Let Y satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ and $(\mathcal{U}_n)$ be a sequence of open covers of $X \times Y$. Since D is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$, $D \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Choose a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite collection $\mathcal{F}$ of nonempty open sets of $D \times Y$ there is a n such that $U \cap (\cup \mathcal{V}_n) \neq \emptyset$ for all $U \in \mathcal{F}$. Since D is dense in X, the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that $X \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Hence X is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

(2). Let Y satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ and $(\mathcal{U}_n)$ be a sequence of open covers of $X \times Y$. Since D is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$, $D \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. Then there exist a sequence $(\mathcal{V}_n)$ and a dense subset Z of $D \times Y$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each finite subset F of $D \times Y$ is contained in $\cup \mathcal{V}_n$ for some n. Observe that $D \times Y$ is dense in $X \times Y$. It follows that Z is dense in $X \times Y$ since Z is dense in $D \times Y$. Then the sequence $(\mathcal{V}_n)$ and the set Z guarantee for $(\mathcal{U}_n)$ that $X \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. Thus X is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. □

Theorem 3.16. *A H-closed space X is*
(1) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

Proof. We furnish proof for the productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ case and the other case can be carried out similarly. Let Y satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Consider a sequence $(\mathcal{U}_n)$ of open covers of $X \times Y$. Without loss of generality assume that $\mathcal{U}_n = \mathcal{V}_n \times \mathcal{W}_n$ for each n, where $\mathcal{V}_n$ and $\mathcal{W}_n$ are respectively open covers of X and Y. Fix $y \in Y$. Since $X \times \{y\}$ is H-closed, there is a sequence $(\mathcal{V}_n^y \times \mathcal{W}_n^y)$ such that for each n $\mathcal{V}_n^y \times \mathcal{W}_n^y$ is a finite subset of $\mathcal{U}_n$ and $\cup(\mathcal{V}_n^y \times \mathcal{W}_n^y)$ is dense in $X \times \{y\}$. Consider the open cover $\mathcal{U}_n' = \{U_n^y : y \in Y\}$ of Y, where for each n $U_n^y = \cap \mathcal{W}_n^y$. Apply the $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ property of Y to $(\mathcal{U}_n')$ to obtain a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n'$ and for each finite collection $\mathcal{F}$ of nonempty open sets of Y there is a n such that $U \cap (\cup \mathcal{H}_n) \neq \emptyset$ for all $U \in \mathcal{F}$. For each n choose $\mathcal{H}_n = \{U_n^{y_1}, U_n^{y_2}, \ldots, U_n^{y_{k_n}}\}$. Now for each n $\mathcal{K}_n = \cup_{i=1}^{k_n} (\mathcal{V}_n^{y_i} \times \mathcal{W}_n^{y_i})$ is a finite subset of $\mathcal{U}_n$. Clearly the sequence $(\mathcal{K}_n)$ witnesses that $X \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Thus X is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. □

Theorem 3.17. *If X satisfies* $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$*, then X is productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

Proof. Let Y satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. To show that $X \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ we pick a sequence $(\mathcal{U}_n)$ of open covers of $X \times Y$. Without loss of generality we assume that for each n $\mathcal{U}_n$ is closed under finite unions. For each compact set $C \subseteq X$ and for each n we find a G_δ set $G_n(C) \subseteq X$ such that $C \subseteq G_\delta(C)$. Then for each $y \in Y$ and for each n there exists a $U \in \mathcal{U}_n$ such that $G_n(C) \times \{y\} \subseteq U$. For each n $\mathcal{G}_n = \{G_n(C) : C \text{ is a compact subset of } X\} \in \mathcal{G}_K$. Apply the $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$ property of X to $(\mathcal{G}_n)$ to obtain a sequence (C_n) of compact subsets of X such that for each nonempty open set $U \subseteq X$, the set $\{n \in \mathbb{N} : U \cap G_n(C) = \emptyset\}$ is finite. For each n let $\mathcal{W}_n = \{V : V \text{ is an open set in } Y \text{ and there is a } U \in \mathcal{U}_n \text{ such that } G_n(C_n) \times V \subseteq U\}$. Clearly $(\mathcal{W}_n)$ is a sequence of open covers of Y. Since Y satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$, there exists a sequence $(\mathcal{K}_n)$ such that for each n $\mathcal{K}_n$ is a finite subset of $\mathcal{W}_n$ and for every finite collection $\mathcal{F}$ of nonempty open sets of Y, the set $\{n \in \mathbb{N} : U \cap (\cup \mathcal{K}_n) \neq \emptyset$ for all $U \in \mathcal{F}\}$ is infinite. For each n and each $V \in \mathcal{K}_n$ we pick a $U_V \in \mathcal{U}_n$ such that $G_n(C_n) \times V \subseteq U_V$. For each n let $\mathcal{V}_n = \{U_V : V \in \mathcal{K}_n\}$. Observe that the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that $X \times Y$ satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Thus X is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. □

Using similar technique we can prove the following.

Theorem 3.18. *If X satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{\Gamma_D})$, then X is productively $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.*

The following result is an improvement of [3, Lemma 44].

Theorem 3.19. *If any dense subspace of X satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$, then X satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{\Gamma_D})$.*

Proof. Let Y be a dense subset of X having the property $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$. We pick a sequence $(\mathcal{U}_n)$ of members of $\mathcal{G}_K$ to show that X has the property $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{\Gamma_D})$. Since Y satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$, there exists a sequence (U_n) such that for each n $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\} \in \mathcal{G}_\Gamma$ for Y. It follows that $\{U_n : n \in \mathbb{N}\} \in \mathcal{G}_{\Gamma_D}$ for X as Y is dense in X. Thus X satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{\Gamma_D})$. □

Since every σ-compact space satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ (see [3, Theorem 22]), we have the following.

Corollary 3.6. *If a space X has a dense σ-compact subset, then X satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{\Gamma_D})$.*

Corollary 3.7. *Every separable space satisfies $\mathrm{S}_1(\mathcal{G}_K, \mathcal{G}_{\Gamma_D})$.*

Using Theorem 3.17, Theorem 3.18 (and also Figure 2) we have the following.

Corollary 3.8. *A separable space X is*

(1) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$
(2) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

Corollary 3.9. *For each cardinal κ, $\mathbb{R}^\kappa$ is*

(1) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$
(2) *productively* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.

