GLOBAL REGULARITY FOR THE 3D AXISYMMETRIC NON-RESISTIVE MHD SYSTEM WITH NONLINEAR DAMPING

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ABSTRACT. This paper is devoted to the study of the Cauchy problem of the 3D non-resistive MHD system with a velocity damping term $|u|^{\beta-1}u$. We prove that the solution of this system with $1 \leq \beta \leq \frac{7}{3}$ is globally well-posed if the initial data is axisymmetric and the swirl component of the velocity and the magnetic vorticity vanish. It should be pointed out that we develop the technique about the local-in-space estimate for solutions and the special axisymmetric initial data can be arbitrarily large.

Keywords: Non-resistive MHD system; Damping term; Axisymmetric solutions; Global regularity

AMS Subject Classification (2000): 35Q35, 76W05, 35B65.

1. Introduction

In this paper, we consider the following 3D non-resistive MHD system with nonlinear damping

$$
\begin{cases}
\partial_t u + u\cdot\nabla u + \nabla P - \Delta u + |u|^{\beta-1}u = b\cdot\nabla b, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t b + u\cdot\nabla b - b\cdot\nabla u = 0,\\
\operatorname{div} u = \operatorname{div} b = 0,\\
(u,b)|_{t=0} = (u_0,b_0), \quad x\in\mathbb{R}^3.
\end{cases} \tag{1.1}
$$

Here u and b denote the velocity and the magnetic field respectively. P is the scalar pressure. The term $|u|^{\beta-1}u$ with $\beta \geq 1$ is called as damping term, which comes from the resistance to the motion of the flow. It describes various physical phenomena such as porous media flow, drag or friction effects, and some dissipative mechanisms. For more physical explanations about (1.1), see [3, 4, 18] and references therein.

Before proceeding, let us introduce the definition of the axisymmetric vector fields. We call a vector field f is axisymmetric if it has the form:

$$f(t,x) = f^r(t,r,z)e_r + f^\theta(t,r,z)e_\theta + f^z(t,r,z)e_z,$$

where $x=(x_1,x_2,z)$, $r=\sqrt{x_1^2+x_2^2}$ and

$$e_r = (\frac{x_1}{r},\frac{x_2}{r},0), \qquad e_\theta = (-\frac{x_2}{r},\frac{x_1}{r},0), \qquad e_z=(0,0,1).$$

We say f^θ is the swirl component and f is axisymmetric without swirl if $f^\theta = 0$.

When the damping term is absent, the system (1.1) reduces to the non-resistive MHD equations. About the well-posedness of non-resistive MHD equations, Fefferman et al. showed the local-in-time existence of strong solutions with the initial data $(u_0,b_0)\in H^s(\mathbb{R}^n)$, $s>\frac{n}{2}$ $(n=2,3)$ in [8], and $(u_0,b_0)\in H^{s-1+\varepsilon}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)$, $s>\frac{n}{2}$ $(n=2,3)$, $0<\varepsilon<1$ in

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[9]. However, the global regularity problem remains unsolved. Thus there are many works devoted to the study of solutions with the geometric structure axisymmetric. Lei in [11] gave an important result that he proved the non-resistive MHD equations exist a unique global solution if the initial data satisfies $u_0^\theta = b_0^r = b_0^z$, $(u_0, b_0) \in H^2(\mathbb{R}^3)$ and $\frac{b_0^\theta}{r} \in L^\infty(\mathbb{R}^3)$. Later on, the initial data regularity was weakened to $(u_0, b_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ and $\frac{\nabla \times u_0}{r} \in L^2(\mathbb{R}^3)$ by Ai and Li in [2]. Other interesting results on the non-resistive MHD equations can be found in [6, 13, 15, 16, 17, 20].

If the magnetic field $b = 0$, (1.1) is reduced to the incompressible damped Navier-Stokes equations. Such system was studied firstly by Cai and Jiu in [5], they obtained the existence of a global weak solution for any $\beta \geq 1$ and global strong solutions for any $\beta \geq \frac{7}{2}$. Moreover, the strong solution is unique for any $\frac{7}{2} \leq \beta \leq 5$. In [22], Zhou showed the strong solution exists globally for $\beta \geq 3$ and established the regularity criteria for any $1 \leq \beta < 3$. Later on, Zhong in [21] proved the global well-posedness of strong solution for $1 \leq \beta < 3$ under some smallness condition. It should be pointed out that the good effect of the damping becomes weaker in the case $1 < \beta < 3$. Until now, the global well-posedness of the 3D damped Navier-Stokes system with large initial data for $1 \leq \beta < 3$ is still unsolved. Recently, under axisymmetric without swirl assumptions, Yu in [19] established the global well-posedness of the damped Navier-Stokes system for $1 \leq \beta \leq \frac{7}{3}$ with large initial data.

For the case of (1.1) with magnetic resistivity, in the periodic domain, Titi and Trabelsi in [18] investigated the existence of global weak solutions for any $\beta \geq 1$ and well-posedness of global smooth solutions for any $\beta \geq 4$.

Inspired by [2, 18, 19], the main purpose of this paper is to establish the global well-posedness for the non-resistive MHD system with nonlinear damping corresponding to large axisymmetric data. Our result reads as follows.

Theorem 1.1. *Assume that u_0 and b_0 are two axisymmetric divergence free vector fields with $u_0^\theta = b_0^r = b_0^z = 0$. Let $(u_0, b_0) \in H^2(\mathbb{R}^3)$ and $\frac{b_0^\theta}{r} \in L^\infty(\mathbb{R}^3)$. Then there exists a unique global solution (u, b) to system (1.1) with $1 \leq \beta \leq \frac{7}{3}$ satisfying*

$$\begin{aligned} &u \in L^\infty(0, T; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)), \\ &b \in L^\infty(0, T; H^2(\mathbb{R}^3)), \end{aligned}$$

for any $0 < T < \infty$.

Remark 1.1. (i). *If $b = 0$, (1.1) is reduced to the incompressible Navier-Stokes equations with damping in $\mathbb{R}^3$. Our result can be regarded as a generalisation a result proved by Yu in* [19].
(ii). *We emphasize that the $L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^\infty(0, T; H^1(\mathbb{R}^3))$ estimate of $\frac{\omega^\theta}{r}$ can not be directly obtain due to the appearance of the velocity damping term. Thus, we need to assume the condition $\frac{b_0^\theta}{r} \in L^\infty(\mathbb{R}^3)$, which is only used in* (3.9).
(iii). *It should be noted that we only prove the global regularity when $1 \leq \beta \leq \frac{7}{3}$. For the case of $\frac{7}{3} < \beta < 3$, since the nonlinearity of $|u|^{\beta-1}u$ is much stronger, it is not enough to control the nonlinear term only by the dissipative term. In the future, we will further study how to use the good effect of $|u|^{\beta-1}u$ to obtain the global regularity with $\frac{7}{3} < \beta < 3$.*

Let us now explain the scheme of the proof. The proof of the main result deeply depends on the special structure of the system (1.1) in axisymmetric case whose the swirl component of velocity and magnetic vorticity are trivial. First, we need to get the $L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^\infty(0, T; H^1(\mathbb{R}^3))$ estimate of $(\frac{\omega^\theta}{r}, \omega^\theta)$. Motivated by [19], we develop the local-in-space estimate for solutions to deal with nonlinear terms due to the appearance of the magnetic field, for more details see Proposition 3.2 below. Next, we derive the $L^1(0, T; L^\infty(\mathbb{R}^3))$ estimate of ∇u by using the maximal regularity result of the heat flow, then the $L^\infty(0, T; L^6(\mathbb{R}^3))$ estimate of ∇b follow, see Proposition 3.3 below. Finally, based on those preparations, the H^2 estimate of the solution can be established by deeply using commutator estimates.

