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GENERALIZED TRANSFORMS AND CONVOLUTIONS OF BOUNDED CYLINDER FUNCTIONALS ON WIENER SPACE

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ABSTRACT. In this paper we study the generalized Fourier–Feynman transform (GFFT) and the generalized convolution product (GCP) associated with Gaussian processes $\mathscr{Z}_h$ for functionals on Wiener space. To do this we first establish the existences of the GFFT and the GCP of bounded cylinder functionals F having the form $F(x)=\widehat{\mu}(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle)$ where $\widehat{\mu}$ is the Fourier–Stieltjes transform of a complex measure μ on $\mathbb{R}^n$ and the pair $\langle\alpha,x\rangle$ denotes the Paley–Wiener–Zygmund (PWZ) integral $\int_0^T\alpha(t)dx(t)$. It turned out that the structure of the cylinder functionals combined with Gaussian processes makes establishing the relationship between the GFFT and the GCP very difficult. We thus clarify a class of the kernel functions h of the processes $\mathscr{Z}_h$ in order to obtain relationships between them.

1. Introduction

The theory of the analytic Fourier–Feynman transform (FFT) suggested by Brue [1] now is playing a central role in the analytic Feynman integration theory and its applications. The classical FFT and several analogies have been improved in various research articles. For instance, see [2, 8, 9, 10, 12, 17].

Let $C_0[0,T]$ be the Wiener space, the space of all real-valued continuous functions x on the time interval $[0,T]$ with $x(0)=0$. The Wiener space $C_0[0,T]$ can be considered as the space of all continuous sample paths of a Wiener process with the time interval $[0,T]$. In [8, 9, 10, 17], Huffman, Park, Skoug and Storvick established basic relationships between the FFT and the convolution product (CP) for various functionals F and G on $C_0[0,T]$, as follows:

$$T_q^{(p)}\big((F*G)_q\big)(y)=T_q^{(p)}(F)\left(\frac{y}{\sqrt{2}}\right)T_q^{(p)}(G)\left(\frac{y}{\sqrt{2}}\right) \tag{1.1}$$

and

$$\big(T_q^{(p)}(F)*T_q^{(p)}(G)\big)_{-q}(y)=T_q^{(p)}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y) \tag{1.2}$$

for scale-almost every $y\in C_0[0,T]$, where $T_q^{(p)}(F)$ and $(F*G)_q$ indicate the L_p analytic FFT and the CP of functionals F and G on $C_0[0,T]$. In view of equations (1.1) and (1.2), we can see that the FFT $T_q^{(p)}$ acts like a homomorphism with convolution $*$. For a basic survey of the FFT and the corresponding CP, see [19].

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In [3, 11], the authors developed the relations (1.1) and (1.2) for the GFFT and the GCP for functionals in Gaussian processes defined on $C_0[0,T]$. The GFFT and the GCP studied in [3, 11] are defined via the generalized Wiener integral. The generalized Wiener integral [7] was defined by

$$\int_{C_0[0,T]} F(\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x)$$

where $\mathfrak{m}$ denote the Wiener measure and $\mathscr{Z}_h$ is a stochastic process on $C_0[0,T]\times[0,T]$ defined by the PWZ integral $\mathscr{Z}_h(x,t)=\int_0^t h(s)\tilde{d}x(s)$, see [14, 15, 16], and where h is a nonzero function in $L_2[0,T]$.

The stochastic process $\mathscr{Z}_h$ is Gaussian with mean zero and variance function $\beta_h(t)=\int_0^t h^2(s)ds$. If we choose h to be constant, say $h=c$, then it follows that given $t_1,t_2,t_3,t_4\in[0,T]$ with $|t_2-t_1|=|t_4-t_3|$, the increments $\mathscr{Z}_h(x,t_2)-\mathscr{Z}_h(x,t_1)$ and $\mathscr{Z}_h(x,t_4)-\mathscr{Z}_h(x,t_3)$ have a same normal distribution $N(0,c^2|t_2-t_1|)$, and so the process $\{\mathscr{Z}_c(x,t):t\in[0,T]\}$ is stationary in time. But if h is not constant, then $\beta_h(t_2)-\beta_h(t_1)\neq\beta_h(t_4)-\beta_h(t_3)$ in spite of $|t_2-t_1|=|t_4-t_3|$. Thus in this case,, the Gaussian process $\{\mathscr{Z}_h(x,t):t\in[0,T]\}$ is not stationary in time.

In [3, 11], the authors used a single L_2 function in order to extend the concept of the CP for functionals on $C_0[0,T]$. But in this paper, we adopt the modified definition of the CP from [4, 5, 18] in order to obtain more general relationships between our transforms and convolutions.

Our main results are summarized as follows. Let F and G be Wiener integrable functionals in a certain class of bounded functionals on $C_0[0,T]$. Then it follows that

$$T_{q,h}^{(p)}\big((F*G)_q^{(k_1,k_2)}\big)(y)=T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F)\left(\frac{y}{\sqrt{2}}\right)T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(G)\left(\frac{y}{\sqrt{2}}\right) \tag{1.3}$$

and

$$\left(T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F)*T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(G)\right)_{-q}^{(k_1,k_2)}(y)=T_{q,h}^{(p)}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y) \tag{1.4}$$

for scale-almost every $y\in C_0[0,T]$, where $T_{q,h}^{(p)}(F)$ and $(F*G)_q^{(k_1,k_2)}$ denote the GFFT and the GCP, respectively, studied in this paper, and h, k_1, k_2, $s(h,k_1)$ and $s(h,k_1)$ are functions in $L_2[0,T]$ which satisfy the relations

$$s(h,k_1)^2(t)=h^2(t)+k_1^2(t)\ \text{and}\ s(h,k_2)^2(t)=h^2(t)+k_2^2(t)$$

for m_L-a.e. $t\in[0,T]$, respectively, and where m_L indicates the Lebesgue measure on $[0,T]$.