Proof. Since $\{f \in \mathbb{R}^\kappa : |\{i : f(i) \neq 0\}| < \omega\}$ is a dense σ-compact subset of $\mathbb{R}^\kappa$ (see [6, Proposition 4]), by Theorem 3.17, Theorem 3.18, Corollary 3.6 and Figure 2, $\mathbb{R}^\kappa$ has the claimed properties. □

By Corollary 3.8, we can say that $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ ($\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$) is preserved under finite products (in the case of sets of reals). But for arbitrary spaces $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ is not preserved under finite products (see Example 3.4). We need the following observations on the Pixley-Roy spaces (from [3, 7]).

Let CH denote the continuum hypothesis, which states that there is no set whose cardinality is strictly between that of the integers and the real numbers, or equivalently, that any subset of the reals is finite, is countably infinite, or has the same cardinality as the reals. There many well known equivalent reformulations of CH. Consider the set of integers $\mathbb{Z}$ equipped with the discrete topology and consider the Tychonoff product ${}^\omega\mathbb{Z}$ equipped with the product topology. In this context note that CH can be used to construct subsets X and Y of ${}^\omega\mathbb{Z}$ such that each has the property $\mathrm{S}_1(\Omega, \Omega)$, but $(X \cup Y) \oplus (X \cup Y) = {}^\omega\mathbb{Z}$ (see [3, 20]).

Lemma 3.3 ([3, Proposition 4]). *For any two spaces X and Y, $\mathsf{PR}(X) \times \mathsf{PR}(Y)$ is homeomorphic to $\mathsf{PR}(X \oplus Y)$.*

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Lemma 3.4 (cf. [3, Proposition 5]). *Assume CH. There exist separable metrizable spaces X and Y such that both satisfy $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ but $X\oplus Y$ does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.*

Theorem 3.20 ([7, Theorem 2A]). *If $\mathsf{PR}(X)$ satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$, then every finite power of X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.*

Theorem 3.21 ([7, Theorem 2B]). *If X is a metrizable space such that every finite power of X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, then $\mathsf{PR}(X)^\kappa$ satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$ for each cardinal κ.*

Example 3.4. *Assume CH. There are $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ spaces X and Y such that $X\times Y$ does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$.*
By Lemma 3.4, there are two separable metrizable spaces X and Y such that both X and Y satisfy $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ but $X\oplus Y$ does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Now every finite power of X and also of Y satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ (see [10, Theorem 3.9]). By Theorem 3.21, every finite power of $\mathsf{PR}(X)$ and also of $\mathsf{PR}(Y)$ satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$. It follows that both $\mathsf{PR}(X)$ and $\mathsf{PR}(Y)$ satisfy $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$ (see Theorem 3.3). Also by Theorem 3.20, $\mathsf{PR}(X\oplus Y)$ does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$ and so $\mathsf{PR}(X)\times\mathsf{PR}(Y)$ does not satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}_D)$ (see Lemma 3.3).

$$\begin{tikzcd}
 & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma^D) \arrow[dr] & \\
\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) \arrow[r] \arrow[urr] & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Gamma}) \arrow[rr] & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma_D) \\
 & & \mathrm{S}_1(\mathcal{G}_K,\mathcal{G}_{\Gamma_D}) \arrow[uu] \arrow[dr] & \\
\mathrm{S}_1(\mathcal{G}_K,\mathcal{G}_\Gamma) \arrow[r] \arrow[uu] \arrow[urr] \arrow[d] & \mathrm{S}_1(\mathcal{G}_K,\mathcal{G}_{\overline{\Gamma}}) \arrow[rr] \arrow[uu] \arrow[d] & & \mathrm{S}_1(\mathcal{G}_K,\mathcal{G}_{D_\Gamma}) \arrow[uu] \arrow[d] \\
\mathrm{S}_1(\mathcal{G}_K,\mathcal{G}) \arrow[r] \arrow[d] & \mathrm{S}_1(\mathcal{G}_K,\overline{\mathcal{G}}) \arrow[rr] \arrow[d] & & \mathrm{S}_1(\mathcal{G}_K,\mathcal{G}_D) \arrow[d] \\
\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega) \arrow[r] & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\overline{\Omega}) \arrow[rr] & & \mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)
\end{tikzcd}$$

FIGURE 2. Weaker variants of $\mathrm{S}_1(\mathcal{G}_K,\mathcal{G})$, $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$ and $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$

4. Weak groupability and games

4.1. **Weakly groupable covers**

Theorem 4.1. *For a space X the following properties are equivalent.*

(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega_D)$.
(2) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}^{wgp}{}_D)$.
(3) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Lambda^{wgp}{}_D)$.

Proof. Since $\Omega_D\subseteq\Lambda_D$ and every countable member of Ω_D is also a member of $\mathcal{O}^{wgp}{}_D$, the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) hold. To show that (2) $\Rightarrow$ (1) we choose a sequence $(\mathcal{U}_n)$ of open covers of X. For each n we define $\mathcal{W}_n=\wedge_{i\leq n}\mathcal{U}_n$. Observe that $(\mathcal{W}_n)$ is a sequence of open covers of X and since X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}^{wgp}{}_D)$, there exists a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$ and

$\{\cup\mathcal{H}_n : n \in \mathbb{N}\} \in \mathcal{O}^{wgp}{}_D$. We consider a sequence $n_1 < n_2 < \cdots$ of members of $\mathbb{N}$ which witnesses that $\{\cup\mathcal{H}_n : n \in \mathbb{N}\} \in \mathcal{O}^{wgp}{}_D$ i.e. for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a k such that $U \cap (\cup\{\cup\mathcal{H}_i : n_k \leq i < n_{k+1}\}) \neq \emptyset$ for all $U \in \mathcal{F}$. Let $(\mathcal{K}_n)$ be a sequence which is given by

$$\mathcal{K}_n = \begin{cases} \cup_{i<n_1}\mathcal{H}_i, & \text{for } n < n_1 \\ \cup_{n_k \leq i < n_{k+1}}\mathcal{H}_i, & \text{for } n_k \leq n < n_{k+1}. \end{cases}$$

We now define a sequence $(\mathcal{V}_n)$ as follows. For each n $\mathcal{V}_n$ is the collection of all members of $\mathcal{U}_n$ from the representation of each member of $\mathcal{K}_n$. Clearly for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup\mathcal{K}_n \subseteq \cup\mathcal{V}_n$. Then the sequence $(\mathcal{V}_n)$ guarantees that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. □

Using analogous tactic we obtain the next result.

