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BOUNDEDNESS THEOREMS AND FUNCTION SPACES OF DISCRETE FRACTIONAL CALCULUS

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ABSTRACT. This paper investigates the boundedness of discrete fractional calculus. A finite-dimensional real vector space is considered and the p-norm of finite dimensions is used. By utilizing the Minkowski inequality on an isolated time scale, the boundedness theorems of fractional sums and differences in both nabla and delta types are provided. The h case is also discussed. If the step–size h tends to zero, the result is consistent with the continuous case.

1 Introduction

The boundedness of operators is important in functional analysis. Kilbas [1] gave the classical fractional integral's boundedness theorem in the space L_p which consists of complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{L_p} < \infty$. The norm $\|\cdot\|_{L_p}$ is defined as

$$\|f\|_{L_p} = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p} \, (1 \leqslant p < \infty)$$

and

$$\|f\|_\infty = ess \sup_{a \leqslant t \leqslant b} |f(t)| .$$

The boundedness of the R-L integral was derived as (see Lemma 2.1, pp. 72 in [1])

$$\|{}_aI_t^\alpha f\|_{L_p} \leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L_p}$$

where $\alpha > 0$ is the fractional order.

In view of this point, the boundedness theorem was also discussed for the general fractional calculus in X_c^p [2]. The space $X_c^p(a,b)$ is defined to consist of those complex-valued Lebesgue measurable functions on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, with

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} \, (1 \leqslant p < \infty, c \in \mathbb{R})$$

and

$$\|f\|_{X_c^\infty} = ess \sup_{a \leqslant t \leqslant b} [|t^c f(t)|] \, (p = \infty).$$

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The general fractional integral is defined by

$$ {}_aI_t^{\alpha,g} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} g'(s) f(s) ds, $$

where the kernel function $g(t)$ is chosen according to the boundedness theorem (see Theorem 2.4 of [2]).

It can be concluded that the fractional calculus is called to be well-defined if the boundedness theorem can hold. Recently, the discrete fractional calculus (see Definition 5) is important for fractional difference equations [3, 4]. However, the boundedness of the fractional sums and differences were not provided yet. As a result, this paper tries to give the result and the function spaces.

2 Preliminaries

Suppose $\mathbb{N}_a := \{a, a+1, \ldots\}$ and $(h\mathbb{N})_a := \{a, a+h, \ldots\}$, $h > 0$, $a \in \mathbb{R}$. For any $\nu \in \mathbb{R}$, the falling and rising factorial functions are defined by [4]

$$ t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},\ t \in \mathbb{N}_\nu, $$

$$ t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)},\ t \in \mathbb{N}_1, $$

and the h-falling factorial function is defined [5]

$$ t_h^{\underline{\nu}} = h^\nu \frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\nu)},\ t \in (h\mathbb{N})_{\nu h}, $$

where Γ denotes the famous Gamma function.

The following proposition of the falling factorial function is useful for the study of the paper.

Proposition 1. [6] *Let $a \in \mathbb{R}, b \in \mathbb{N}_a, a < b$ and $\nu > 0$. Then the following equation holds*

$$ \sum_{\tau=a-b+1}^{0} (-\tau+\nu-1)^{\underline{\nu-1}} = \frac{(b-a+\nu-1)^{\underline{\nu}}}{\nu}. $$

The forward and backward differences are defined as follows

$$ \Delta f(t) = f(t+1) - f(t),\ \nabla f(t) = f(t) - f(t-1). $$

If a function $f : (h\mathbb{N})_a \to \mathbb{R}$, the differences are defined as

$$ \Delta_h f(t) = \frac{f(t+h)-f(t)}{h},\ \nabla_h f(t) = \frac{f(t)-f(t-h)}{h}, $$

respectively.

More generally, a time scale $\mathbb{T}$ is defined to be any closed subset of $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by [7]

$$ \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, $$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by [7]

$$ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},\ t \in \mathbb{T}. $$

Theorem 2. [8] *(Holder inequality) Assume f and $g : [a,b) \to \mathbb{R}$ are rd-continuous functions.*
(i) If $a \in \mathbb{R}$, $\mathbb{T} = \mathbb{N}_a$ and $b \in \mathbb{N}_a$, then

$$\sum_{t=a}^{b-1} |f(t)g(t)| \leqslant \left(\sum_{t=a}^{b-1} |f(t)|^p\right)^{\frac{1}{p}} \left(\sum_{t=a}^{b-1} |g(t)|^q\right)^{\frac{1}{q}}, \ t \in \mathbb{N}_a.$$

(ii) If $a \in \mathbb{R}$, $\mathbb{T} = (h\mathbb{N})_a$ and $b \in (h\mathbb{N})_a$, then

$$\sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} |f(th)g(th)|h \leqslant \left(\sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} |f(th)|^p h\right)^{\frac{1}{p}} \left(\sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} |g(th)|^q h\right)^{\frac{1}{q}}, \ t \in (h\mathbb{N})_a.$$

where $p > 1$ and $q = p/(p-1)$.