In [4, 18], equations (1.3) and (1.4) were established for specific bounded functionals defined on Wiener and Yeh–Wiener spaces. The functionals considered in this paper are also bounded on the Wiener space $C_0[0,T]$. But the functionals used in this paper are also characterized as the n-dimensional cylinder functionals

$$f(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle),\quad x\in C_0[0,T]$$

where f is a Lebesgue measurable function on $\mathbb{R}^n$ and the pair $\langle\alpha,x\rangle$ denotes the PWZ integral $\mathscr{Z}_\alpha(x,T)=\int_0^T\alpha(s)\tilde{d}x(s)$. It turns out, as noted in Remark 3.3 below, that the structure of the cylinder functionals combined with Gaussian processes makes establishing the existences of the GFFT and the GCP associated with Gaussian processes $\mathscr{Z}_h$, as well as the equalities in (1.3) and (1.4), very difficult. We thus clarify a class of the kernel functions h of the processes $\mathscr{Z}_h$ in order to obtain the existences of the GFFT and the GCP associated with Gaussian processes.

2. Preliminaries

On the Wiener space $C_0[0,T]$, let $\mathscr{M}$ denote the family of all Wiener measurable subsets of $C_0[0,T]$. It is well-known that $(C_0[0,T],\mathscr{M},\mathfrak{m})$ is a complete probability measure space.

In order to define the GFFT and the GCP, we need the concept of the scale-invariant measurability on $C_0[0,T]$. A subset B of $C_0[0,T]$ is called a scale-invariant measurable (SIM) set if $\rho B\in\mathscr{M}$ for all $\rho>0$, and a SIM set N is called a scale-invariant null set if $\mathfrak{m}(\rho N)=0$ for all $\rho>0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (SI-a.e.). A functional F is said to be SIM provided F is defined on a SIM set and $F(\rho\,\cdot)$ is $\mathscr{M}$-measurable for every $\rho>0$. If two functionals F and G are equal SI-a.e., we write $F\approx G$. For more detailed studies of the scale-invariant measurability, see [6, 13].

For any $h\in L_2[0,T]$ with $\|h\|_2>0$, let $\mathscr{Z}_h : C_0[0,T]\times[0,T]\to\mathbb{R}$ be the stochastic process

$$\mathscr{Z}_h(x,t)=\int_0^t h(s)\tilde{d}x(s)=\int_0^T h(s)\chi_{[0,t]}(s)\tilde{d}x(s)=\langle h\chi_{[0,t]},x\rangle \tag{2.1}$$

where $\chi_{[0,t]}$ denotes the indicator function of the interval $[0,t]$. Let $\beta_h(t)=\int_0^t h^2(s)ds$ for each $t\in[0,T]$. Then the stochastic process $\mathscr{Z}_h$ on $C_0[0,T]$ and the time interval $[0,T]$ is a Gaussian process with mean zero and covariance function

$$\beta_h(\min\{s,t\})=\int_{C_0[0,T]}\mathscr{Z}_h(x,s)\mathscr{Z}_h(x,t)d\mathfrak{m}(x).$$

In addition, by [20, Theorem 21.1], $\mathscr{Z}_h(\cdot,t)$ is stochastically continuous in t on $[0,T]$. If h is of bounded variation on $[0,T]$, then for every $x\in C_0[0,T]$, $\mathscr{Z}_h(x,t)$ is a continuous function of t. Furthermore, for any $h_1,h_2\in L_2[0,T]$,

$$\int_{C_0[0,T]}\mathscr{Z}_{h_1}(x,s)\mathscr{Z}_{h_2}(x,t)d\mathfrak{m}(x)=\int_0^{\min\{s,t\}}h_1(u)h_2(u)du.$$

If $h(t)\equiv1$ on $[0,T]$, then the process $\mathscr{Z}_1$ on $C_0[0,T]\times[0,T]$ given by $\mathscr{Z}_1(x,t)=x(t)$ is a Wiener process. It is known that the Wiener process $\mathscr{Z}_1$ is stationary in time, whereas the stochastic process $\mathscr{Z}_h$ generally is not. For more details, see [5, 7].

Let $\mathbb{C}_+=\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)>0\}$ and let $\widetilde{\mathbb{C}}_+=\{\lambda\in\mathbb{C}\setminus\{0\}:\mathrm{Re}(\lambda)\geq0\}$. Let $F:C_0[0,T]\to\mathbb{C}$ be a SIM functional such that

$$J_F(h;\lambda)=\int_{C_0[0,T]}F(\lambda^{-1/2}\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x)$$

exists as a finite number for all $\lambda>0$. If there exists a function $J_F^*(h;\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(h;\lambda)=J_F(h;\lambda)$ for all $\lambda>0$, then $J_F^*(h;\lambda)$ is defined to be the generalized analytic Wiener integral of F over $C_0[0,T]$ with parameter λ. For $\lambda\in\mathbb{C}_+$, we will write

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda}F(\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x)=J_F^*(h;\lambda).$$

Let $q\neq0$ be a real number and let F be a functional such that $\int_{C_0[0,T]}^{\mathrm{anw}_\lambda}F(\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x)$ exists for all $\lambda\in\mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q, and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x) = \lim_{\lambda\to -iq} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x),$$

where $\lambda \to -iq$ through in $\mathbb{C}_+$.

We are now ready to state the definition of the GFFT.