Theorem 4.2. *For a space X the following properties are equivalent.*

(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.
(2) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}^{wgp D})$.
(3) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Lambda^{wgp D})$.

Theorem 4.3. *For a Lindelöf space X the following properties are equivalent.*

(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Omega_D)$.
(3) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$.
(4) *For each sequence $(\mathcal{U}_n)$ of γ-covers of X there is a sequence $(\mathcal{V}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each finite collection $\mathcal{F}$ of nonempty open subsets of X there is a n such that $U \cap (\cup\mathcal{V}_n) \neq \emptyset$ for all $U \in \mathcal{F}$.*
(5) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \mathcal{O}^{wgp}{}_D)$.
(6) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$.

Proof. (2) ⇒ (1). Let $(\mathcal{U}_n)$ be a sequence of open covers of X. For each n choose $\mathcal{U}_n = \{U_m^{(n)} : m \in \mathbb{N}\}$ and $\mathcal{W}_n = \{V_m^{(n)} : m \in \mathbb{N}\}$ with $V_m^{(n)} = \cup_{1 \leq i \leq m} U_i^{(n)}$. Then $(\mathcal{W}_n)$ is a sequence of γ-covers of X. Apply the $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Omega_D)$ property of X to $(\mathcal{W}_n)$ to obtain a sequence $(V_{m_n}^{(n)})$ such that for each n $V_{m_n}^{(n)} \in \mathcal{W}_n$ and $\{V_{m_n}^{(n)} : n \in \mathbb{N}\} \in \Omega_D$. For each n let $\mathcal{V}_n = \{U_i^{(n)} : 1 \leq i \leq m_n\}$ and then $\cup\mathcal{V}_n = V_{m_n}^{(n)}$. It follows that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.

(2) ⇒ (3). Let $(\mathcal{U}_n)$ be a sequence of γ-covers of X. Without loss of generality we can assume that for $m \neq n$, $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$. Use the $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Omega_D)$ property of X to obtain a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\cup\mathcal{V}_n : n \in \mathbb{N}\} \in \Omega_D$. It follows that $\{\cup\mathcal{V}_n : n \in \mathbb{N}\} \in \Lambda_D$ and hence also $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \Lambda_D$. Now the partition $(\mathcal{V}_n)$ witnesses that $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}^{wgp}{}_D$. Thus $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \Lambda^{wgp}{}_D$.

(3) ⇒ (4). The proof for this implication is modelled in [1, Theorem 2]. Here we present a complete proof for convenience of the reader. Let $(\mathcal{U}_n)$ be a sequence of γ-covers of X. Without loss of generality we can assume that for $m \neq n$, $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$. For each n choose $\mathcal{U}_n = \{U_m^{(n)} : m \in \mathbb{N}\}$. Then for each n define $\mathcal{W}_n = \{U_m^{(1)} \cap \cdots \cap U_m^{(n)} : m \in \mathbb{N}\} \setminus \{\emptyset\}$. Clearly each $\mathcal{W}_n$ is a γ-cover of X. By omitting elements where necessary we can suppose that for $m \neq n$, $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$. We also in advance choose for each element of each $\mathcal{W}_n$ a representation as an intersection as in the definition. Using $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$ we get a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{H}_n \in \Lambda^{wgp}{}_D$. Let $\cup_{n\in\mathbb{N}}\mathcal{H}_n = \cup_{n\in\mathbb{N}}\mathcal{K}_n$, where each $\mathcal{K}_n$ is finite and $\mathcal{K}_m \cap \mathcal{K}_n = \emptyset$ for $m \neq n$, and

for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a n such that $U \cap (\cup \mathcal{K}_n) \neq \emptyset$ for all $U \in \mathcal{F}$.

We now choose a sequence of positive integers $i_1 < i_2 < \cdots$ as follows. Let $i_1 \geq 1$ be so small such that $\mathcal{H}_1 \cap \mathcal{K}_j = \emptyset$ for all $j > i_1$. Now choose $\mathcal{V}_1$ as the set of $U \in \mathcal{U}_1$ that appear as terms (if exist) in the representations of elements of $\mathcal{K}_j, j \leq i_1$.

Next take $i_2 > i_1$ so small such that $\mathcal{H}_2 \cap \mathcal{K}_j = \emptyset$ for all $j > i_2$. Choose $\mathcal{V}_2$ as the set of $U \in \mathcal{U}_2$ that appear as terms (if exist) in the representations of elements of $\mathcal{K}_j, j \leq i_2$.

Proceeding similarly we obtain a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$ for $m \neq n$. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. Then there exists a $n_0 \in \mathbb{N}$ such that $U \cap (\cup \mathcal{K}_{n_0}) \neq \emptyset$ for all $U \in \mathcal{F}$. Let $k_0 \in \mathbb{N}$ be the least such that $n_0 \leq i_{k_0}$. It is easy to see that $(\mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_{k_0-1}) \cap \mathcal{K}_{n_0} = \emptyset$. This implies that for each $V \in \mathcal{K}_{n_0}$ there exists a $U_V \in \mathcal{U}_{k_0}$ such that U_V is a term in the representation of V. Choose $\mathcal{V} = \{U_V : V \in \mathcal{K}_{n_0}\}$ and then we get $\cup \mathcal{K}_{n_0} \subseteq \cup \mathcal{V} \subseteq \cup \mathcal{V}_{k_0}$. Hence X satisfies (4).

(5) $\Rightarrow$ (4). The proof is similar to the proof of (3) $\Rightarrow$ (4).

(6) $\Rightarrow$ (3). Let $(\mathcal{U}_n)$ be a sequence of γ-covers of X. We can assume that $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$ for $m \neq n$. Since X satisfies $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$, there is a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\cup \mathcal{V}_n : n \in \mathbb{N}\} \in \Lambda^{wgp}{}_D$. Clearly $\cup_{n\in\mathbb{N}} \mathcal{V}_n \in \Lambda_D$. Now choose $\{\cup \mathcal{V}_n : n \in \mathbb{N}\} = \cup_{n\in\mathbb{N}} \mathcal{H}_n$, where $(\mathcal{H}_n)$ is a sequence of pairwise disjoint finite sets such that for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a n such that $V \cap (\cup \mathcal{H}_n) \neq \emptyset$ for all $V \in \mathcal{F}$. Using the sequence $(\mathcal{H}_n)$ we can find a sequence $(\mathcal{F}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{F}_n \subseteq \cup_{n\in\mathbb{N}} \mathcal{V}_n$ with $\cup_{n\in\mathbb{N}} \mathcal{V}_n = \cup_{n\in\mathbb{N}} \mathcal{F}_n$ and also $\cup \mathcal{F}_n = \cup \mathcal{H}_n$. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. Then there exists a $n_0 \in \mathbb{N}$ such that $U \cap (\cup \mathcal{H}_{n_0}) = U \cap (\cup \mathcal{F}_{n_0}) \neq \emptyset$ for all $U \in \mathcal{F}$. It follows that $\cup_{n\in\mathbb{N}} \mathcal{V}_n \in \Lambda^{wgp}{}_D$ and consequently X satisfies $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$.