Theorem 3. [9] *(Minkowski inequality) Let $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 = [a,b) \times [c,d) = \{(x,y) : x \in [a,b)$ and $y \in [c,d)\}$. Assume $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function.*
(i) If $a, c \in \mathbb{R}$, $\mathbb{T}_1 = \mathbb{N}_a$, $\mathbb{T}_2 = \mathbb{N}_c$, $b \in \mathbb{N}_a$ and $d \in \mathbb{N}_c$, then

$$\left(\sum_{x=a}^{b} \left|\sum_{y=c}^{d} f(x,y)\right|^p\right)^{\frac{1}{p}} \leqslant \sum_{y=c}^{d} \left(\sum_{x=a}^{b} |f(x,y)|^p\right)^{\frac{1}{p}}, \ x \in \mathbb{N}_a, \ y \in \mathbb{N}_c.$$

(ii) If $a, c \in \mathbb{R}$, $\mathbb{T}_1 = (h\mathbb{N})_a$, $\mathbb{T}_2 = (h\mathbb{N})_c$, $b \in (h\mathbb{N})_a$ and $d \in (h\mathbb{N})_c$, then

$$\left(\sum_{x=\frac{a}{h}}^{\frac{b}{h}-1} \left|\sum_{y=\frac{c}{h}}^{\frac{d}{h}-1} f(xh,yh)h\right|^p h\right)^{\frac{1}{p}} \leqslant \sum_{y=\frac{c}{h}}^{\frac{d}{h}-1} \left(\sum_{x=\frac{a}{h}}^{\frac{b}{h}-1} |f(xh,yh)|^p h\right)^{\frac{1}{p}} h, \ x \in (h\mathbb{N})_a, \ y \in (h\mathbb{N})_c.$$

where $p > 1$ and $q = \frac{p}{p-1}$.

Definition 4. [10] *Let $1 \leqslant p \leqslant \infty$ and $0 < a < b < \infty$. The space $L_p(\mathbb{T})$ is defined to consist of those complex-valued Lebesgue measurable functions. The following norms are defined on $\mathbb{T}$.*
(i) If $\mathbb{T} = \mathbb{N}_a$ and $b \in \mathbb{N}_a$, then

$$\|f\|_{L_p} = \left(\sum_{t=a}^{b-1} |f(t)|^p\right)^{\frac{1}{p}}, \ f \in L_p, \ 1 \leqslant p < \infty.$$

(ii) If $\mathbb{T} = (h\mathbb{N})_a$ and $b \in (h\mathbb{N})_a$, then

$$\|f\|_{L_p} = \left(\sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} |f(th)|^p h\right)^{\frac{1}{p}}, \ f \in L_p, \ 1 \leqslant p < \infty$$

and

$$\|f\|_\infty = ess \sup_{a \leqslant t \leqslant b} |f(t)|, \ f \in L_p, \ p = \infty.$$

3 Boundedness theorem of fractional sums

3.1 *Fractional sums of delta type*

Definition 5. [4] *Let $f:\mathbb{N}_a\to\mathbb{R}$ and $\nu>0$ be given. Then the ν-th order delta fractional sum of f is given by*

$$\Delta_a^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}f(s),\ t\in\mathbb{N}_{a+\nu},$$

where $\sigma(s)=s+1$ and $a\in\mathbb{R}$ is fixed.

Theorem 6. *For $\nu>0$ and $1\leqslant p<\infty$, the delta fractional sum $\Delta_a^{-\nu}f$ is bounded in $L_p(\Omega_1)$*

$$\|\Delta_a^{-\nu}f\|_{L_p}\leqslant\frac{(b-a+\nu-1)^{\underline{\nu}}}{\Gamma(\nu+1)}\|f\|_{L_p},\tag{1}$$

where $\Omega_1=\{a,a+1,\cdots,b-1\}$.