The remainder of the paper is organized as follows. In Section 2, we introduce the system (1.1) in cylindrical coordinates and state some useful facts. Section 3 is devoted to proving Theorem 1.1.

Throughout the paper, we shall use the notation $\int \cdot dx$ for $\int_{\mathbb{R}^3} \cdot dx$ and $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(\mathbb{R}^3)}$. In addition, we denote $\nabla_h := (\partial_1, \partial_2)$ and the letter C to denote a generic positive constant, which may vary from line to line.

2. Preliminaries

At the beginning, we introduce the system (1.1) in cylindrical coordinates and some useful facts. Considering the system does not have the swirl component for velocity field and magnetic vorticity. This means solution of the form:

$$\begin{cases} u(t,x) = u^r(t,r,z)e_r + u^z(t,r,z)e_z, \\ b(t,x) = b^\theta(t,r,z)e_\theta, \\ P(t,x) = P(t,r,z). \end{cases}$$

Similar to the local existence and uniqueness result for the 3D incompressible Navier-Stokes equations in [12], see also [21], it is not difficult to have the following local well-posedness result for the system (1.1).

Lemma 2.1. *Let $(u_0, b_0) \in H^2(\mathbb{R}^3)$ be axisymmetric divergence free vector fields. Then there exists $T > 0$ and a unique solution (u, b) to the system (1.1) with $\beta \geq 1$ such that*

$$u \in L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3)),$$
$$b \in L^\infty(0,T; H^2(\mathbb{R}^3)).$$

From the uniqueness of solutions, we see that $u_0^\theta = b_0^r = b_0^z = 0$ implies $u^\theta = b^r = b^z = 0$ for all later time. Thus, in this case, (1.1) can be written into the following

$$\begin{cases} \partial_t u^r + (u^r\partial_r + u^z\partial_z)u^r + |u|^{\beta-1}u^r + \partial_r P = (\Delta - \frac{1}{r^2})u^r - \frac{(b^\theta)^2}{r}, \\ \partial_t u^z + (u^r\partial_r + u^z\partial_z)u^z + |u|^{\beta-1}u^z + \partial_z P = \Delta u^z, \\ \partial_t b^\theta + (u^r\partial_r + u^z\partial_z)b^\theta = \frac{u^r b^\theta}{r}, \\ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0. \end{cases} \tag{2.1}$$

Notice that the vorticity of the swirl free axisymmetric velocity is given by

$$\omega = \nabla \times u = \omega^\theta e_\theta = (\partial_z u^r - \partial_r u^z)e_\theta$$

and satisfying

$$\begin{aligned} &\partial_t\omega^\theta + u\cdot\nabla\omega^\theta - \Delta\omega^\theta + \frac{1}{r^2}\omega^\theta - \frac{1}{r}\omega^\theta u^r + |u|^{\beta-1}\omega^\theta \\ &\quad + (\beta-1)|u|^{\beta-3}((u^r)^2\partial_z u^r + u^r u^z\partial_z u^z - u^r u^z\partial_r u^r - (u_z)^2\partial_r u^z) = -\partial_z\frac{(b^\theta)^2}{r}. \end{aligned} \tag{2.2}$$

Moreover, we introduce two new quantities $\Gamma := \frac{\omega^\theta}{r}$ and $\Pi := \frac{b^\theta}{r}$. It follows from (2.1) and (2.2) that (Γ, Π) obeys the following system:

$$\begin{cases} \partial_t\Gamma + u\cdot\nabla\Gamma - (\Delta + \frac{2}{r}\partial_r)\Gamma + |u|^{\beta-1}\Gamma + (\beta-1)|u|^{\beta-3}(u^z)^2\Gamma + \partial_z\Pi^2 \\ \qquad +(\beta-1)|u|^{\beta-3}\left((u^r)^2\partial_z(\frac{u^r}{r}) - (u^z)^2\partial_z(\frac{u^r}{r}) - 2u^r u^z\partial_r(\frac{u^r}{r}) - 3(\frac{u^r}{r})^2 u^z\right) = 0, \\ \partial_t\Pi + u\cdot\nabla\Pi = 0. \end{cases} \tag{2.3}$$

In order to obtain the $L^\infty(0,T; L^2(\mathbb{R}^3))$ estimate for $\frac{\omega^\theta}{r}$, we need the following two lemmas.

Lemma 2.2 ([12]). *Let u be a smooth axisymmetric vector field with $u \in L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3))$ and $\omega = \omega^\theta e_\theta$ its curl. Then*
(i). $\frac{\omega^\theta}{r^{2-\varepsilon}}$ and $\frac{1}{r^{1-\varepsilon}}\frac{\partial\omega^\theta}{\partial r}$ belong to $L^2(0,T; L^2(\mathbb{R}^3))$ for all $\varepsilon > 0$.

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(ii). Let $g_1(\eta)=\int_{-\infty}^{\infty}(\eta^\delta|\frac{\omega^\theta}{\eta}|^2)(\eta,z)\,dz$ *and* $g_2(\eta)=\int_{-\infty}^{\infty}(\eta^\delta|\frac{\partial\omega^\theta}{\partial r}|^2)(\eta,z)\,dz$*. Then* g_1 *and* g_2 *are bounded for any* $\delta\in(0,2)$*.*

Lemma 2.3 ([12])**.** *For any* $\varepsilon>0$*, there holds* $\lim\limits_{\eta\to0^+}\int_{-\infty}^{\infty}\frac{\partial\omega^\theta}{\partial r}\frac{\omega^\theta}{\eta^{1-\varepsilon}}(\eta,z)\,dz=0.$

3. Proof of Theorem 1.1

In this section, we will give some a priori estimates and then complete the proof of Theorem 1.1. We first give the basic L^2 estimate of (u,b), which does not need the axisymmetric assumption.

Proposition 3.1. *Suppose* $1\le\beta\le\frac{7}{3}$*. Let* (u,b) *be the smooth solution of* (1.1) *with* $(u_0,b_0)\in L^2$*. Then we have for any* $t\in\mathbb{R}^+$*,*

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\int_0^t\|\nabla u(\tau)\|_{L^2}^2\,d\tau+\int_0^t\int|u(\tau)|^{\beta+1}\,dxd\tau\le\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2.$$

Proof. Taking the L^2 inner product of the first and second equations of (1.1) with u and b respectively, integrating over $\mathbb{R}^3$ and adding up, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\int|u|^{\beta+1}\,dx=0.$$

Integrating on time yields the desired result. □

Next, we intend to give some estimates for Π, Γ and ω^θ.