Also since the implications (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (2) $\Rightarrow$ (6) are routine, all the properties are equivalent. □

Corollary 4.1. *For a Lindelöf space X the following properties are equivalent.*

(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.
(2) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Omega_D)$.
(3) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$.
(4) *For each sequence $(\mathcal{U}_n)$ of γ-covers of X there is a sequence $(\mathcal{V}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each finite collection $\mathcal{F}$ of nonempty open subsets of X there is a n such that $U \cap (\cup \mathcal{V}_n) \neq \emptyset$ for all $U \in \mathcal{F}$.*
(5) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \mathcal{O}^{wgp}{}_D)$.
(6) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}^{wgp}{}_D)$.
(7) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Lambda^{wgp}{}_D)$.
(8) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp}{}_D)$.

In analogy to Theorem 4.3 we can prove the following.

Theorem 4.4. *For a Lindelöf space X the following properties are equivalent.*

(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.
(2) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Omega^D)$.
(3) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^{wgpD})$.
(4) *For each sequence $(\mathcal{U}_n)$ of γ-covers of X there is a dense set $Y \subseteq X$ and a sequence $(\mathcal{V}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$ and each finite set $F \subseteq Y$ is contained in $\cup \mathcal{V}_n$ for some n.*
(5) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \mathcal{O}^{wgpD})$.

(6) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp D})$.

Corollary 4.2. *For a Lindelöf space X the following properties are equivalent.*
(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$.
(2) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Omega^D)$.
(3) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp D})$.
(4) *For each sequence $(\mathcal{U}_n)$ of γ-covers of X there is a dense set $Y \subseteq X$ and a sequence $(\mathcal{V}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$ and each finite set $F \subseteq Y$ is contained in $\cup \mathcal{V}_n$ for some n.*
(5) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \mathcal{O}^{wgp D})$.
(6) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}^{wgp D})$.
(7) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Lambda^{wgp D})$.
(8) $\mathrm{U}_{\mathrm{fin}}(\Gamma, \Lambda^{wgp D})$.

We use the following lemma in subsequent observations.

Lemma 4.1. *For a space X the following assertions hold.*
(1) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda_D) = \mathrm{S}_{\mathrm{fin}}(\Omega, \Lambda_D)$
(2) $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^D) = \mathrm{S}_{\mathrm{fin}}(\Omega, \Lambda^D)$.

Proof. (1). It is easy to check that $\mathrm{S}_{\mathrm{fin}}(\Omega, \Lambda_D)$ implies $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda_D)$. Let X satisfy $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda_D)$. Let $(\mathcal{U}_n)$ be a sequence of ω-covers of X. Without loss of generality we may assume that for each finite $\mathcal{F} \subseteq \cup_{n\in\mathbb{N}}\mathcal{U}_n$, $\mathcal{U}_k \cap \mathcal{F} = \emptyset$ for all but finitely many k. For each n enumerate $\mathcal{U}_n$ bijectively as $\{U_m^{(n)} : m \in \mathbb{N}\}$. Next for each n and each m define $V_m^{(n)} = \cup_{1\leq i\leq m} U_i^{(n)}$. Then for each n $\mathcal{W}_n = \{V_m^{(n)} : m \in \mathbb{N}\}$ is a γ-cover of X. Apply the $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda_D)$ property of X to $(\mathcal{W}_n)$ to obtain a sequence $(\mathcal{H}_n)$ such that for each n $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{H}_n \in \Lambda_D$. Clearly the sequence $(\mathcal{H}_n)$ produces a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup\mathcal{V}_n = \cup\mathcal{H}_n$. Since each $\mathcal{V}_n$ is disjoint from $\mathcal{U}_k$ for all but finitely many k, $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \Lambda_D$. Thus X has the property $\mathrm{S}_{\mathrm{fin}}(\Omega, \Lambda_D)$.

The proof of (2) is similar to the proof of (1). □

Theorem 4.5. *Let X be a Lindelöf space satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Then X satisfies $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda_D)$ and each large cover of X is a member of $\mathcal{O}^{wgp}{}_D$.*

Proof. Let $(\mathcal{U}_n)$ be a sequence of γ-covers of X. Without loss of generality we can assume that $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$ for $m \neq n$. Since X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$, we apply Theorem 4.3(4). Thus we get a sequence $(\mathcal{V}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each finite collection $\mathcal{F}$ of nonempty open sets of X there is a n such that $U \cap (\cup\mathcal{V}_n) \neq \emptyset$ for all $U \in \mathcal{F}$. It is easy to observe that $\{\cup\mathcal{V}_n : n \in \mathbb{N}\} \in \Omega_D$ and hence $\{\cup\mathcal{V}_n : n \in \mathbb{N}\} \in \Lambda_D$. Thus we can conclude that $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \Lambda_D$ and consequently X satisfies $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda_D)$.

For the next part, first we pick a large cover $\mathcal{U}$ of X. We then enumerate $\mathcal{U}$ bijectively as $\{U_n : n \in \mathbb{N}\}$. Now $(\mathcal{W}_n)$ is a sequence of γ-covers of X, where $\mathcal{W}_n = \{\cup_{n<j\leq m} U_j : m \in \mathbb{N}\}$ for each n. We assume that $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$ for $m \neq n$. Again we use Theorem 4.3(4) to $(\mathcal{W}_n)$ to obtain a sequence $(\mathcal{H}_n)$ of pairwise disjoint finite sets such that for each n $\mathcal{H}_n \subseteq \mathcal{W}_n$ and for each finite collection $\mathcal{F}$ of nonempty open sets of X there is a n such that $U \cap (\cup\mathcal{H}_n) \neq \emptyset$ for all $U \in \mathcal{F}$.