Definition 2.1. *Let $F: C_0[0,T] \to \mathbb{C}$ be a SIM functional such that the analytic Wiener integral*

$$T_{\lambda,h}(F)(y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(y+\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x)$$

exists for all $\lambda \in \mathbb{C}_+$ and for SI-a.e. $y \in C_0[0,T]$. Let q be a nonzero real number. For $p \in (1,2]$, we define the L_p analytic GFFT (with respect to the process $\mathscr{Z}_h$), $T_{q,h}^{(p)}(F)$ of F, by the formula,

$$T_{q,h}^{(p)}(F)(y) = \underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\, T_{\lambda,h}(F)(y),$$

whenever this limit exists; i.e., for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \int_{C_0[0,T]} \big|T_{\lambda,h}(F)(\rho y) - T_{q,h}^{(p)}(F)(\rho y)\big|^{p'} d\mathfrak{m}(y) = 0,$$

where $1/p+1/p' = 1$. We define the L_1 analytic GFFT, $T_{q,h}^{(1)}(F)$ of F, by the formula

$$T_{q,h}^{(1)}(F)(y) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,h}(F)(y)$$

for SI-a.e. $y \in C_0[0,T]$, if the limit exists.

Next we give the definition of the GCP [4, 5, 18].

Definition 2.2. *Let F and G be SIM functionals on $C_0[0,T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$ and $h_1,h_2 \in L_2[0,T]$, we define their GCP with respect to $\{\mathscr{Z}_{h_1}, \mathscr{Z}_{h_2}\}$ (if it exists) by*

$$(2.2)\qquad (F*G)_\lambda^{(h_1,h_2)}(y) = \begin{cases} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(\dfrac{y+\mathscr{Z}_{h_1}(x,\cdot)}{\sqrt{2}}\Big) G\Big(\dfrac{y-\mathscr{Z}_{h_2}(x,\cdot)}{\sqrt{2}}\Big) d\mathfrak{m}(x), & \lambda\in\mathbb{C}_+ \\ \int_{C_0[0,T]}^{\mathrm{anf}_q} F\Big(\dfrac{y+\mathscr{Z}_{h_1}(x,\cdot)}{\sqrt{2}}\Big) G\Big(\dfrac{y-\mathscr{Z}_{h_2}(x,\cdot)}{\sqrt{2}}\Big) d\mathfrak{m}(x), & \lambda=-iq, q\in\mathbb{R}\setminus\{0\}. \end{cases}$$

When $\lambda = -iq$, we denote $(F*G)_\lambda^{(h_1,h_2)}$ by $(F*G)_q^{(h_1,h_2)}$.

Remark 2.3. *The processes $\mathscr{Z}_{h_1}$ and $\mathscr{Z}_{h_2}$ in (2.2) are Gaussian processes on $C_0[0,T]\times[0,T]$ which are not stationary in time. Furthermore, one can see that for each $t \in [0,T]$,*

$$\mathscr{Z}_{h_1}(x,t) \sim N\Big(0, \int_0^t h_1^2(s)ds\Big) \text{ and } \mathscr{Z}_{h_2}(x,t) \sim N\Big(0, \int_0^t h_2^2(s)ds\Big).$$

That is, the processes $\mathscr{Z}_{h_1}$ and $\mathscr{Z}_{h_2}$ have different Gaussian distributions.

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3. Cylinder Functionals

In this section, we introduce a class of certain bounded cylinder functionals. We then establish the existence of the GFFT of such cylinder functionals. A functional F on $C_0[0,T]$ is called a cylinder functional, if the functional F is represented by

$$F(x) = \phi(\langle w_1, x\rangle, \dots, \langle w_m, x\rangle), \tag{3.1}$$

where $\phi : \mathbb{R}^m \to \mathbb{C}$ is a Lebesgue measurable function and $\{w_1, \dots, w_m\}$ is a linearly independent set of functions in the Hilbert space $L_2[0,T]$. The functional F given by (3.1) is Wiener measurable if and only if f is Lebesgue measurable [6]. It is easy to show that given a cylinder functional F of the form (3.1), there exists an orthogonal set $\{g_1, \dots, g_n\}$ of functions in $L_2[0,T] \setminus \{0\}$ and a complex-valued Lebesgue measurable function f on $\mathbb{R}^n$ such that F is expressed as

$$F(x) = f(\langle g_1, x\rangle, \dots, \langle g_n, x\rangle), \quad x \in C_0[0,T]. \tag{3.2}$$

Thus we lose no generality in assuming that every cylinder functional on $C_0[0,T]$ is of the form (3.2).

In order to simplify many expressions in this paper, we use the following conventions: for $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and a set $\{g_1, \dots, g_n\}$ of functions in $L_2[0,T]$, let $f(\vec{u}) \equiv f(u_1, \dots, u_n)$, $\langle \vec{g}, x\rangle \equiv (\langle g_1, x\rangle, \dots, \langle g_n, x\rangle)$, and $f(\langle \vec{g}, x\rangle) \equiv f(\langle g_1, x\rangle, \dots, \langle g_n, x\rangle)$.

Equation (3.3) below can be easily obtained by the change of variables theorem.

Lemma 3.1. *Let $\mathscr{G} = \{g_1, \dots, g_n\}$ be an orthogonal set of nonzero functions in $L_2[0,T]$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a Lebesgue measurable function. Then*

$$\int_{C_0[0,T]} f(\langle \vec{g}, x\rangle) d\mathfrak{m}(x) \overset{*}{=} \left(\prod_{j=1}^{n} 2\pi \|g_j\|_2^2 \right)^{-1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{ -\sum_{j=1}^{n} \frac{u_j^2}{2\|g_j\|_2^2} \right\} d\vec{u}, \tag{3.3}$$

where by $\overset{*}{=}$ we mean that if either side exists, both sides exist and equality holds.