Proposition 3.2. *Suppose* $1\le\beta\le\frac{7}{3}$*. Let* (u,b) *be the smooth solution of* (1.1) *with* $(u_0,b_0)\in H^2(\mathbb{R}^3)$ *and* $\frac{b_0^\theta}{r}\in L^\infty(\mathbb{R}^3)$*, which satisfy the conditions of Theorem 1.1. Then there holds*

$$\|\Pi(t)\|_{L^p}\le\|\Pi_0\|_{L^p},\quad\forall\,2\le p\le\infty,\tag{3.1}$$

$$\|\Gamma(t)\|_{L^2}^2+\int_0^T\|\nabla\Gamma(t)\|_{L^2}^2\,dt\le C,\tag{3.2}$$

and

$$\|\omega^\theta(t)\|_{L^2}^2+\int_0^T\left(\|\nabla\omega^\theta(t)\|_{L^2}^2+\|\frac{\omega^\theta}{r}(t)\|_{L^2}^2\right)dt\le C.\tag{3.3}$$

Here the constant C *depends on* T*,* $\|u_0\|_{H^2}$*,* $\|b_0\|_{H^2}$ *and* $\|\frac{b_0^\theta}{r}\|_{L^\infty}$*.*

Proof. Since Π satisfies the homogeneous transport equation, then we can get by standard process that

$$\|\Pi(t)\|_{L^p}\le\|\Pi_0\|_{L^p},\quad\forall\,2\le p<\infty,$$

which along with the maximum principle gives rise to the inequality (3.1).

Taking the L^2 inner product of the first equation of (2.3) with $\frac{\omega^\theta}{r^{1-\varepsilon}}$ and integrating by parts, we infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2+\|\nabla(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}})\|_{L^2}^2+(\varepsilon-\frac{\varepsilon^2}{4})\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\|_{L^2}^2+\int|u|^{\beta-1}|\frac{\omega^\theta}{r^{1-\varepsilon}}|^2\,dx\\&+(\beta-1)\int|u|^{\beta-3}(u^z)^2|\frac{\omega^\theta}{r^{1-\varepsilon}}|^2\,dx=\frac{\varepsilon}{2}\int\frac{u^r}{r}\frac{(\omega^\theta)^2}{r^{2-\varepsilon}}\,dx-\int\partial_z\Pi^2\frac{\omega^\theta}{r^{1-\varepsilon}}\,dx\\&-(\beta-1)\int|u|^{\beta-3}\left((u^r)^2\partial_z(\frac{u^r}{r})-(u^z)^2\partial_z(\frac{u^r}{r})-2u^ru^z\partial_r(\frac{u^r}{r})-3(\frac{u^r}{r})^2u^z\right)\frac{\omega^\theta}{r^{1-\varepsilon}}\,dx\\&:=\sum_{i=1}^6I_i,\end{aligned}\tag{3.4}$$

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where we have used Lemma 2.2 and Lemma 2.3.

In the following, we estimate I_i term by term. The term I_1 is from the convective term, it follows from Hölder's inequality, Young's inequality and the Agmon inequality (see [1])

$$\|f\|_{L^\infty} \le C\|\nabla f\|_{L^2}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2}^{\frac{1}{2}}$$

that

$$\begin{aligned}
I_1 &\le \frac{\varepsilon}{2}\int |u^r||\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}||\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}|\,dx \\
&\le \frac{\varepsilon}{2}\|u\|_{L^\infty}\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\|_{L^2} \\
&\le \frac{\varepsilon}{4}\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\|_{L^2}^2 + C\varepsilon\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2.
\end{aligned}$$

For I_2,

$$I_2 = \int \Pi^2\partial_z\left(\frac{\omega^\theta}{r^{1-\varepsilon}}\right)dx \le (\int_{r\le 1} + \int_{r>1})\Pi^2|\partial_z\frac{\omega^\theta}{r^{1-\varepsilon}}|\,dx := I_{21} + I_{22}.$$

From the Hölder inequality and Young inequality, we get

$$I_{21} \le \int_{r\le 1}\Pi^2|\frac{\partial_z\omega^\theta}{r}|\,dx \le C\||\Pi|^2\|_{L^2}\|\partial_z\frac{\omega^\theta}{r}\|_{L^2} \le C\|\Pi\|_{L^4}^4 + \frac{1}{10}\|\nabla\frac{\omega^\theta}{r}\|_{L^2}^2,$$

and

$$I_{22} \le \int_{r>1}\Pi^2|\partial_z\omega^\theta|\,dx \le C\||\Pi|^2\|_{L^2}\|\partial_z\omega^\theta\|_{L^2} \le C\|\Pi\|_{L^4}^4 + \frac{1}{8}\|\nabla\omega^\theta\|_{L^2}^2.$$

Thus,

$$I_2 \le C\|\Pi\|_{L^4}^4 + \frac{1}{8}\|\nabla\omega^\theta\|_{L^2}^2 + \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2.$$

For I_3, we recall the following estimate, see Lemma 2.5 of [14],

$$\|\nabla\frac{u^r}{r}\|_{L^p} \le C(p)\|\frac{\omega^\theta}{r}\|_{L^p}, \quad \forall\, 1<p<\infty.$$

This together with Hölder's inequality, Young's inequality and the interpolation estimate

$$\|f\|_{L^{\frac{4}{3-\beta}}} \le \|f\|_{L^2}^{\frac{7-3\beta}{4}}\|\nabla f\|_{L^2}^{\frac{3\beta-3}{4}}, \quad \forall\, 1\le\beta\le\frac{7}{3}$$