For $p=\infty$, the delta fractional sum $\Delta_a^{-\nu}f$ is bounded in L_∞

$$\|\Delta_a^{-\nu}f\|_\infty\leqslant\frac{(t-a)^{\underline{\nu}}}{\Gamma(\nu+1)}\|f\|_\infty.\tag{2}$$

Proof. According to the domain of the fractional sum $\Delta_a^{-\nu}f(t)$, let $t\in\{a+\nu,a+1+\nu,\cdots,b-1+\nu\}$, then

$$\begin{aligned}\|\Delta_a^{-\nu}f\|_{L_p}&=\left(\sum_{t=a+\nu}^{b-1+\nu}\left|\sum_{s=a}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}f(s)\right|^p\right)^{\frac{1}{p}}\\&=\left(\sum_{t=a}^{b-1}\left|\sum_{s=a}^{t}\frac{(t+\nu-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}f(s)\right|^p\right)^{\frac{1}{p}}.\end{aligned}\tag{3}$$

Replace the variable with $\tau=s-t$

$$\|\Delta_a^{-\nu}f\|_{L_p}=\left(\sum_{t=a}^{b-1}\left|\sum_{\tau=a-t}^{0}\frac{(-\tau+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)}f(t+\tau)\right|^p\right)^{\frac{1}{p}}.$$

Then, using the Minkowski inequality of Theorem 3, we give

$$\begin{aligned}\|\Delta_a^{-\nu}f\|_{L_p}&\leqslant\sum_{\tau=a-b+1}^{0}\left(\sum_{t=a-\tau}^{b-1}\left|\frac{(-\tau+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)}f(t+\tau)\right|^p\right)^{\frac{1}{p}}\\&=\sum_{\tau=a-b+1}^{0}\frac{(-\tau+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)}\left(\sum_{t=a-\tau}^{b-1}|f(t+\tau)|^p\right)^{\frac{1}{p}}.\end{aligned}\tag{4}$$

By interchange of variables again, we obtain

$$\begin{aligned}\|\Delta_a^{-\nu} f\|_{L_p} &\leqslant \sum_{\tau=a-b+1}^{0} \frac{(-\tau+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)} \left(\sum_{s=a}^{b-1+\tau} |f(s)|^p\right)^{\frac{1}{p}} \\ &\leqslant \sum_{\tau=a-b+1}^{0} \frac{(-\tau+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)} \left(\sum_{s=a}^{b-1} |f(s)|^p\right)^{\frac{1}{p}}.\end{aligned} \tag{5}$$

From Proposition 1, we arrive at

$$\|\Delta_a^{-\nu} f\|_{L_p} \leq \frac{(b-a+\nu-1)^{\underline{\nu}}}{\Gamma(\nu+1)} \|f\|_{L_p}.$$

For $p=\infty$,

$$\begin{aligned}\left|\Delta_a^{-\nu} f(t)\right| &= \left|\sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} f(s)\right| \\ &\leqslant \sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} |f(s)| \\ &\leqslant \sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} \|f\|_{\infty}.\end{aligned} \tag{6}$$

Due to Proposition 1,

$$\sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} = \frac{(t-a)^{\underline{\nu}}}{\Gamma(\nu+1)}, \tag{7}$$

consequently, we give

$$\|\Delta_a^{-\nu} f\|_{\infty} \leqslant \frac{(t-a)^{\underline{\nu}}}{\Gamma(\nu+1)} \|f\|_{\infty},\ t \in \mathbb{N}_{a+\nu}$$

from which the proof is completed. □

Since the proof of the case $p=\infty$ is relatively easy, we only discuss the case $1 \leqslant p < \infty$ in the rest of this study.

Goodrich studied the continuity of solutions to discrete fractional initial value problems [12] where the norm is the absolute value $|\cdot|$. We investigate the boundedness theorems with the norm $\|\cdot\|_{L_p}$. They are clearly different. A concept of l^p solution was given in [13] and the norm $\|\cdot\|_{L_p}$ should be used. In addition, we can compare two norms' roles through the solutions' dependence.