Lemma 3.2 ([7]). *For each $v \in L_2[0,T]$ and each $h \in L_\infty[0,T]$ with $\|h\|_2 > 0$, it follows that*

$$\langle u, \mathscr{Z}_h(x, \cdot)\rangle = \langle uh, x\rangle \tag{3.4}$$

for SI-a.e. $x \in C_0[0,T]$.

Remark 3.3. *In view of Lemma 3.2, we require h to be in $L_\infty[0,T]$ rather than simply in $L_2[0,T]$ throughout this paper. For a nonzero function $h \in L_\infty[0,T]$, let $\mathscr{Z}_h$ be the Gaussian process defined by* (2.1) *above and let F be given by* (3.2)*. Then by equation* (3.4)*, we observe that*

$$F(\mathscr{Z}_h(x, \cdot)) = f(\langle g_1, \mathscr{Z}_h(x, \cdot)\rangle, \dots, \langle g_n, \mathscr{Z}_h(x, \cdot)\rangle) = f(\langle g_1 h, x\rangle, \dots, \langle g_n h, x\rangle).$$

Even though the subset $\mathscr{A} = \{g_1, \dots, g_n\}$ of $L_2[0,T]$ is orthogonal, the subset $\mathscr{A}h \equiv \{gh : g \in \mathscr{A}\}$ of $L_2[0,T]$ might not be orthogonal. Thus the equality in the equation

$$\int_{C_0[0,T]} f(\langle \vec{g}, \mathscr{Z}_h(x, \cdot)\rangle) d\mathfrak{m}(x) = \left(\prod_{j=1}^{n} 2\pi \|g_j h\|_2^2 \right)^{-1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{ -\sum_{j=1}^{n} \frac{u_j^2}{2\|g_j h\|_2^2} \right\} d\vec{u}$$

does not hold true. This observation is critical to the development of the relationships between the GFFT and the GCP of cylinder functionals. Thus, throughout remainder of this paper, we will need to *put additional restrictions on the kernel function h of the Gaussian process $\mathscr{Z}_h$ in order to establish the formulas involving GFFTs and GCPs.*

As mentioned in Remark 3.3, we clearly need to impose additional restrictions on the functionals used in this paper.

Definition 3.4. *Given an orthogonal set $\mathscr{A} = \{g_1, \dots, g_n\}$ of functions in $L_2[0,T] \setminus \{0\}$, let $\mathscr{O}_\infty(\mathscr{A})$ be the class of all nonzero elements $h \in L_\infty[0,T]$ such that $\mathscr{A}h$ is orthogonal in $L_2[0,T]$.*

Example 3.5. *For every $\rho \in \mathbb{R} \setminus \{0\}$, the constant function ρ is an element of $\mathscr{O}_\infty(\mathscr{A})$ for every orthogonal subset $\mathscr{A}$ of $L_2[0,T]$.*

Example 3.6. *For each $j \in \mathbb{N}$, let*

$$g_j(t) = \frac{\sqrt{2}}{\sqrt[4]{T}} \cos\left(\frac{(2j-1)\pi}{2T} t\right)$$

and

$$\alpha_j(t) = \frac{1}{\sqrt[4]{T}} g_j(t) = \frac{\sqrt{2}}{\sqrt[2]{T}} \cos\left(\frac{(2j-1)\pi}{2T} t\right) \tag{3.5}$$

on $[0,T]$. Then $\mathscr{S} = \{g_j\}_{j=1}^\infty$ is an orthogonal sequence of functions in $L_2[0,T]$. In addition, $\widetilde{\mathscr{S}} = \{\alpha_j\}_{j=1}^\infty$ is a complete orthonormal sequence in $L_2[0,T]$. In this case we have the following assertions.

(i) *For every $j \in \mathbb{N}$, $\|g_j\|_2^2 = \sqrt{T} > 0$ and $g_j \in L_\infty[0,T]$.*
(ii) *Let $n \in \mathbb{N}$ be fixed and let L be a positive integer with $L > n$. Then for any $i, j \in \{1, \dots, n\}$, $\int_0^T g_i(t) g_j(t) g_L^2(t) dt = \delta_{ij}$ (the Kronecker delta). In other words, for each integer L with $L > n$, the set $\{g_1 g_L, \dots, g_n g_L\}$ is an orthonormal set of functions in $L_2[0,T]$.*

Let $\mathscr{A}_{\cos} = \{\alpha_1, \dots, \alpha_n\}$ where α_i's are given by (3.5) for each $i \in \{1, \dots, n\}$. Then, from the observations above, it follows that

(i) *$\mathscr{A}_{\cos}$ is an orthonormal set of functions in $L_2[0,T]$,*
(ii) *$\{g_{n+1}, g_{n+2}, \dots\} \subset \mathscr{O}_\infty(\mathscr{A}_{\cos})$,*
(iii) *$\{\alpha_{n+1}, \alpha_{n+2}, \dots\} = \widetilde{\mathscr{S}} \setminus \mathscr{A}_{\cos} \subset \mathscr{O}_\infty(\mathscr{A}_{\cos})$.*

Next, we introduce a class of bounded cylinder functionals on $C_0[0,T]$. Let $\mathscr{M}(\mathbb{R}^n)$ be the space of complex-valued Borel measures on $\mathscr{B}(\mathbb{R}^n)$, the Borel class on $\mathbb{R}^n$. It is known that a complex-valued Borel measure μ has a finite total variation $\|\mu\|$, and the class $\mathscr{M}(\mathbb{R}^n)$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication.