ensure

$$\begin{aligned}
I_3 &\le C\int |u|^{\beta-1}|\partial_z\frac{u^r}{r}||\frac{\omega^\theta}{r^{1-\varepsilon}}|\,dx \\
&\le C(\int_{r\le 1} + \int_{|u|\le 1, r>1} + \int_{|u|>1, r>1})|u|^{\beta-1}|\partial_z\frac{u^r}{r}||\frac{\omega^\theta}{r^{1-\varepsilon}}|\,dx \\
&\le C\||u|^{\beta-1}\|_{L^{\frac{2}{\beta-1}}}\|\partial_z\frac{u^r}{r}\|_{L^{\frac{4}{3-\beta}}}\|\frac{\omega^\theta}{r}\|_{L^{\frac{4}{3-\beta}}} + C\|\omega^\theta\|_{L^2}\|\partial_z\frac{u^r}{r}\|_{L^2} + C\||u|^{\frac{4}{3}}\|_{L^3}\|\partial_z\frac{u^r}{r}\|_{L^6}\|\omega^\theta\|_{L^2} \\
&\le C\|u\|_{L^2}^{\beta-1}\|\frac{\omega^\theta}{r}\|_{L^{\frac{4}{3-\beta}}}^2 + C\|\nabla u\|_{L^2}\|\frac{\omega^\theta}{r}\|_{L^2} + C\|u\|_{L^4}^{\frac{4}{3}}\|\frac{\omega^\theta}{r}\|_{L^6}\|\omega^\theta\|_{L^2} \\
&\le C\|u\|_{L^2}^{\beta-1}\|\Gamma\|_{L^2}^{\frac{7-3\beta}{2}}\|\nabla\Gamma\|_{L^2}^{\frac{3\beta-3}{2}} + C\|\nabla u\|_{L^2}\|\Gamma\|_{L^2} + C\|u\|_{L^2}^{\frac{1}{3}}\|\nabla u\|_{L^2}\|\nabla\Gamma\|_{L^2}\|\omega^\theta\|_{L^2} \\
&\le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{4(\beta-1)}{7-3\beta}}\|\Gamma\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\Gamma\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{2}{3}}\|\nabla u\|_{L^2}^2\|\omega^\theta\|_{L^2}^2,
\end{aligned}$$

which along with Proposition 3.1 implies that

$$I_3 \le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|\Gamma\|_{L^2}^2 + C(1+\|\omega^\theta\|_{L^2}^2)\|\nabla u\|_{L^2}^2.$$

Following the same line as the estimate of I_3, one can conclude that

$$I_4+I_5\le\frac{1}{10}\|\nabla\Gamma\|_{L^2}^2+C\|\Gamma\|_{L^2}^2+C(1+\|\omega^\theta\|_{L^2}^2)\|\nabla u\|_{L^2}^2.$$

For I_6, recall the following estimate, see (2.3) of [7],

$$\|\frac{u^r}{r}\|_{L^\infty}\le C\|\frac{\omega^\theta}{r}\|_{L^2}^{\frac12}\|\nabla\frac{\omega^\theta}{r}\|_{L^2}^{\frac12}. \tag{3.5}$$

This together with Hölder's inequality and Young's inequality lead to

$$\begin{aligned}
I_6&\le C\int|u|^{\beta-2}|\frac{u^r}{r}|^2|\frac{\omega^\theta}{r^{1-\varepsilon}}|\,dx\\
&\le C(\int_{|u|>1,r>1}+\int_{|u|>1,r\le1}+\int_{|u|\le1,r>1}+\int_{|u|\le1,r\le1})|u|^{\beta-2}|\frac{u^r}{r}|^2|\frac{\omega^\theta}{r^{1-\varepsilon}}|\,dx\\
&\le C\int_{|u|>1,r>1}|u|^{\beta-1}|\frac{u^r}{r}||\omega^\theta|\,dx+C\int_{|u|>1,r\le1}|u|^{\beta-2}|\frac{u^r}{r}|^2|\frac{\omega^\theta}{r}|\,dx+C\int_{|u|\le1,r>1}|u|^{\beta-2}|\frac{u^r}{r}|^2|\omega^\theta|\,dx\\
&\quad+C\int_{|u|\le1,r\le1}|u|^{\beta-2}|\frac{u^r}{r}|^2|\frac{\omega^\theta}{r}|\,dx\\
&\le C\int_{|u|>1,r>1}|u|^{\frac43}|\frac{u^r}{r}||\omega^\theta|\,dx+C\int_{|u|>1,r\le1}|u|^{\frac13}|\frac{u^r}{r}|^2|\frac{\omega^\theta}{r}|\,dx+C\int_{|u|\le1,r>1}|u|^{\beta-1}|\frac{u^r}{r}||\omega^\theta|\,dx\\
&\quad+C\int_{-\infty}^{+\infty}(\int_{|u|\le1,r\le1}|\frac{u^r}{r}||\frac{\omega^\theta}{r}|\frac{1}{r}\,dx_1dx_2)\,dx_3\\
&\le C\|\frac{u^r}{r}\|_{L^\infty}\|u\|_{L^{\frac83}}^{\frac43}\|\omega^\theta\|_{L^2}+C\|u\|_{L^2}^{\frac13}\|\frac{u^r}{r}\|_{L^\infty}\|\frac{u^r}{r}\|_{L^2}\|\frac{\omega^\theta}{r}\|_{L^3}+C\|\frac{u^r}{r}\|_{L^2}\|\omega^\theta\|_{L^2}\\
&\quad+C\int_{-\infty}^{+\infty}\|\frac{u^r}{r}\|_{L^6(\mathbb{R}^2)}\|\frac{\omega^\theta}{r}\|_{L^6(\mathbb{R}^2)}\|\frac1r\|_{L^{\frac32}(\mathbb{R}^2)\cap\{r\le1\}}\,dx_3\\
&\le C\|\frac{\omega^\theta}{r}\|_{L^2}^{\frac12}\|\nabla\frac{\omega^\theta}{r}\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac56}\|\nabla u\|_{L^2}^{\frac12}\|\omega^\theta\|_{L^2}+C\|u\|_{L^2}^{\frac13}\|\frac{\omega^\theta}{r}\|_{L^2}\|\nabla\frac{\omega^\theta}{r}\|_{L^2}\|\frac{u^r}{r}\|_{L^2}+C\|\nabla u\|_{L^2}^2\\
&\quad+C\int_{-\infty}^{+\infty}\|\frac{u^r}{r}\|_{L^2(\mathbb{R}^2)}^{\frac13}\|\nabla_h\frac{u^r}{r}\|_{L^2(\mathbb{R}^2)}^{\frac23}\|\frac{\omega^\theta}{r}\|_{L^2(\mathbb{R}^2)}^{\frac13}\|\nabla_h\frac{\omega^\theta}{r}\|_{L^2(\mathbb{R}^2)}^{\frac23}\,dx_3\\
&\le C\|u\|_{L^2}^{\frac56}\|\nabla u\|_{L^2}^{\frac32}\|\Gamma\|_{L^2}^{\frac12}\|\nabla\Gamma\|_{L^2}^{\frac12}+C\|u\|_{L^2}^{\frac13}\|\Gamma\|_{L^2}\|\nabla\Gamma\|_{L^2}\|\nabla u\|_{L^2}+C\|\nabla u\|_{L^2}^2\\
&\quad+C\|\frac{u^r}{r}\|_{L^2}^{\frac13}\|\nabla_h\frac{u^r}{r}\|_{L^2}^{\frac23}\|\Gamma\|_{L^2}^{\frac13}\|\nabla_h\Gamma\|_{L^2}^{\frac23}\\
&\le\frac{1}{10}\|\nabla\Gamma\|_{L^2}^2+C\|u\|_{L^2}^{\frac{10}{9}}\|\nabla u\|_{L^2}^2\|\Gamma\|_{L^2}^{\frac23}+C\|u\|_{L^2}^{\frac23}\|\Gamma\|_{L^2}^2\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\frac{u^r}{r}\|_{L^2}^{\frac12}\|\Gamma\|_{L^2}^{\frac32}.
\end{aligned}$$