Suppose there exists a unique solution of the initial value problem of the fractional difference equation

$$\begin{cases} {}^C\Delta_a^{\nu} x(t) = F(x(t+\nu-1),\ t+\nu-1),\ t \in \mathbb{N}_{a+1-\nu},\ 0<\nu\leqslant 1, \\ x(a) = C. \end{cases} \tag{8}$$

$F: \mathbb{R}\times\mathbb{N}_a \to \mathbb{R}$, $F(x,t)$ is continuous with respect to t and x. It satisfies the Lipschitz condition

$$\|F(x,\ t) - F(y,\ t)\|_{L_p} \leq L\|x-y\|_{L_p}.$$

The solution satisfies the fractional sum equation

$$x(t) = x(a) + \Delta^{-\nu}_{a+1-\nu} F(x(t+\nu-1),\, t+\nu-1),\ t \in \mathbb{N}_{a+1}.$$

Considering a minor change in $x(a)$, we have a new initial value $\tilde{x}(a)$ and

$$\tilde{x}(t) = \tilde{x}(a) + \Delta^{-\nu}_{a+1-\nu} F(\tilde{x}(t+\nu-1),\, t+\nu-1),\ t \in \mathbb{N}_{a+1}.$$

The differences between the two solutions from a to $b-1$ are estimated by

$$\|x(t)-\tilde{x}(t)\|_{L_p} \leqslant \|x(a)-\tilde{x}(a)\|_{L_p} + \|\Delta^{-\nu}_{a+1-\nu}(F(x,\, t+\nu-1) - F(\tilde{x},\, t+\nu-1))\|_{L_p}. \tag{9}$$

According to Theorem 6 and the Lipschitz condition, we give

$$\|x(t)-\tilde{x}(t)\|_{L_p} \leqslant \|x(a)-\tilde{x}(a)\|_{L_p} + KL\|x(t)-\tilde{x}(t)\|_{L_p}$$

where $K = \frac{(b-a+\nu-1)^{\nu}}{\Gamma(\nu+1)}$ and $0 < KL < 1$. As a result, we arrive at the global estimation from a to $b-1$

$$\|x(t)-\tilde{x}(t)\|_{L_p} = \left(\sum_{t=a}^{b-1} |x(t)-\tilde{x}(t)|^p\right)^{\frac{1}{p}} \leq \frac{\|x(a)-\tilde{x}(a)\|_{L_p}}{1-KL},\ t \in \mathbb{N}_{a+1}. \tag{10}$$

On the other hand, if we use the absolute value norm, we have

$$|x(t)-\tilde{x}(t)| \leqslant |x(a)-\tilde{x}(a)| + L\Delta^{-\nu}_{a+1-\nu}|x(t+\nu-1)-\tilde{x}(t+\nu-1)|. \tag{11}$$

With the delay discrete–time Mittag-Leffler function

$$e_\nu(\lambda, (t-\sigma(a))^{(\nu)}) := \sum_{k=0}^{\infty} \frac{\lambda^k (t-a+k\nu-k)^{(k\nu)}}{\Gamma(k\nu+1)},\ 0<\nu\leq 1,\ t \in \mathbb{N}_{a+1},$$

we give the following Gronwall inequality for the delayed fractional difference equation (8).

Lemma 7. [14] *Let η and L be two non-negative constants. If $u : \mathbb{N}_a \to \mathbb{R}$ satisfies*

$$u(t) \leqslant \eta + L\Delta^{-\nu}_{a+1-\nu} u(t+\nu-1),\ t \in \mathbb{N}_{a+1},$$

then $u(t)$ is bounded by

$$u(t) \leqslant \eta e_\nu(L, (t-\sigma(a))^{(\nu)}).$$

As a result, we obtain

$$|x(t)-\tilde{x}(t)| \leqslant |x(a)-\tilde{x}(a)| e_\nu(L, (t-\sigma(a))^{(\nu)}),\ t \in \mathbb{N}_{a+1}$$

which is a point-wise estimation result for each time t. It can be concluded that they are different and both of the two norms are useful in real–world applications of fractional difference equations.

3.2 *Fractional sums of nabla type*

Definition 8. [4, 11] *Let $\nu > 0$ be given. Then the ν-th order nabla fractional sum of f is given by*

$$\nabla_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\nu-1}} f(s),\ t \in \mathbb{N}_{a+1}.$$

Theorem 9. *For $\nu > 0$, $1 \leqslant p < \infty$, the nabla fractional sum $\nabla_a^{-\nu} f$ is bounded in $L_p(\Omega_2)$*

$$\|\nabla_a^{-\nu} f\|_{L_p} \leqslant \frac{(b-a)^{\overline{\nu}}}{\Gamma(\nu+1)} \|f\|_{L_p},$$

where $\Omega_2 = \{a+1, a+2, \cdots, b\}$.