Now define $k_0 = m_0 = n_0 = 1$ and continue as follows.

Choose $m_1 = 2$. Observe that $\mathcal{H}_1 \subseteq \cup_{j\leq m_1}\mathcal{H}_j$. Next choose $n_1 \geq m_1$ so small in such a way that if U_i is a term in the representation of an element of $\cup_{j\leq m_1}\mathcal{H}_j$, then $i < n_1$. Again choose $k_1 > n_1$ so that if $j \geq k_1$, then the following conditions are satisfied.

(1) If U_i is a term in the representation of an element of $\mathcal{H}_j$, then $i \geq n_1$;
(2) k_1 is minimal subject to 1 and $k_1 > n_1$.

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Next choose $m_2 = k_1 + 1$. Now take $n_2 \geq m_2$ so small such that if U_i is a term in the representation of an element of $\cup_{j \leq m_2} \mathcal{H}_j$, then $i < n_2$. Again choose $k_2 > n_2$ so that if $j \geq k_2$, then the following conditions are satisfied.

(1) If U_i is a term in the representation of an element of $\mathcal{H}_j$, then $i \geq n_2$;
(2) k_2 is minimal subject to 1 and $k_2 > n_2$.

For the general case, choose $m_{j+1} = k_j + 1$. Next take $n_{j+1} \geq m_{j+1}$ so small such that if U_l is a term in the representation of an element of $\cup_{i \leq m_{j+1}} \mathcal{H}_i$, then $l < n_{j+1}$. Again choose $k_{j+1} > n_{j+1}$ such that if $l \geq k_{j+1}$, then the following conditions are satisfied.

(1) If U_i is a term in the representation of an element of $\mathcal{H}_l$, then $i \geq n_{j+1}$;
(2) k_{j+1} is minimal subject to 1 and $k_{j+1} > n_{j+1}$.

For each n let $\mathcal{K}_n = \cup_{k_{n-1}+1 \leq j \leq k_n} \mathcal{H}_j$. It is easy to observe that for each m $\cup\mathcal{K}_m \subseteq \cup_{n_{m-1} \leq i \leq n_{m+1}} U_i$.

By the construction of $\mathcal{H}_i$'s we get for each finite collection $\mathcal{F}$ of nonempty open sets of X either there is a n such that $U \cap (\cup\mathcal{K}_{2n-1}) \neq \emptyset$ for all $U \in \mathcal{F}$, or for each finite collection $\mathcal{F}$ of nonempty open sets of X there is a n such that $U \cap (\cup\mathcal{K}_{2n}) \neq \emptyset$ for all $U \in \mathcal{F}$. In the first case the partition

$$(\{U_i : n_{2k-2} \leq i < n_{2k}, k \in \mathbb{N}\})$$

guarantees that $\mathcal{U} \in \mathcal{O}^{wgp}{}_D$.

In the latter case the partition

$$(\{U_i : n_{2k-1} \leq i < n_{2k+1}, k \in \mathbb{N}\})$$

guarantees that $\mathcal{U} \in \mathcal{O}^{wgp}{}_D$. Hence the result. □

Corollary 4.3. *If X is a Lindelöf $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ space, then X satisfies $\mathrm{S}_{\mathrm{fin}}(\Omega, \Lambda_D)$ and each large cover of X is a member of $\mathcal{O}^{wgp}{}_D$.*

The proof of the following theorem uses similar technique of Theorem 4.5 and so we omit it.

Theorem 4.6. *Let X be a Lindelöf space satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. Then X satisfies $\mathrm{S}_{\mathrm{fin}}(\Gamma, \Lambda^D)$ and each large cover of X is a member of $\mathcal{O}^{wgp D}$.*

Corollary 4.4. *Let X be a Lindelöf space satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$. Then X satisfies $\mathrm{S}_{\mathrm{fin}}(\Omega, \Lambda^D)$ and each large cover of X is a member of $\mathcal{O}^{wgp D}$.*

4.2. **Game theoretic observations**

We begin with the following game theoretic observation which will be used subsequently.

Theorem 4.7 ([19, 23]). *For a space X the following assertions are equivalent.*

(1) *X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ on X.*
(3) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda)$ on X.*

Let $\mathcal{O}^{\uparrow}$ denote the collection of all countable and increasing open covers of a space X.

Theorem 4.8. *For a Lindelöf space X the following games are equivalent.*

(1) $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$
(2) $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$
(3) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$.

Proof. (1) ⇔ (3). We only prove that winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ implies the winning strategy for ONE in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ and winning strategy for TWO in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ implies the winning strategy for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$.

Suppose that ONE has a winning strategy σ in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ on X. Let us define a strategy τ for ONE in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ on X as follows. Let $\sigma(\emptyset)$ be the first move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$. Since X is Lindelöf, we choose $\sigma(\emptyset) = \{U_m^{(1)} : m \in \mathbb{N}\}$. Consider $\tau(\emptyset) = \{\cup_{k=1}^m U_k^{(1)} : m \in \mathbb{N}\}$ as the first move of ONE in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ and TWO responds by choosing a $V_1 \in \tau(\emptyset)$. Then V_1 gives a finite subset $\mathcal{V}_1$ of $\sigma(\emptyset)$ such that $\cup\mathcal{V}_1 = V_1$ and choose $\mathcal{V}_1$ as the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$. Let $\sigma(\mathcal{V}_1)$ be the second move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ and so on. Thus we obtain a winning strategy τ for ONE in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ on X.

Suppose that TWO has a winning strategy σ in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ on X. We now define a strategy τ for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ on X as follows. Let $\mathcal{U}_1 = \{U_m^{(1)} : m \in \mathbb{N}\}$ be the first move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$. Choose $\mathcal{W}_1 = \{\cup_{k=1}^m U_k^{(1)} : m \in \mathbb{N}\}$ as the first move of ONE in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ and TWO responds by selecting $\sigma(\mathcal{W}_1) \in \mathcal{W}_1$. Clearly $\sigma(\mathcal{W}_1)$ gives a finite subset $\mathcal{V}_1$ of $\mathcal{U}_1$ such that $\sigma(\mathcal{W}_1) = \cup\mathcal{V}_1$ and consider $\tau(\mathcal{U}_1) = \mathcal{V}_1$ as the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$. Let $\mathcal{U}_2 = \{U_m^{(2)} : m \in \mathbb{N}\}$ be the second move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ and so on. This defines a winning strategy τ for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ on X.