Given a complex measure μ in $\mathscr{M}(\mathbb{R}^n)$, the Fourier–Stieltjes transform $\widehat{\mu}$ of μ is a complex-valued function on $\mathbb{R}^n$ defined by

$$\widehat{\mu}(\vec{u}) = \int_{\mathbb{R}^n} \exp\left\{ i \sum_{j=1}^n u_j v_j \right\} d\mu(\vec{v}). \tag{3.6}$$

Given an orthonormal set $\mathscr{A}=\{\alpha_1,\dots,\alpha_n\}$ of functions in $L_2[0,T]$, let $\widehat{\mathfrak{T}}_{\mathscr{A}}$ be the class of functionals F_μ on $C_0[0,T]$ defined by

$$F_\mu(x)=\widehat{\mu}(\langle\vec{\alpha},x\rangle)=\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^n\langle\alpha_j,x\rangle v_j\Big\}d\mu(\vec{v}) \tag{3.7}$$

for SI-a.e. $x\in C_0[0,T]$, where $\widehat{\mu}$ is the Fourier–Stieltjes transform of μ in $\mathscr{M}(\mathbb{R}^n)$. Given any $\mu\in\mathscr{M}(\mathbb{R}^n)$, the function $\widehat{\mu}$ corresponding to μ by (3.6) is bounded (and so is F_μ), because $|\widehat{\mu}(\vec{u})|\le\|\mu\|<+\infty$ for every $\vec{u}\in\mathbb{R}^n$. Note that the functional F_μ having the form (3.7) is SIM on $C_0[0,T]$.

In Sections 4 and 5 below, we will use the following integration formula in order to verify the existences of the GFFT and the GCP of functionals in the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$:

$$\int_{\mathbb{R}}\exp\{-av^2+bv\}dv=\sqrt{\frac{\pi}{a}}\exp\Big\{\frac{b^2}{4a}\Big\} \tag{3.8}$$

for $a,b\in\mathbb{C}$ with $\mathrm{Re}(a)>0$.

4. Generalized Fourier–Feynman transform

In this section, we will provide the existence of the L_p analytic GFFT of functionals in the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$.

Theorem 4.1. *Let $F_\mu\in\widehat{\mathfrak{T}}_{\mathscr{A}}$ be given by (3.7). Then for all nonzero real number q, and any $h\in\mathscr{O}_\infty(\mathscr{A})$, the L_1 analytic GFFT $T_{q,h}^{(1)}$ of F_μ exists and is given by the formula*

$$T_{q,h}^{(1)}(F_\mu)(y)=\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j-\frac{i}{2q}\sum_{j=1}^n\|\alpha_jh\|_2^2v_j^2\Big\}d\mu(\vec{v}) \tag{4.1}$$

for SI-a.e. $y\in C_0[0,T]$.

Proof. Using (3.7) with x replaced with $y+\lambda^{-1/2}x$, (3.4), the Fubini theorem, (3.3), and (3.8), it follows that for all $\lambda>0$,

$$\begin{aligned}
&J_{F_\mu(y+\cdot)}(h;\lambda)\\
&=\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j\Big\}\Big[\int_{C_0[0,T]}\exp\Big\{i\lambda^{-1/2}\sum_{j=1}^n\langle\alpha_jh,x\rangle v_j\Big\}d\mathfrak{m}(x)\Big]d\mu(\vec{v})\\
&=\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j\Big\}\prod_{j=1}^n\Big[\big(2\pi\|\alpha_jh\|_2^2\big)^{-1/2}\int_{\mathbb{R}}\exp\Big\{i\lambda^{-1/2}u_jv_j-\frac{u_j^2}{2\|\alpha_jh\|_2^2}\Big\}du_j\Big]d\mu(\vec{v})\\
&=\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j-\frac{1}{2\lambda}\sum_{j=1}^n\|\alpha_jh\|_2^2v_j^2\Big\}d\mu(\vec{v}).
\end{aligned}$$

Now, for each $\lambda\in\mathbb{C}$, let

$$J^*_{F_\mu(y+\cdot)}(h;\lambda)\equiv\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j-\frac{1}{2\lambda}\sum_{j=1}^n\|\alpha_jh\|_2^2v_j^2\Big\}d\mu(\vec{v}). \tag{4.2}$$

Because the function $\Psi(\lambda)=\exp\{-\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2/(2\lambda)\}$ is analytic on $\mathbb{C}_+$, applying the Morera theorem, it follows that

$$\int_\triangle J^*_{F_\mu(y+\cdot)}(h;\lambda)d\lambda=\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j\bigg\}\bigg(\int_\triangle\Psi(\lambda)d\lambda\bigg)d\mu(\vec{v})=0$$

where $\triangle$ is a simple closed contour lying in $\mathbb{C}_+$. Thus the analytic transform $T_{\lambda,h}(F_\mu)(y)=J^*_{F_\mu(y+\cdot)}(h;\lambda)$ exists and is given by the right-hand side of (4.2).

Next, we note that for each $\lambda\in\mathbb{C}_+$, $\mathrm{Re}(\lambda)>0$. From this, we see that for all $\lambda\in\mathbb{C}_+$,

$$\begin{aligned}|T_{\lambda,h}(F_\mu)(y)|&\le\int_{\mathbb{R}^n}\bigg|\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j-\frac{1}{2\lambda}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}\bigg|d|\mu|(\vec{v})\\&=\int_{\mathbb{R}^n}\bigg|\exp\bigg\{-\frac{\mathrm{Re}(\lambda)}{2|\lambda|^2}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}\bigg|d|\mu|(\vec{v})\le\|\mu\|<+\infty.\end{aligned}$$