Proof. Use a change of variable $\tau = s - t$, then

$$\|\nabla_a^{-\nu} f\|_{L_p} = \left(\sum_{t=a+1}^{b} \left| \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \right|^p \right)^{\frac{1}{p}}$$
$$= \left(\sum_{t=a+1}^{b} \left| \sum_{\tau=a+1-t}^{0} \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} f(t+\tau) \right|^p \right)^{\frac{1}{p}}.$$

By using of the Minkowski inequality,

$$\|\nabla_a^{-\nu} f\|_{L_p} \leqslant \sum_{\tau=a+1-b}^{0} \left(\sum_{t=a+1-\tau}^{b} \left| \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} f(t+\tau) \right|^p \right)^{\frac{1}{p}}$$
$$= \sum_{\tau=a+1-b}^{0} \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \left(\sum_{t=a+1-\tau}^{b} |f(t+\tau)|^p \right)^{\frac{1}{p}}.$$

With $s = t + \tau$, we have

$$\|\nabla_a^{-\nu} f\|_{L_p} \leqslant \sum_{\tau=a+1-b}^{0} \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \left(\sum_{s=a+1}^{b+\tau} |f(s)|^p \right)^{\frac{1}{p}}$$
$$\leqslant \sum_{\tau=a+1-b}^{0} \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \left(\sum_{s=a+1}^{b} |f(s)|^p \right)^{\frac{1}{p}}$$
$$= \frac{(b-a)^{\overline{\nu}}}{\Gamma(\nu+1)} \|f\|_{L_p},$$

where $\sum_{\tau=a+1-b}^{0} \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)}$ is a fractional sum and its result reads

$$\sum_{\tau=a+1-b}^{0} \frac{(-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} = \frac{(b-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

As a result,

$$\|\nabla_a^{-\nu} f\|_{L_p} \leqslant \frac{(b-a)^{\overline{\nu}}}{\Gamma(\nu+1)} \|f\|_{L_p},$$

from which the proof is completed. □

4 Boundedness theorem of fractional differences

Let us revisit the definitions of the fractional differences.

Definition 10. [4] *Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ and $n-1 < \nu \leqslant n$. The ν-th order R-L difference of f is defined by*

$$\begin{aligned}\Delta_a^\nu f(t) &= \Delta^n \Delta_a^{-(n-\nu)} x(t)\\ &= \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} x(s),\ t \in \mathbb{N}_{a+n-\nu}.\end{aligned}$$

Definition 11. [4] *Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ and $n-1 < \nu \leqslant n$. The ν-th order Caputo difference of f is defined as*

$$\begin{aligned}{}^C\Delta_a^\nu f(t) &= \Delta_a^{-(n-\nu)} \Delta^n f(t)\\ &= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-\sigma(s))^{\underline{n-\nu-1}} \Delta^n f(s),\ t \in \mathbb{N}_{a+n-\nu}.\end{aligned}$$

Theorem 12. *For $n-1 < \nu \leqslant n$ and $1 \leqslant p < \infty$, the R-L difference $\Delta_a^\nu f$ is bounded in $L_p(\Omega_3)$*

$$\|\Delta_a^\nu f\|_{L_p} \leqslant \frac{(b-a-\nu-1)^{\underline{-\nu}}}{\Gamma(-\nu+1)} \|f\|_{L_p},$$

where $\Omega_3 = \{a, a+1, \ldots, b-1-n\}$.

Proof. Similarly, the R-L difference can be rewritten as

$$\begin{aligned}\|\Delta_a^\nu f\|_{L_p} &= \left(\sum_{t=a+n-\nu}^{b-1-\nu} \left| \sum_{s=a}^{t+\nu} \frac{(t+n-\nu-\sigma(s))^{\underline{-\nu-1}}}{\Gamma(-\nu)} f(s) \right|^p \right)^{\frac{1}{p}}\\ &= \left(\sum_{t=a}^{b-1-n} \left| \sum_{s=a}^{t+n} \frac{(t+n-\nu-\sigma(s))^{\underline{-\nu-1}}}{\Gamma(-\nu)} f(s) \right|^p \right)^{\frac{1}{p}}\\ &= \left(\sum_{t=a}^{b-1-n} \left| \sum_{\tau=a-t}^{n} \frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)} f(t+\tau) \right|^p \right)^{\frac{1}{p}}.\end{aligned}$$