Using the same technique as in the proof of (1) ⇔ (3) one can readily prove that (1) ⇔ (2). □

We intimately follow the proof of (1) ⇔ (3) of Theorem 4.8 to obtain the following.

Theorem 4.9. *For a Lindelöf space X the games $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \Lambda_D)$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ are equivalent.*

Theorem 4.10 ([2, Theorem 28])**.** *For a Lindelöf space X the following assertions are equivalent.*

(1) *X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on X.*

Theorem 4.11. *For a Lindelöf space X the following assertions are equivalent.*

(1) *X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X.*

Proof. (1) ⇒ (2). We closely follow the technique of [23, Corollary 4]. Let σ be a strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X. We now define a strategy τ for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on $X \times \mathbb{N}$ as follows. Let $\sigma(\emptyset)$ be the first move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X. We choose $\tau(\emptyset) = \{U \times \{n\} : U \in \sigma(\emptyset) \text{ and } n \in \mathbb{N}\}$ as the first move of ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on $X \times \mathbb{N}$ and then TWO responds by selecting a finite subset $\mathcal{V}_1$ of $\tau(\emptyset)$. Consider $\mathcal{H}_1 = \{U \in \sigma(\emptyset) : U \times \{n\} \in \mathcal{V}_1 \text{ for some } n \in \mathbb{N}\}$ as the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X. Let $\sigma(\mathcal{H}_1)$ be the second move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X and so on. Thus we get a legitimate strategy τ for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on $X \times \mathbb{N}$. Since $X \times \mathbb{N}$ satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$, by Theorem 4.10, τ is not a winning for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on $X \times \mathbb{N}$. We pick a τ-play $\tau(\emptyset), \mathcal{V}_1, \tau(\mathcal{V}_1), \mathcal{V}_2, \tau(\mathcal{V}_1, \mathcal{V}_2), \dots$ which is lost by ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on $X \times \mathbb{N}$. Thus $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}_D$ for $X \times \mathbb{N}$. The corresponding σ-play is given by $\sigma(\emptyset), \mathcal{H}_1, \sigma(\mathcal{H}_1), \mathcal{H}_2, \sigma(\mathcal{H}_1, \mathcal{H}_2), \dots$. We claim that $\{\cup\mathcal{H}_n : n \in \mathbb{N}\} \in \Lambda_D$ for X. Let V be a nonempty open set in X. Choose a $n_1 \in \mathbb{N}$ such that $(V \times \{1\}) \cap (\cup\mathcal{V}_{n_1}) \neq \emptyset$ and $V \cap (\cup\mathcal{H}_{n_1}) \neq \emptyset$. Clearly the set

$$F = \{m \in \mathbb{N} : U \times \{m\} \in \cup_{i=1}^{n_1}\mathcal{V}_i \text{ for some } U \in \cup_{i=1}^{n_1}\sigma(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{i-1})\}$$

is finite. Let $k = \max F + 1$. Choose a $n_2 \in \mathbb{N}$ such that $(V \times \{k\}) \cap (\cup\mathcal{V}_{n_2}) \neq \emptyset$. Accordingly $V \cap (\cup\mathcal{H}_{n_2}) \neq \emptyset$ and $n_1 < n_2$. Proceeding similarly we can say that

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$V \cap (\cup \mathcal{H}_n) \neq \emptyset$ for infinitely many n. Thus $\{\cup \mathcal{H}_n : n \in \mathbb{N}\} \in \Lambda_D$ for X. It follows that ONE loses the above σ-play and σ is not a winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X. Thus ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X. □

Corollary 4.5. *For a Lindelöf space X the following assertions are equivalent.*

(1) *X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ on X.*
(3) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \mathcal{O}_D)$ on X.*
(4) *ONE does not have a winning strategy in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \mathcal{O}_D)$ on X.*
(5) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda_D)$ on X.*
(6) *ONE does not have a winning strategy in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \Lambda_D)$ on X.*

The proof of the following two theorems can be obtained using the similar approach of (1) $\Leftrightarrow$ (3) of Theorem 4.8.

Theorem 4.12. *For a Lindelöf space X the games $\mathrm{G}_1(\mathcal{O}^{\uparrow}, \Omega_D)$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega_D)$ are equivalent.*

Theorem 4.13. *For a Lindelöf space X the games $\mathrm{G}_1(\mathcal{O}^{\uparrow}, {\Lambda^{wgp}}_D)$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ are equivalent.*

Theorem 4.14. *Let X satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$. Then the following assertions are equivalent.*

(1) *X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ on X.*

Proof. Suppose that X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Let σ be a strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ on X. Let us define a strategy τ for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda)$ on X as follows. In each inning, the move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda)$ is equal to the move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ and the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ is equal to the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda)$. Since X satisfies $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, by Theorem 4.7, τ is not a winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Lambda)$ on X, i.e., there exists a τ-play $\tau(\emptyset), \mathcal{V}_1, \tau(\mathcal{V}_1), \mathcal{V}_2, \tau(\mathcal{V}_1, \mathcal{V}_2), \mathcal{V}_3, \ldots$ such that $\{\cup \mathcal{V}_n : n \in \mathbb{N}\} \in \Lambda$. Since X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$, by Theorem 4.5, $\{\cup \mathcal{V}_n : n \in \mathbb{N}\} \in {\Lambda^{wgp}}_D$. Thus the corresponding σ-play $\sigma(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \sigma(\mathcal{V}_1, \mathcal{V}_2), \mathcal{V}_3, \ldots$ is lost by ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$. It follows that σ is not a winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$. Hence ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ on X. The other implication follows from Theorem 4.1. □

Corollary 4.6. *Let X satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$. Then the following assertions are equivalent.*

(1) *X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_1(\mathcal{O}^{\uparrow}, {\Lambda^{wgp}}_D)$ on X.*
(3) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ on X.*

Problem 4.1. *Let X be a Lindelöf space satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Is it true that ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega_D)$ on X?*

Problem 4.2. *Let X be a Lindelöf space satisfying $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$. Is it true that ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, {\Lambda^{wgp}}_D)$ on X?*

Theorem 4.15. *For a P-space X the games $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ and $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \overline{\mathcal{O}})$ are equivalent.*

Proof. It is enough to prove that winning strategy for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \overline{\mathcal{O}})$ implies the winning strategy for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$ and winning strategy for TWO in