By the bounded convergence theorem, it thus follows that

$$\begin{aligned}T^{(1)}_{q,h}(F_\mu)(y)&=\lim_{\substack{\lambda\to-iq\\\lambda\in\mathbb{C}_+}}\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j-\frac{1}{2\lambda}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}d\mu(\vec{v})\\&=\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle v_j-\frac{i}{2q}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}d\mu(\vec{v})\end{aligned}$$

as desired. □

Theorem 4.2. *Let $F_\mu\in\widehat{\mathfrak{T}}_{\mathscr{A}}$ be given by (3.7). Then for each $p\in(1,2]$, all nonzero real number q, and any $h\in O_\infty(\mathscr{A})$, the L_p analytic GFFT of F_μ, $T^{(p)}_{q,h}(F_\mu)$ exists and is given by the right-hand side of (4.1) for SI-a.e. $y\in C_0[0,T]$.*

Proof. It was shown in the proof of Theorem 4.1 that $T_{\lambda,h}(F)(y)$ is an analytic function of λ throughout the right-half complex plane $\mathbb{C}_+$.

In view of Definition 2.1, it will suffice to show that for each $\rho>0$,

$$\lim_{\substack{\lambda\to-iq\\\lambda\in\mathbb{C}_+}}\int_{C_0[0,T]}\big|T_{\lambda,h}(F_\mu)(\rho y)-T^{(1)}_{q,h}(F_\mu)(\rho y)\big|^{p'}d\mathfrak{m}(y)=0,$$

where $1/p+1/p'=1$.

Fixing $p\in(1,2]$, it follows that for all $\rho>0$ and all $\lambda\in\mathbb{C}_+$,

$$\begin{aligned}&\big|T_{\lambda,h}(F_\mu)(\rho y)-T^{(1)}_{q,h}(F_\mu)(\rho y)\big|^{p'}\\&=\bigg|\int_{\mathbb{R}^n}\exp\bigg\{i\rho\sum_{j=1}^n\langle\alpha_j,y\rangle\bigg\}\bigg[\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}-\exp\bigg\{-\frac{i}{2q}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}\bigg]d\mu(\vec{v})\bigg|^{p'}\\&\le\bigg(\int_{\mathbb{R}^n}\bigg[\bigg|\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^n\|\alpha_j h\|_2^2v_j^2\bigg\}\bigg|+1\bigg]d|\mu|(\vec{v})\bigg)^{p'}\end{aligned}$$

$$\leq \left(2\int_{\mathbb{R}^n} d|\mu|(\vec{v})\right)^{p'} = (2\|\mu\|)^{p'} < +\infty.$$

Hence, by the bounded convergence theorem, we see that for each $p \in (1,2]$ and all $\rho > 0$,

$$\begin{aligned}
&\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \int_{C_0[0,T]} \left|T_{\lambda,h}(F_\mu)(\rho y) - T^{(1)}_{q,h}(F_\mu)(\rho y)\right|^{p'} d\mathfrak{m}(y)\\
&= \int_{C_0[0,T]} \bigg| \int_{\mathbb{R}^n} \exp\bigg\{ i\rho\sum_{j=1}^n \langle\alpha_j, y\rangle \bigg\}\\
&\quad \times \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \bigg[\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^n \|\alpha_j h\|_2^2 v_j^2\bigg\} - \exp\bigg\{-\frac{i}{2q}\sum_{j=1}^n \|\alpha_j h\|_2^2 v_j^2\bigg\}\bigg] d\mu(\vec{v})\bigg|^{p'} d\mathfrak{m}(y)\\
&= 0
\end{aligned}$$

which concludes the proof of Theorem 4.2. □

Theorem 4.3. *Let $F_\mu \in \widehat{\mathfrak{T}}_{\mathscr{A}}$ be given by (3.7). Then for each $p \in [1,2]$, all $q \in \mathbb{R}\setminus\{0\}$, and any $h \in \mathscr{O}_\infty(\mathscr{A})$, the L_p analytic GFFT $T^{(p)}_{q,h}(F_\mu)$ of F is in the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$.*

Proof. Let $F_\mu \in \widehat{\mathfrak{T}}_{\mathscr{A}}$ be given by (3.7) with corresponding complex measure $\mu \in \mathscr{M}(\mathbb{R}^n)$. Given any function h in $\mathscr{O}_\infty(\mathscr{A})$, define a set function $\mu^h_{t,q} : \mathscr{B}(\mathbb{R}^n) \to \mathbb{C}$ by the formula.

$$\mu^h_{t,q}(U) = \int_U \exp\bigg\{-\frac{i}{2q}\sum_{j=1}^n \|\alpha_j h\|_2^2 v_j^2\bigg\} d\mu(\vec{v})$$

for $U \in \mathscr{B}(\mathbb{R}^n)$. It is obvious that the set function $\mu^h_{t,q}$ satisfies the countable additivity on the class $\mathscr{B}(\mathbb{R}^n)$. We also note that

$$\begin{aligned}
\|\mu^h_{t,q}\| \equiv |\mu^h_{t,q}|(\mathbb{R}^n) &\leq \int_{\mathbb{R}^n} \bigg|\exp\bigg\{-\frac{i}{2q}\sum_{j=1}^n \|\alpha_j h\|_2^2 v_j^2\bigg\}\bigg| d|\mu|(\vec{v})\\
&= \int_{\mathbb{R}^n} d|\mu|(\vec{v}) = |\mu|(\mathbb{R}^n) = \|\mu\| < +\infty.
\end{aligned}$$

Thus the set function $\mu^h_{t,q}$ is in the Banach algebra $\mathscr{M}(\mathbb{R}^n)$. Furthermore, it follows that

$$\begin{aligned}
T^{(p)}_{q,h}(F_\mu)(y) &= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^n \langle\alpha_j, y\rangle v_j - \frac{i}{2q}\sum_{j=1}^n \|\alpha_j h\|_2^2 v_j^2\bigg\} d\mu(\vec{v})\\
&= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^n \langle\alpha_j, y\rangle v_j\bigg\} d\mu^h_{t,q}(\vec{v}) = F_{\mu^h_{t,q}}(y)
\end{aligned}$$

for SI-a.e. $y \in C_0[0,T]$. Thus we see that $T^{(p)}_{q,h}(F_\mu)$ is an element of the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$ for all $q \in \mathbb{R}\setminus\{0\}$ and any function $h \in \mathscr{O}_\infty(\mathscr{A})$. □

The following corollary is a simple consequence of Theorems 4.1, 4.2, and 4.3, and will be very useful to prove our main theorem (namely, Theorem 6.3 below).