The Minkowski inequality is employed to give

$$\begin{aligned}\|\Delta_a^\nu f\|_{L_p} \leqslant & \sum_{\tau=a-b+1+n}^{0}\left(\sum_{t=a-\tau}^{b-1-n}\left|\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}f(t+\tau)\right|^p\right)^{\frac{1}{p}}+\\ & \sum_{\tau=1}^{n}\left(\sum_{t=a}^{b-1-n}\left|\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}f(t+\tau)\right|^p\right)^{\frac{1}{p}}\\ = & \sum_{\tau=a-b+1+n}^{0}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{t=a-\tau}^{b-1-n}|f(t+\tau)|^p\right)^{\frac{1}{p}}+\\ & \sum_{\tau=1}^{n}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{t=a}^{b-1-n}|f(t+\tau)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

By interchange of variables, it can be presented as

$$\begin{aligned}\|\Delta_a^\nu f\|_{L_p} \leqslant & \sum_{\tau=a-b+1+n}^{0}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{s=a}^{b-1-n+\tau}|f(s)|^p\right)^{\frac{1}{p}}+\\ & \sum_{\tau=1}^{n}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{s=a+\tau}^{b-1-n+\tau}|f(s)|^p\right)^{\frac{1}{p}}\\ \leqslant & \sum_{\tau=a-b+1+n}^{0}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{s=a}^{b-1-n}|f(s)|^p\right)^{\frac{1}{p}}+\\ & \sum_{\tau=1}^{n}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{s=a}^{b-1}|f(s)|^p\right)^{\frac{1}{p}}\\ \leqslant & \sum_{\tau=a-b+1+n}^{n}\frac{(-\tau+n-\nu-1)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\left(\sum_{s=a}^{b-1}|f(s)|^p\right)^{\frac{1}{p}}\\ = & \frac{(b-a-\nu-1)^{\underline{-\nu}}}{\Gamma(-\nu+1)}\|f\|_{L_p},\end{aligned}$$

the proof is completed. □

Theorem 13. *For $n-1<\nu\leqslant n$ and $1\leqslant p<\infty$, the Caputo difference ${}^{C}\Delta_a^\nu f$ is bounded in $L_p(\Omega_3)$*

$$\|{}^{C}\Delta_a^\nu f\|_{L_p} \leqslant \frac{(b-a+n-\nu-1)^{\underline{n-\nu}}}{\Gamma(n-\nu+1)}\|\Delta^n f\|_{L_p}.$$

5 Boundedness theorem of h-discrete fractional calculus

Definition 14. [5, 15] *Let $f:(h\mathbb{N})_a \to \mathbb{R}$ and $\nu > 0$ be given. Then the ν-th order h sum of f is given by*

$$_h\Delta_a^{-\nu} f(t) = \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu} (t-\sigma(sh))_h^{\underline{\nu-1}} f(sh),\ \sigma(sh) = (s+1)h,\ t \in (h\mathbb{N})_{a+\nu h}.$$

Definition 15. [5, 15] *Let $f:(h\mathbb{N})_a \to \mathbb{R}$ and $n-1 < \nu \leqslant n$. Then the ν-th order R-L h-difference of f is defined by*

$$_h\Delta_a^{\nu} f(t) = \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t-\sigma(sh))_h^{\underline{-\nu-1}} f(sh),\ \sigma(sh) = (s+1)h,\ t \in (h\mathbb{N})_{a+(n-\nu)h}.$$

Definition 16. [5, 15] *Let $f:(h\mathbb{N})_a \to \mathbb{R}$ and $n-1 < \nu \leqslant n$. Then the ν-th order Caputo h-difference of f is defined by*

$$^C_h\Delta_a^{\nu} f(t) = \frac{h}{\Gamma(n-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(n-\nu)} (t-\sigma(sh))_h^{\underline{n-\nu-1}} \Delta_h^n f(sh),\ \sigma(sh) = (s+1)h,\ t \in (h\mathbb{N})_{a+(n-\nu)h}.$$

We use the same idea for boundedness of the discrete fractional calculus on the isolate time scale $\mathbb{N}_a$. So we extend it to the case of $(h\mathbb{N})_a$ directly and give the following theorems without proof.