$\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ implies the winning strategy for TWO in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$. Let σ be a winning strategy for ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ on X. Let us define a strategy τ for ONE in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ on X as follows. In each inning, the move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ is equal to the move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ and the response of TWO in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ is equal to the response of TWO in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$. Let $\tau(\emptyset),\mathcal{V}_1,\tau(\mathcal{V}_1),\mathcal{V}_2,\tau(\mathcal{V}_1,\mathcal{V}_2),\mathcal{V}_3,\dots$ be a τ-play in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ and $\sigma(\emptyset),\mathcal{V}_1,\sigma(\mathcal{V}_1),\mathcal{V}_2,\sigma(\mathcal{V}_1,\mathcal{V}_2),\mathcal{V}_3,\dots$ be the corresponding σ-play in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$. Since σ is a winning strategy for ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ on X, $\{\overline{V}: V\in\cup_{n\in\mathbb{N}}\mathcal{V}_n\}$ does not cover X. Observe that $\cup\{\overline{V}: V\in\cup_{n\in\mathbb{N}}\mathcal{V}_n\}=\overline{\cup(\cup_{n\in\mathbb{N}}\mathcal{V}_n)}$. It follows that $\cup_{n\in\mathbb{N}}\mathcal{V}_n\notin\mathcal{O}_D$ and hence the τ-play is won by ONE. Thus τ is a winning strategy for ONE in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ on X.

Let σ be a winning strategy for TWO in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ on X. Let us define a strategy τ for TWO in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ on X as follows. In each inning, the move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ is equal to the move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ and the response of TWO in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ is equal to the response of TWO in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$. Let $\mathcal{U}_1,\tau(\mathcal{U}_1),\mathcal{U}_2,\tau(\mathcal{U}_1,\mathcal{U}_2),\mathcal{U}_3,\dots$ be a τ-play in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ and $\mathcal{U}_1,\sigma(\mathcal{U}_1),\mathcal{U}_2,\sigma(\mathcal{U}_1,\mathcal{U}_2),\mathcal{U}_3,\dots$ be the corresponding σ-play in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$. Since σ is a winning strategy for TWO in $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$ on X, $\cup_{n\in\mathbb{N}}\sigma(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_n)\in\mathcal{O}_D$. Also since $\cup\{\overline{V}: V\in\cup_{n\in\mathbb{N}}\sigma(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_n)\}=\overline{\cup\sigma(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_n)}$ and for each n $\sigma(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_n)=\tau(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_n)$, $\cup_{n\in\mathbb{N}}\tau(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_n)\in\overline{\mathcal{O}}$. Thus the τ-play is won by TWO and hence τ is a winning strategy for TWO in $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ on X. □

Theorem 4.16 ([5]). *For a regular space X the games $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$ and $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O})$ are equivalent.*

Proof. It is easy to observe that ONE has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X implies that ONE has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X. Also TWO has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X implies that TWO has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X. We now show that if ONE has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X, then ONE has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X. Let σ be a winning strategy for ONE in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X. Let us define a strategy τ for ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X as follows. Suppose that $\sigma(\emptyset)$ is the first move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$. We can obtain an open cover $\mathcal{U}_0$ of X such that $\{\overline{U}: U\in\mathcal{U}_0\}$ refines $\sigma(\emptyset)$. Consider $\tau(\emptyset)=\mathcal{U}_0$ as the first move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ and TWO responds by choosing a finite subset $\mathcal{V}_1\subseteq\tau(\emptyset)$. For each $V\in\mathcal{V}_1$ choose a $U_V\in\sigma(\emptyset)$ such that $\overline{V}\subseteq U_V$ and put $\mathcal{H}_1=\{U_V: V\in\mathcal{V}_1\}$. Define $\mathcal{H}_1$ as the response of TWO in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$. Let $\sigma(\mathcal{H}_1)$ be the second move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ and so on. This defines a winning strategy τ for ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X.

Next we observe that if TWO has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X, then TWO has a winning strategy in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X. Let σ be a winning strategy for TWO in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ on X. We now define a strategy τ for TWO in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X as follows. Let $\mathcal{U}_1$ be the first move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$. Let $\mathcal{W}_1$ be an open cover of X such that $\{\overline{U}: U\in\mathcal{W}_1\}$ refines $\mathcal{U}_1$. Consider $\mathcal{W}_1$ as the first move of ONE in $\mathrm{G_{fin}}(\mathcal{O},\overline{\Omega})$ and TWO responds by choosing a finite subset $\sigma(\mathcal{W}_1)\subseteq\mathcal{W}_1$. For each $V\in\sigma(\mathcal{W}_1)$ choose a $U_V\in\mathcal{U}_1$ such that $\overline{V}\subseteq U_V$ and define $\tau(\mathcal{U}_1)=\{U_V: V\in\sigma(\mathcal{W}_1)\}$ as the response of TWO in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ and so on. Thus we get a winning strategy τ for TWO in $\mathrm{G_{fin}}(\mathcal{O},\Omega)$ on X. Hence the result. □

Corollary 4.7. *For a regular P-space X the following games are equivalent.*

(1) $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O}_D)$
(2) $\mathrm{G_{fin}}(\mathcal{O},\overline{\mathcal{O}})$
(3) $\mathrm{G_{fin}}(\mathcal{O},\mathcal{O})$.

Theorem 4.17. *For a P-space X the games $\mathrm{G_{ufin}}(\mathcal{O},\Omega_D)$ and $\mathrm{G_{ufin}}(\mathcal{O},\overline{\Omega})$ are equivalent.*