Corollary 4.4. *Let $F_\mu \in \widehat{\mathfrak{T}}_{\mathscr{A}}$ be given by (3.7). Then for each $p \in [1,2]$, all $q \in \mathbb{R}\setminus\{0\}$, and any $h \in \mathscr{O}_\infty(\mathscr{A})$,*

$$T^{(p)}_{-q,h}\big(T^{(p)}_{q,h}(F_\mu)\big) \approx F_\mu. \tag{4.3}$$

In other words, the L_p analytic GFFT, $T^{(p)}_{q,h}$ has the inverse transform $\{T^{(p)}_{q,h}\}^{-1} = T^{(p)}_{-q,h}$.

5. Generalized convolution product

In this section, we establish the existence of the GCP of functionals in the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$. But in order to ensure the existence of the GCP of functionals in $\widehat{\mathfrak{T}}_{\mathscr{A}}$, we have to provide a condition for functions in $\mathscr{O}_\infty(\mathscr{A})$.

Given an orthonormal set $\mathscr{A} = \{\alpha_1,\dots,\alpha_n\}$ of functions in $L_2[0,T]$, consider the class $\mathscr{O}_\infty(A)$ defined in Section 3 above. In evaluation of the GCP $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$ of functionals F_{μ_1} and F_{μ_2} in $\widehat{\mathfrak{T}}_{\mathscr{A}}$, it arises the question that for all vectors $\vec{u} = (u_1,\dots,u_n)$ and $\vec{v} = (v_1,\dots,v_n)$ in $\mathbb{R}^n$, whether the PWZ integrals

$$\mathscr{G}_{(\vec{u},\vec{v};k_1,k_2)} = \{\langle u_1\alpha_1k_1 - v_1\alpha_1k_2, x\rangle,\dots,\langle u_n\alpha_nk_1 - v_n\alpha_nk_2, x\rangle\} \tag{5.1}$$

form a set of independent Gaussian random variables or not. Precisely speaking, when we evaluate the GCP $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$, it might not be able to use Lemma 3.1, because the Gaussian random variables in the set $\mathscr{G}_{(\vec{u},\vec{v};k_1,k_2)}$ are generally not independent. Consequently, in order to apply Lemma 3.1 to the calculation of the GCP $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$, we have to apply the Gram–Schmidt process to the set

$$\mathscr{A}_{(\vec{u},\vec{v};k_1,k_2)} = \{u_1\alpha_1k_1 - v_1\alpha_1k_2,\dots,u_n\alpha_nk_1 - v_n\alpha_nk_2\}. \tag{5.2}$$

In view of these situation, we will consider the class of ordered pairs of functions in $\mathscr{O}_\infty(\mathscr{A})$ throughout the remainder of this paper.

Definition 5.1. *Given an orthonormal set $\mathscr{A} = \{\alpha_1,\dots,\alpha_n\}$ of functions in $L_2[0,T]\setminus\{0\}$, let*

$$\mathscr{P}_\infty(\mathscr{A}) = \left\{(k_1,k_2) \in \mathscr{O}_\infty(\mathscr{A})\times\mathscr{O}_\infty(\mathscr{A}) : \int_0^T \alpha_i(t)\alpha_j(t)k_1(t)k_2(t)dt = 0 \text{ for } i \neq j\right\}.$$

Clearly, for any $h \in \mathscr{O}_\infty(\mathscr{A})$, $(h,h) \in \mathscr{P}_\infty(\mathscr{A})$.

Following example tells us that the class $\mathscr{P}_\infty(\mathscr{A}_{\cos})$ is not empty for the orthonormal set $\mathscr{A}_{\cos}$ discussed in Example 3.6.

Example 5.2. *Consider the orthonormal sequence $\widetilde{\mathscr{S}} = \{\alpha_j\}_{j=1}^\infty$ in $L_2[0,T]$ presented in Example 3.6. Let $\mathscr{A}_{\cos} = \{\alpha_1,\dots,\alpha_n\}$. As shown in Example 3.6, one can see that for any functions k_1 and k_2 in $\widetilde{\mathscr{S}}\setminus\mathscr{A}_{\cos}$, the pair (k_1,k_2) of the functions k_1 and k_2 is in the class $\mathscr{P}_\infty(\mathscr{A}_{\cos})$.*

Lemma 5.3. *Given an orthonormal set $\mathscr{A} = \{\alpha_1,\dots,\alpha_n\}$ in $L_2[0,T]$, let (k_1,k_2) be in the class $\mathscr{P}_\infty(\mathscr{A})$. Then for any vectors $\vec{u} = (u_1,\dots,u_n)$ and $\vec{v} = (v_1,\dots,v_n)$ in $\mathbb{R}^n$, the set of functions in $\mathscr{A}_{(\vec{u},\vec{v};k_1,k_2)}$ defined by (5.2) is an orthogonal set in $L_2[0,T]$. Thus it follows that the Gaussian random variables in the set $\mathscr{G}_{(\vec{u},\vec{v};k_1,k_2)}$ given by (5.1) form a set of independent Gaussian random variables.*