We deduce from Proposition 3.1 that

$$I_6\le\frac{1}{10}\|\nabla\Gamma\|_{L^2}^2+C\|\Gamma\|_{L^2}^2(1+\|\nabla u\|_{L^2}^2)+C\|\nabla u\|_{L^2}^2.$$

Substituting the above estimates into (3.4), we get from (3.1) that

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2+\|\nabla(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}})\|_{L^2}^2+(\frac{3\varepsilon}{4}-\frac{\varepsilon^2}{4})\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\|_{L^2}^2\\
&\le C\varepsilon\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2+C\|\Pi_0\|_{L^4}^4+\frac18\|\nabla\omega^\theta\|_{L^2}^2+\frac12\|\nabla\Gamma\|_{L^2}^2\\
&\quad+C(1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2+\|\omega^\theta\|_{L^2}^2)+C\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Consequently, integrating the above estimate with respect to time from 0 to t with $0 \leq t \leq T$, we have

$$\begin{aligned}
&\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2 + 2\int_0^T \|\nabla(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}})\|_{L^2}^2\,dt \\
&\leq C\|\frac{\omega_0^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2 + C\varepsilon\int_0^T \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2}^2\,dt + C\|\Pi_0\|_{L^4}^4 T \\
&\quad + \frac{1}{4}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt + \int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt + C\int_0^T \|\nabla u\|_{L^2}^2\,dt \\
&\quad + C\int_0^T (1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2 + \|\omega^\theta\|_{L^2}^2)\,dt.
\end{aligned} \tag{3.6}$$

Noting that

$$|\nabla b|^2 = |\nabla b^\theta|^2 + |\Pi|^2,$$

and using Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(2 \leq p \leq 6)$, one has

$$\|\Pi_0\|_{L^p} \leq C\|\nabla b_0\|_{L^p} \leq C\|b_0\|_{H^2}, \qquad \forall\, 2 \leq p \leq 6.$$

In addition, we know that $|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}| \leq |\frac{\omega^\theta}{r}| + |\omega^\theta|$, then for $t \geq 0$,

$$\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\|_{L^2} \leq \|\frac{\omega^\theta}{r}\|_{L^2} + \|\omega^\theta\|_{L^2}.$$

Therefore, combing with Lemma 2.1, we can pass to the limit as $\varepsilon \to 0$ at the right hand side of (3.6). Passing $\varepsilon \to 0$ in (3.6) and by virtue of the Lebesgue dominated theorem, it leads to

$$\begin{aligned}
\|\Gamma\|_{L^2}^2 + \int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt &\leq C\|u_0\|_{H^2}^2 + CT + \frac{1}{4}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt \\
&\quad + C\int_0^T (1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2 + \|\omega^\theta\|_{L^2}^2)\,dt.
\end{aligned} \tag{3.7}$$

On the other hand, we multiply the equation (2.2) by ω^θ and integrate over $\mathbb{R}^3$ to derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega^\theta\|_{L^2}^2 + \|\nabla\omega^\theta\|_{L^2}^2 + \|\frac{\omega^\theta}{r}\|_{L^2}^2 + \int |u|^{\beta-1}(\omega^\theta)^2\,dx \\
&= -(\beta-1)\int |u|^{\beta-3}\big((u^r)^2\partial_z u^r + u^r u^z\partial_z u^z - u^r u^z\partial_r u^r - (u_z)^2\partial_r u^z\big)\omega^\theta\,dx \\
&\quad + \int \frac{u^r}{r}|\omega^\theta|^2\,dx - \int \partial_z\frac{(b^\theta)^2}{r}\omega^\theta\,dx.
\end{aligned} \tag{3.8}$$

Thanks to the following well-known Biot-Savart law, for the proof see (3.8) of [2],

$$\|\nabla u\|_{L^p} \leq C\frac{p^2}{p-1}\|\omega\|_{L^p}, \quad \forall\, 1 < p < +\infty,$$

we get

$$\begin{aligned}
&(\beta-1)\int |u|^{\beta-3}\left((u^r)^2\partial_z u^r + u^r u^z\partial_z u^z - u^r u^z\partial_r u^r - (u_z)^2\partial_r u^z\right)\omega^\theta\,dx\\
&\le (\beta-1)\int |u|^{\beta-1}|\nabla u||\omega^\theta|\,dx\\
&\le C\||u|^{\beta-1}\|_{L^{\frac{\beta+1}{\beta-1}}}\|\nabla u\|_{L^{\beta+1}}\|\omega^\theta\|_{L^{\beta+1}}\\
&\le C\|u\|_{L^{\beta+1}}^{\beta-1}\|\omega^\theta\|_{L^{\beta+1}}^2\\
&\le C\|u\|_{L^{\beta+1}}^{\beta-1}\|\omega^\theta\|_{L^2}^{\frac{5-\beta}{\beta+1}}\|\nabla\omega^\theta\|_{L^2}^{\frac{3(\beta-1)}{\beta+1}}\\
&\le \frac{1}{8}\|\nabla\omega^\theta\|_{L^2}^2 + C\|\omega^\theta\|_{L^2}^2\|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}}.
\end{aligned}$$

Applying Hölder's inequality, Young's inequality and the interpolation estimate yield

$$\begin{aligned}
\int \frac{u^r}{r}|\omega^\theta|^2\,dx &\le C\|u^r\|_{L^3}\|\frac{\omega^\theta}{r}\|_{L^2}\|\omega^\theta\|_{L^6}\\
&\le C\|u^r\|_{L^2}^{\frac12}\|\nabla u^r\|_{L^2}^{\frac12}\|\frac{\omega^\theta}{r}\|_{L^2}\|\nabla\omega^\theta\|_{L^2}\\
&\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\frac{\omega^\theta}{r}\|_{L^2}^2 + \frac{1}{8}\|\nabla\omega^\theta\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
&-\int \partial_z\frac{(b^\theta)^2}{r}\omega^\theta\,dx = \int \frac{(b^\theta)^2}{r}\partial_z\omega^\theta\,dx\\
&\le C\|b^\theta\|_{L^2}\|\frac{b^\theta}{r}\|_{L^\infty}\|\partial_z\omega^\theta\|_{L^2} \le C\|\Pi\|_{L^\infty}^2\|b^\theta\|_{L^2}^2 + \frac{1}{8}\|\nabla\omega^\theta\|_{L^2}^2.
\end{aligned}\tag{3.9}$$

Substituting the above estimates into (3.8), we have

$$\begin{aligned}
&\frac{d}{dt}\|\omega^\theta\|_{L^2}^2 + \frac{5}{4}\|\nabla\omega^\theta\|_{L^2}^2 + 2\|\frac{\omega^\theta}{r}\|_{L^2}^2 + 2\int |u|^{\beta-1}(\omega^\theta)^2\,dx\\
&\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\frac{\omega^\theta}{r}\|_{L^2}^2 + C\|\Pi\|_{L^\infty}^2\|b^\theta\|_{L^2}^2 + C\|\omega^\theta\|_{L^2}^2\|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}}.
\end{aligned}$$