Proof. We only prove that winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ implies the winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ and winning strategy for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ implies the winning strategy for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$. Let σ be a winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ on X. Let us define a strategy τ for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ on X as follows. In each inning, the move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ is equal to the move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ and the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ is equal to the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$. Let $\tau(\emptyset),\mathcal{V}_1,\tau(\mathcal{V}_1),\mathcal{V}_2,\tau(\mathcal{V}_1,\mathcal{V}_2),\mathcal{V}_3,\dots$ be a τ-play in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ and $\sigma(\emptyset),\mathcal{V}_1,\sigma(\mathcal{V}_1),\mathcal{V}_2,\sigma(\mathcal{V}_1,\mathcal{V}_2),\mathcal{V}_3,\dots$ be the corresponding σ-play in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$. Since σ is a winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ on X, $\{\cup\mathcal{V}_n : n\in\mathbb{N}\}\notin\overline{\Omega}$. We claim that $\{\cup\mathcal{V}_n : n\in\mathbb{N}\}\notin\Omega_D$. Let $F=\{x_i : 1\le i\le k\}$ be a finite subset of X such that for each n $F\nsubseteq\overline{\cup\mathcal{V}_n}$. Then we can choose $\mathbb{N}=\cup_{i=1}^k N_i$, where N_i's are pairwise disjoint such that for each $n\in N_i$, $x_i\notin\overline{\cup\mathcal{V}_n}$, $i=1,2,\dots,k$. For each $i=1,2,\dots,k$ and each $n\in N_i$ let $V_i^{(n)}$ be an open set in X containing x_i such that $V_i^{(n)}\cap(\cup\mathcal{V}_n)=\emptyset$. Since X is a P-space, for each $i=1,2,\dots,k$, $V_i=\cap_{n\in N_i}V_i^{(n)}$ is an open set in X with $V_i\cap(\cup\mathcal{V}_n)=\emptyset$ for all $n\in N_i$. Thus we get a family $\mathcal{F}=\{V_i : 1\le i\le k\}$ of nonempty open sets of X such that there does not exist any $n\in\mathbb{N}$ with $U\cap(\cup\mathcal{V}_n)\neq\emptyset$ for all $U\in\mathcal{F}$. Hence $\{\cup\mathcal{V}_n : n\in\mathbb{N}\}\notin\Omega_D$. It follows that the τ-play is won by ONE and consequently τ is a winning strategy for ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ on X.

Let σ be a winning strategy for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ on X. Let us define a strategy τ for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ on X as follows. In each inning, the move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$ is equal to the move of ONE in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ and the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ is equal to the response of TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$. Similarly we can observe that τ is a winning strategy for TWO in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ on X. $\square$

The proof of the following result is similar to Theorem 4.16 and so we omit it.

Theorem 4.18 ([5]). *For a regular space X the games $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega)$ are equivalent.*

Corollary 4.8. *For a regular P-space X the following games are equivalent.*

(1) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega_D)$
(2) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\overline{\Omega})$
(3) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega)$.

Following the same line of proof of Theorems 4.12, 4.13 and 4.14 with necessary modifications we can prove the next three theorems respectively.

Theorem 4.19. *For a Lindelöf space X the games $\mathrm{G}_1(\mathcal{O}^{\uparrow},\Omega^D)$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Omega^D)$ are equivalent.*

Theorem 4.20. *For a Lindelöf space X the games $\mathrm{G}_1(\mathcal{O}^{\uparrow},\Lambda^{wgpD})$ and $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Lambda^{wgpD})$ are equivalent.*

Theorem 4.21. *Let X satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Then the following assertions are equivalent.*

(1) *X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Lambda^{wgpD})$ on X.*

Corollary 4.9. *Let X satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Then the following assertions are equivalent.*

(1) *X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O},\Omega^D)$.*
(2) *ONE does not have a winning strategy in $\mathrm{G}_1(\mathcal{O}^{\uparrow},\Lambda^{wgpD})$ on X.*
(3) *ONE does not have a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O},\Lambda^{wgpD})$ on X.*

Since $\Omega^D\subseteq\Omega_D$, the next result follows directly from Theorem 4.17.

Theorem 4.22. *For a P-space X the following assertions hold.*

(1) *If ONE has a winning strategy in* $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \overline{\Omega})$ *on* X*, then ONE has a winning strategy in* $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ *on* X*.*

(2) *If TWO has a winning strategy in* $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ *on* X*, then TWO has a winning strategy in* $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \overline{\Omega})$ *on* X*.*

Theorem 4.23. *For a regular P-space X the games* $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ *and* $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$ *are equivalent.*

Proof. If ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ on X, then ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$ on X since $\Omega \subseteq \Omega^D$. Let us suppose that ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$ on X. By Theorem 4.18, ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \overline{\Omega})$ on X. Then by Theorem 4.22(1), ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ on X. Thus ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ on X if and only if ONE has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$ on X. Similarly using Theorems 4.18 and 4.22(2) we can see that TWO has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$ on X if and only if TWO has a winning strategy in $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$ on X. Hence the result. □

Corollary 4.10. *For a regular P-space X the following games are equivalent.*

(1) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega_D)$
(2) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \overline{\Omega})$
(3) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega^D)$
(4) $\mathrm{G}_{\mathrm{ufin}}(\mathcal{O}, \Omega)$.

5. Open Problems

We give a short proof of the following result in the context of almost Lindelöf spaces.

Theorem 5.1 ([5]). *Every almost Lindelöf space X with cardinality less than* $\mathfrak{d}$ *satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$*.*

Proof. Let $(\mathcal{U}_n)$ be a sequence of open covers of X. For each n choose a countable set $\mathcal{W}_n = \{V_m^{(n)} : m \in \mathbb{N}\} \subseteq \mathcal{U}_n$ such that $\cup_{V \in \mathcal{W}_n} \overline{V} = X$. Now for each $x \in X$ define a $f_x \in \mathbb{N}^{\mathbb{N}}$ by $f_x(n) = \min\{m \in \mathbb{N} : x \in \overline{V_m^{(n)}}\}$, $n \in \mathbb{N}$. Since the cardinality of $Y = \{f_x : x \in X\}$ is less than $\mathfrak{d}$, $\mathsf{maxfin}(Y)$ is also of cardinality less than $\mathfrak{d}$. Consequently there are a $g \in \mathbb{N}^{\mathbb{N}}$ and a $n_F \in \mathbb{N}$ corresponding to each finite set $F \subseteq X$ such that $f_F(n_F) < g(n_F)$ with $f_F \in \mathsf{maxfin}(Y)$, where $f_F(n) = \max\{f_x(n) : x \in F\}$ for all $n \in \mathbb{N}$. We use the convention that if $F = \{x\}$, $x \in X$, then we write f_x instead of f_F. The sequence $(\mathcal{V}_n)$ now witnesses for X to be $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \overline{\Omega})$. □

So the following question naturally arises.

Problem 5.1. *If X is a weakly Lindelöf space with cardinality less than* $\mathfrak{d}$*, then does X satisfy* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$ *(or,* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$*,* $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$*)?*

We present two more open problems for further investigation.

Problem 5.2. *Give an example of a space which satisfies* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ *but does not satisfy* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega^D)$*.*

Problem 5.3. *Does every productively weakly Lindelöf space satisfy* $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Omega_D)$ *(or,* $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$*)?*

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