Theorem 17. *For $0 < \nu$ and $1 \leqslant p < \infty$, the ν-th order h-sum $_h\Delta_a^{-\nu} f$ is bounded in $L_p(\Omega_4)$*

$$\|_h\Delta_a^{-\nu} f\|_{L_p} \leqslant \frac{(b-a+\nu h-h)_h^{\underline{\nu}}}{\Gamma(\nu+1)} \|f\|_{L_p},$$

where $\Omega_4 = \{a, a+h, \ldots, b-h\}$.

Theorem 18. *For $n-1 < \nu \leqslant n$, $1 \leqslant p < \infty$, the R-L h-difference $_h\Delta_a^{\nu} f$ is bounded in $L_p(\Omega_5)$*

$$\|_h\Delta_a^{\nu} f\|_{L_p} \leqslant \frac{(b-a-\nu h-h)_h^{\underline{-\nu}}}{\Gamma(-\nu+1)} \|f\|_{L_p},$$

where $\Omega_5 = \{a, a+h, \ldots, b-h-nh\}$.

Theorem 19. *For $n-1 < \nu \leqslant n$ and $1 \leqslant p < \infty$, the Caputo h-difference $^C_h\Delta_a^{\nu} f$ is bounded in $L_p(\Omega_5)$*

$$\|^C_h\Delta_a^{\nu} f\|_{L_p} \leqslant \frac{(b-a+(n-\nu)h-h)_h^{\underline{n-\nu}}}{\Gamma(n-\nu+1)} \|\Delta_h^n f\|_{L_p}.$$

6 Boundedness theorem of the continuous fractional calculus

The boundedness results can be reduced to that of the continuous fractional calculus (see Lemma 2.1 of [1]).

Theorem 20. *For $n-1 < \nu \leqslant n$, $h \to 0$ and $1 \leqslant p < \infty$, the R-L integral is bounded in $L_p(\Omega_6)$*

$$\|_aI_t^{\nu} f\|_{L_p} \leqslant \frac{(b-a)^{\nu}}{\Gamma(\nu+1)} \|f\|_{L_p}.$$

Proof. We have

$$\lim_{h\to 0} {}_h\Delta_a^{-\nu} f(t) = {}_aI_t^\nu f(t)$$

and

$$\|{}_aI_t^\nu f\|_{L_p} \leqslant \lim_{h\to 0} \frac{(b-a+\nu h-h)^{\frac{\nu}{h}}}{\Gamma(\nu+1)} \|f\|_{L_p}.$$

The approximation formula of the Beta function holds

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \sim \Gamma(x)y^{-x},$$

when y is large and x is fixed.

The approximation formula can be rewritten as

$$\frac{\Gamma(x+y)}{\Gamma(y)} \sim y^x,$$

therefore

$$\begin{aligned}\lim_{h\to 0}(b-a+\nu h-h)^{\frac{\nu}{h}} =& \lim_{h\to 0} h^\nu \frac{\Gamma(\frac{b-a+\nu h-h}{h}+1)}{\Gamma(\frac{b-a+\nu h-h}{h}+1-\nu)}\\ =& \lim_{h\to 0} h^\nu (\frac{b-a+\nu h-h}{h}+1-\nu)^\nu\\ =&(b-a)^\nu.\end{aligned}$$

As a result, we obtain

$$\|{}_aI_t^\nu f\|_{L_p} \leqslant \frac{(b-a)^\nu}{\Gamma(\nu+1)} \|f\|_{L_p},$$

from which the proof is completed. □

Theorem 21. *For $n-1<\nu\leqslant n$ and $1\leqslant p<\infty$, the Caputo derivative ${}_a^CD_t^\nu f$ is bounded in $L_p(\Omega_6)$*

$$\|{}_a^CD_t^\nu f\|_{L_p} \leqslant \frac{(b-a)^{n-\nu}}{\Gamma(n-\nu+1)} \|f^{(n)}\|_{L_p}.$$

Conclusion

The boundedness of discrete fractional calculus is given in this paper. It is discussed in space $L_p(\mathbb{T})$ on an isolated time scale which unifies both the continuous and discrete–time cases: For $h=1$, the results can be reduced to the standard discrete fractional calculus; For h tends to zero, the boundedness theorem meets that of the fractional calculus [1] in $L_p[a,b]$ space. The discrete fractional calculus's definitions are provided with the function space $L_p(\mathbb{T})$ in which the bounded theorems can hold. In addition, we use the boundedness theorem in dependence of solutions on initial values. These results are useful for numerical analysis and stability theory of fractional difference equations. We will consider these possible applications in future work.

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