Integrating the above estimate on t yields

$$\begin{aligned}
&\|\omega^\theta\|_{L^2}^2 + \frac{5}{4}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt + 2\int_0^T \|\frac{\omega^\theta}{r}\|_{L^2}^2\,dt\\
&\le C\|u_0\|_{H^1}^2 + C\sup_{0\le t\le T}\|u\|_{L^2}\int_0^T \|\nabla u\|_{L^2}\|\frac{\omega^\theta}{r}\|_{L^2}^2\,dt + C\|\Pi_0\|_{L^\infty}^2\sup_{0\le t\le T}\|b^\theta\|_{L^2}^2 T\\
&\quad + C\int_0^T \|\omega^\theta\|_{L^2}^2\|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}}\,dt.
\end{aligned}$$

From this and (3.7), we get

$$\begin{aligned}
&\|\omega^\theta\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 + \int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt + \int_0^T \|\frac{\omega^\theta}{r}\|_{L^2}^2\,dt + \int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt\\
&\le C\|u_0\|_{H^2}^2 + C\int_0^T \left(1 + \|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}} + \|\nabla u\|_{L^2}^2\right)\left(\|\Gamma\|_{L^2}^2 + \|\omega^\theta\|_{L^2}^2\right)dt + CT,
\end{aligned}$$

where we have used Proposition 3.1 and (3.1). Note that $\frac{2(\beta-1)(\beta+1)}{5-\beta} \le \beta+1$ when $1\le\beta\le\frac{7}{3}$, the Gronwall inequality and Proposition 3.1 allow us to get the desired estimates (3.2) and (3.3). Hence Proposition 3.2 is proved. □

With the help of the above proposition, we immediately have the following corollary.

Corollary 3.1. *Under the assumptions of Proposition 3.2, there holds*

$$\sup_{0\le t\le T}\|\nabla u(t)\|_{L^2}^2+\int_0^T\|\nabla^2 u(t)\|_{L^2}^2\,dt\le C, \tag{3.10}$$

and

$$\int_0^T\|\frac{u^r}{r}(t)\|_{L^\infty}\,dt\le C. \tag{3.11}$$

Here the constant C depends on T, $\|u_0\|_{H^2}$, $\|b_0\|_{H^2}$ and $\|\frac{b_0^\theta}{r}\|_{L^\infty}$.

Proof. We obtain from the Biot-Savart law that

$$\|\nabla u\|_{L^2}\le C\|\omega^\theta\|_{L^2},\quad\text{and}\quad \|\nabla^2 u\|_{L^2}\le C(\|\nabla\omega^\theta\|_{L^2}+\|\frac{\omega^\theta}{r}\|_{L^2}),$$

which along with (3.3) leads to the desired estimate (3.10). Using Hölder's inequality and (3.5) yield

$$\int_0^T\|\frac{u^r}{r}(t)\|_{L^\infty}\,dt\le C\sup_{0\le t\le T}\|\Gamma(t,\cdot)\|_{L^2}^{\frac12}\Big(\int_0^T\|\nabla\Gamma(t)\|_{L^2}^2\,dt\Big)^{\frac14}T^{\frac34}\le C.$$

□

Subsequently, we will establish the $L^1(0,T;\mathrm{Lip}(\mathbb{R}^3))$ estimate for u and $L^\infty(0,T;L^6(\mathbb{R}^3))$ estimate for b, which plays the key role in our proof.

Proposition 3.3. *Suppose $1\le\beta\le\frac73$. Let (u,b) be the smooth solution of (1.1) with $(u_0,b_0)\in H^2$ and $\frac{b_0^\theta}{r}\in L^\infty(\mathbb{R}^3)$ satisfying the assumptions of Theorem 1.1. Then one has*

$$\int_0^T\|\nabla u(t)\|_{L^\infty}\,dt+\sup_{0\le t\le T}\|\nabla b(t)\|_{L^6}\le C,$$

where the constant C depends on T, $\|u_0\|_{H^2}$, $\|b_0\|_{H^2}$ and $\|\frac{b_0^\theta}{r}\|_{L^\infty}$.

Proof. Firstly, we get from the interpolation estimate $\|f\|_{L^\infty}\le C\|\nabla f\|_{L^2}^{\frac12}\|\nabla^2 f\|_{L^2}^{\frac12}$ and (3.10) that

$$\int_0^T\|u(t)\|_{L^\infty}^4\,dt\le C\int_0^T\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\,dt\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\int_0^T\|\nabla^2 u\|_{L^2}^2\,dt\le C. \tag{3.12}$$

For $2\le p<\infty$, we multiply the b^θ equation in (2.1) by $|b^\theta|^{p-2}b^\theta$ and integrating by parts to obtain

$$\frac1p\frac{d}{dt}\|b^\theta\|_{L^p}^p\le C\int|b^\theta|^p|\frac{u^r}{r}|\,dx\le C\|b^\theta\|_{L^p}^p\|\frac{u^r}{r}\|_{L^\infty},$$

which along with Gronwall's inequality and (3.11) imply

$$\|b^\theta\|_{L^p}\le C\|b^\theta\|_{L^p}\exp\Big(\int_0^T\|\frac{u^r}{r}\|_{L^\infty}\,dt\Big)\le C.$$

Passing $p\to\infty$ in the above estimate, we have

$$\|b^\theta\|_{L^p}\le C,\qquad\forall\, 2\le p\le\infty. \tag{3.13}$$

Moreover, recalling the equation for vorticity $\omega=\nabla\times u$, one has

$$\partial_t\omega-\Delta\omega=-\nabla\times(u\cdot\nabla u)-\nabla\times(|u|^{\beta-1}u)+\nabla\times(b\cdot\nabla b).$$

Applying Hölder's inequality, (3.10) and (3.12) lead to

$$\begin{aligned}\int_0^T \|u\cdot\nabla u\|_{L^4}^{\frac43}\,dt &\le \int_0^T \|u\|_{L^\infty}^{\frac43}\|\nabla u\|_{L^4}^{\frac43}\,dt\\ &\le \left(\int_0^T \|u\|_{L^\infty}^4\,dt\right)^{\frac13}\left(\int_0^T \|\nabla u\|_{L^4}^2\,dt\right)^{\frac23}\\ &\le C\left(\int_0^T \|u\|_{L^\infty}^4\,dt\right)^{\frac13}\left(\int_0^T \|\nabla^2 u\|_{L^2}^2\,dt\right)^{\frac23}\\ &\le C.\end{aligned}$$

For any $1\le\beta\le\frac73$, we have the following interpolation estimate

$$\|f\|_{L^{4\beta}}\le C\|f\|_{L^4}^{\frac1\beta}\|f\|_{L^\infty}^{1-\frac1\beta}.$$

From this, together with (3.10) and (3.12), we get

$$\begin{aligned}\int_0^T \||u|^{\beta-1}u\|_{L^4}^{\frac43}\,dt &\le \int_0^T \|u\|_{L^{4\beta}}^{\frac43\beta}\,dt\\ &\le C\int_0^T \|u\|_{L^4}^{\frac43}\|u\|_{L^\infty}^{\frac43\beta-\frac43}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac43}\int_0^T \|u\|_{L^\infty}^{\frac43\beta-\frac43}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac43}\left(\int_0^T \|u\|_{L^\infty}^4\,dt\right)^{\frac{\beta-1}{3}}T^{\frac{4-\beta}{3}}\\ &\le C.\end{aligned}$$

A routine computation gives rise to

$$b\cdot\nabla b=-\frac{(b^\theta)^2}{r}e_r=-\Pi b^\theta e_r.$$

We apply (3.1) and (3.13) to get

$$\|\Pi b^\theta\|_{L^4}\le C\|\Pi\|_{L^4}\|b^\theta\|_{L^\infty}\le C,$$

which implies

$$b\cdot\nabla b\in L^\infty(0,T;L^4(\mathbb{R}^3))\subset L^{\frac43}(0,T;L^4(\mathbb{R}^3)).$$

In order to improve the regulairty of the velocity field, we cite the $L^q_TL^p$-estimates for the heat flow.

Lemma 3.1 (Lemma 2.8 of [14]). *Let us define the operator $\mathcal{L}$ by the formula*

$$\mathcal{L}: f\mapsto \int_0^t \nabla^2 e^{(t-s)\Delta}f(s,\cdot)\,ds.$$

Then $\mathcal{L}$ is bounded from $L^q(0,T;L^p(\mathbb{R}^3))$ to $L^q(0,T;L^p(\mathbb{R}^3))$ for every $T\in(0,\infty]$ and $1<p,q<\infty$. Besides,

$$\|\mathcal{L}f\|_{L^q(0,T;L^p(\mathbb{R}^3))}\le C\|f\|_{L^q(0,T;L^p(\mathbb{R}^3))}.$$

Based on the above estimates, we obtain from Lemma 3.1 that

$$\nabla\omega\in L^{\frac43}(0,T;L^4(\mathbb{R}^3)).$$

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Thanks to the interpolation estimate $\|f\|_{L^\infty} \le C\|f\|_{L^2}^{\frac{1}{7}}\|\nabla f\|_{L^4}^{\frac{6}{7}}$ and (3.10), we have

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^\infty}\,dt &\le C\int_0^T \|\nabla u\|_{L^2}^{\frac{1}{7}}\|\nabla^2 u\|_{L^4}^{\frac{6}{7}}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac{1}{7}}\int_0^T \|\nabla \omega\|_{L^4}^{\frac{6}{7}}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac{1}{7}}\left(\int_0^T \|\nabla \omega\|_{L^4}^{\frac{4}{3}}\,dt\right)^{\frac{9}{14}} T^{\frac{5}{14}}\\ &\le C.\end{aligned}$$

On the other hand, a direct calculation shows

$$\nabla(\frac{1}{r})u^r b = -\frac{1}{r^2}e_r u^r b = -\frac{b^\theta}{r^2}u^r e_r\otimes e_\theta.$$

And applying the operator ∇ to the b equation of (2.1) yields

$$\partial_t\nabla b + \nabla u\cdot\nabla b + u\cdot\nabla\nabla b - \frac{u^r}{r}\nabla b - \nabla u^r\frac{b}{r}e_\theta + \frac{u^r}{r}\Pi e_r\otimes e_\theta = 0.$$

For $2\le p\le 6$, multiplying the above equation by $|\nabla b|^{p-2}\nabla b$ and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{p}\frac{d}{dt}\|\nabla b\|_{L^p}^p \le C\left(\|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty}\right)\|\nabla b\|_{L^p}^p + C\left(\|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty}\right)\|\Pi\|_{L^p}\|\nabla b\|_{L^p}^{p-1}.$$

This leads to

$$\frac{d}{dt}\|\nabla b\|_{L^p} \le C\left(\|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty}\right)\|\nabla b\|_{L^p} + C\left(\|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty}\right)\|\Pi\|_{L^p},$$

which along with Gronwall's inequality implies

$$\begin{aligned}\|\nabla b(t)\|_{L^p} &\le C\exp\int_0^T\left(\|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty}\right)dt\\ &\quad\times\left(\|\nabla b_0\|_{L^p} + \|\Pi_0\|_{L^p}\int_0^T(\|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty})\,dt\right)\\ &\le C.\end{aligned}$$

This completes the proof of Proposition 3.3. □

Finally, our task is to establish the H^2 estimate for (u, b) and then complete the proof of Theorem 1.1.

Proposition 3.4. *Suppose $1\le\beta\le\frac{7}{3}$. Let (u,b) be the smooth solution of (1.1) with $(u_0,b_0)\in H^2$ and $\frac{b_0^\theta}{r}\in L^\infty(\mathbb{R}^3)$ satisfying the assumptions of Theorem 1.1. Then there holds*

$$\sup_{0\le t\le T}\left(\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2\right) + \int_0^T\|\nabla u(t)\|_{H^2}^2\,dt \le C,$$

where the constant C depends on T, $\|u_0\|_{H^2}$, $\|b_0\|_{H^2}$ and $\|\frac{b_0^\theta}{r}\|_{L^\infty}$.

Proof. Applying the operator ∇^2 to the u and b equations in (1.1), one derives

$$\begin{cases}\partial_t\nabla^2 u + u\cdot\nabla\nabla^2 u - b\cdot\nabla\nabla^2 b + \nabla\nabla^2 P - \Delta\nabla^2 u = -\nabla^2(|u|^{\beta-1}u) - [\nabla^2, u\cdot\nabla]u + [\nabla^2, b\cdot\nabla]b,\\ \partial_t\nabla^2 b + u\cdot\nabla\nabla^2 b - b\cdot\nabla\nabla^2 u = -[\nabla^2, u\cdot\nabla]b + [\nabla^2, b\cdot\nabla]u,\end{cases}$$

where $[\mathcal{A},\mathcal{B}] := \mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$ denotes the commutator between $\mathcal{A}$ and $\mathcal{B}$. Performing the L^2 energy estimate of the above system, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2\right)+\|\nabla^3 u\|_{L^2}^2\\
&\quad=-\int \nabla^2(|u|^{\beta-1}u)\cdot\nabla^2 u\,dx-\int[\nabla^2,u\cdot\nabla]u\cdot\nabla^2 u\,dx+\int[\nabla^2,b\cdot\nabla]b\cdot\nabla^2 u\,dx\\
&\qquad-\int[\nabla^2,u\cdot\nabla]b\cdot\nabla^2 b\,dx+\int[\nabla^2,b\cdot\nabla]u\cdot\nabla^2 b\,dx\\
&\quad:=\sum_{i=1}^{5} II_i.
\end{aligned}
\tag{3.14}
$$

For II_1, we obtain from Hölder's inequality and Young's inequality that

$$
\begin{aligned}
II_1&=-\int \nabla^2|u|^{\beta-1}u\cdot\nabla^2 u\,dx-2\int\nabla|u|^{\beta-1}\nabla u\cdot\nabla^2 u\,dx-\int|u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx\\
&=\int(\nabla|u|^{\beta-1}\nabla u\cdot\nabla^2 u+\nabla|u|^{\beta-1}u\cdot\nabla^3 u)\,dx+2\int|u|^{\beta-1}(\nabla^2 u\cdot\nabla^2 u+\nabla u\cdot\nabla^3 u)\,dx\\
&\quad-\int|u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx\\
&=-\int|u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx-\int|u|^{\beta-1}\nabla u\cdot\nabla^3 u\,dx+\int\nabla|u|^{\beta-1}u\cdot\nabla^3 u\,dx\\
&\quad+2\int|u|^{\beta-1}(\nabla^2 u\cdot\nabla^2 u+\nabla u\cdot\nabla^3 u)\,dx-\int|u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx\\
&\le\int|u|^{\beta-1}|\nabla u||\nabla^3 u|\,dx+(\beta-1)\int|u|^{\beta-2}|\nabla u||u||\nabla^3 u|\,dx\\
&\le C\|u\|_{L^\infty}^{\beta-1}\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^2}\\
&\le C\|u\|_{L^\infty}^{2(\beta-1)}\|\nabla u\|_{L^2}^2+\frac{1}{8}\|\nabla^3 u\|_{L^2}^2\\
&\le C\|u\|_{L^\infty}^4+C\|\nabla u\|_{L^2}^{\frac{4}{3-\beta}}+\frac{1}{8}\|\nabla^3 u\|_{L^2}^2.
\end{aligned}
$$

In order to obtain the estimates of II_2-II_5, we need the following commutator estimate.

Lemma 3.2 (Lemma 2.10 of [10]). *Let $s>0$ and $1<p<\infty$, then there exists a constant $C>0$ such that*

$$
\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p(\mathbb{R}^3)}\le C\left(\|\nabla f\|_{L^{p_1}(\mathbb{R}^3)}\|\Lambda^{s-1}g\|_{L^{p_2}(\mathbb{R}^3)}+\|\Lambda^s f\|_{L^{p_3}(\mathbb{R}^3)}\|g\|_{L^{p_4}(\mathbb{R}^3)}\right),
$$

where $\Lambda^s:=(-\Delta)^{\frac{s}{2}}$ and $1<p_2,p_3<\infty$ satisfying

$$
\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.
$$

Thanks to Lemma 3.2 and Hölder's inequality, we have

$$
|II_2|\le\|[\nabla^2,u\cdot\nabla]u\|_{L^2}\|\nabla^2 u\|_{L^2}\le C\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2.
$$

Applying the interpolation estimate $\|f\|_{L^3}\le C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}$ and Young's inequality yield

$$
\begin{aligned}
|II_3|&\le\|[\nabla^2,b\cdot\nabla]b\|_{L^{\frac{3}{2}}}\|\nabla^2 u\|_{L^3}\\
&\le C\|\nabla b\|_{L^6}\|\nabla^2 b\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\\
&\le C\|\nabla b\|_{L^6}^{\frac{4}{3}}\|\nabla^2 b\|_{L^2}^{\frac{4}{3}}\|\nabla^2 u\|_{L^2}^{\frac{2}{3}}+\frac{1}{8}\|\nabla^3 u\|_{L^2}^2\\
&\le C\|\nabla b\|_{L^6}^{\frac{4}{3}}\left(\|\nabla^2 b\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\right)+\frac{1}{8}\|\nabla^3 u\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
|II_4| &\leq \|[\nabla^2, u\cdot\nabla]b\|_{L^2}\|\nabla^2 b\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}^2 + C\|\nabla^2 u\|_{L^3}\|\nabla b\|_{L^6}\|\nabla^2 b\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}^2 + C\|\nabla b\|_{L^6}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}^2 + C\|\nabla b\|_{L^6}^{\frac{4}{3}}\left(\|\nabla^2 b\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) + \frac{1}{8}\|\nabla^3 u\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|II_5| &\leq \|[\nabla^2, b\cdot\nabla]u\|_{L^2}\|\nabla^2 b\|_{L^2}\\
&\leq C\left(\|\nabla b\|_{L^6}\|\nabla^2 u\|_{L^3} + \|\nabla^2 b\|_{L^2}\|\nabla u\|_{L^\infty}\right)\|\nabla^2 b\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}^2 + C\|\nabla b\|_{L^6}\|\nabla^2 u\|_{L^3}\|\nabla^2 b\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}^2 + C\|\nabla b\|_{L^6}^{\frac{4}{3}}\left(\|\nabla^2 b\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) + \frac{1}{8}\|\nabla^3 u\|_{L^2}^2.
\end{aligned}
$$

Substituting all the above estimates into (3.14), we deduce

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2\right) + \|\nabla^3 u\|_{L^2}^2\\
&\quad\leq C\|u\|_{L^\infty}^4 + C\|\nabla u\|_{L^2}^{\frac{4}{3-\beta}} + C\left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^6}^{\frac{4}{3}}\right)\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2\right).
\end{aligned}
$$

Therefore, thanks to Gronwall's inequality, we get from (3.10), (3.12) and Proposition 3.3 that

$$
\begin{aligned}
&\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 b(t)\|_{L^2}^2 + \int_0^T \|\nabla^3 u(t)\|_{L^2}^2\,dt\\
&\quad\leq C\left(\|\nabla^2 u_0\|_{L^2}^2 + \|\nabla^2 b_0\|_{L^2}^2 + C\int_0^T (\|u\|_{L^\infty}^4 + \|\nabla u\|_{L^2}^{\frac{4}{3-\beta}})\,dt\right)\\
&\qquad\times\exp\left(\int_0^T (\|\nabla u(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^6}^{\frac{4}{3}})\,dt\right)\\
&\quad\leq C.
\end{aligned}
$$

From this, together with Proposition 3.1 and (3.10), this ends the proof of Proposition 3.4. □

Proof of Theorem 1.1. With Proposition 3.4 in hand, we complete the proof of Theorem 1.1 by standard continuity argument for local strong solutions, as in [11]. □

Acknowledgments. The authors would like to thank the anonymous referees for their careful reading and valuable suggestions. Li is partially supported by the National Natural Science Foundation of China (Program No. 12201491), Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 22JK0475), Shaanxi Fundamental Science Research Project for Mathematics and Physics (Program No. 22JSQ031) and Young Talent Fund of Association for Science and Technology in Shaanxi, China (Program No. 20230525). Liu is partially supported by the National Natural Science Foundation of China (Program No. 12301292), Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 23JK0762).

Conflict of interests. The authors have no relevant financial or non-financial interests to disclose. The authors have no competing interests to declare that are relevant to the content of this article.

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