Proof. Let $\vec{u}=(u_1,\dots,u_n)$ and $\vec{v}=(v_1,\dots,v_n)$ be any vectors in $\mathbb{R}^n$. Then, for $i,j\in\{1,\dots,n\}$ with $i\neq j$, it follows that

$$\begin{aligned}&\int_0^T \big(u_i\alpha_i(t)k_1(t)-v_i\alpha_i(t)k_2(t)\big)\big(u_j\alpha_j(t)k_1(t)-v_j\alpha_j(t)k_2(t)\big)dt\\ &=u_iu_j\int_0^T \alpha_i(t)\alpha_j(t)k_1^2(t)dt-u_iv_j\int_0^T \alpha_i(t)\alpha_j(t)k_1(t)k_2(t)dt\\ &-v_iu_j\int_0^T \alpha_i(t)\alpha_j(t)k_1(t)k_2(t)dt+v_iv_j\int_0^T \alpha_i(t)\alpha_j(t)k_2^2(t)dt.\end{aligned}$$

From the condition for the functions k_1 and k_2, it follows that

$$\int_0^T \big(u_i\alpha_i(t)k_1(t)-v_i\alpha_i(t)k_2(t)\big)\big(u_j\alpha_j(t)k_1(t)-v_j\alpha_j(t)k_2(t)\big)dt=0$$

and the lemma is proved. □

Theorem 5.4. *Let F_{μ_1} and F_{μ_2} be functionals in $\widehat{\mathfrak{T}}_{\mathscr{A}}$ and let (k_1,k_2) be in the class $\mathscr{P}_\infty(\mathscr{A})$. Then for all real $q\in\mathbb{R}\setminus\{0\}$, the GCP $(F_{\mu_1}*F_{\mu_2})_q^{(k_1,k_2)}$ of F_{μ_1} and F_{μ_2} exists and is given by the formula*

$$\begin{aligned}&(F_{\mu_1}*F_{\mu_2})_q^{(k_1,k_2)}(y)\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle\bigg(\frac{u_j+v_j}{\sqrt{2}}\bigg)-\frac{i}{4q}\sum_{j=1}^n\|\alpha_j(u_jk_1-v_jk_2)\|_2^2\bigg\}d\mu_1(\vec{u})d\mu_2(\vec{v})\end{aligned}\tag{5.3}$$

for SI-a.e. $y\in C_0[0,T]$.

Proof. Using (2.2), (3.4), and the Fubini theorem, it first follows that for all $\lambda>0$ and SI-a.e. $y\in C_0[0,T]$,

$$\begin{aligned}(F_{\mu_1}*F_{\mu_2})_\lambda^{(k_1,k_2)}(y)&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle\frac{(u_j+v_j)}{\sqrt{2}}\bigg\}\\ &\qquad\times\bigg[\int_{C_0[0,T]}\exp\bigg\{\frac{i}{\sqrt{2\lambda}}\sum_{j=1}^n\langle u_j\alpha_jk_1-v_j\alpha_jk_2,x\rangle\bigg\}d\mathfrak{m}(x)\bigg]d\mu_1(\vec{u})d\mu_2(\vec{v}).\end{aligned}$$

Next, applying Lemma 5.3 and using (3.3), the Fubini theorem, and (3.8), it follows that

$$\begin{aligned}&(F_{\mu_1}*F_{\mu_2})_\lambda^{(k_1,k_2)}(y)\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle\frac{(u_j+v_j)}{\sqrt{2}}\bigg\}\prod_{j=1}^n\bigg[\Big(2\pi\|u_j\alpha_jk_1-v_j\alpha_jk_2\|_2^2\Big)^{-1/2}\\ &\quad\times\int_{\mathbb{R}}\exp\bigg\{\frac{i}{\sqrt{2\lambda}}r_j-\frac{r_j^2}{2\|u_j\alpha_jk_1-v_j\alpha_jk_2\|_2^2}\bigg\}dr_j\bigg]d\mu_1(\vec{u})d\mu_2(\vec{v})\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle\frac{(u_j+v_j)}{\sqrt{2}}-\frac{1}{4\lambda}\sum_{j=1}^n\|u_j\alpha_jk_1-v_j\alpha_jk_2\|^2\bigg\}d\mu_1(\vec{u})d\mu_2(\vec{v}).\end{aligned}$$

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Now, for each $\lambda \in \mathbb{C}$, let

$$
\begin{aligned}
&J^*_{(F_{\mu_1}*F_{\mu_2})}(k_1,k_2;\lambda)\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{j=1}^n\langle\alpha_j,y\rangle\frac{(u_j+v_j)}{\sqrt{2}}-\frac{1}{4\lambda}\sum_{j=1}^n\|u_j\alpha_jk_1-v_j\alpha_jk_2\|^2\bigg\}d\mu_1(\vec{u})d\mu_2(\vec{v}).
\end{aligned}
$$

Using similar methods as those used in the proof of Theorem 4.1, we can show that the function $J^*_{(F_{\mu_1}*F_{\mu_2})}(k_1,k_2;\lambda)$ is an analytic function of λ throughout $\mathbb{C}_+$, and is bounded as a function of λ in $\mathbb{C}_+$. Hence, in view of Definition 2.2 and by the bounded convergence theorem, the GCP $(F_{\mu_1}*F_{\mu_2})_q^{(k_1,k_2)}$ of F_{μ_1} and F_{μ_2} exists and is given by the right-hand side of (5.3) for SI-a.e. $y \in C_0[0,T]$. □

Theorem 5.5. *Let F_{μ_1}, F_{μ_2}, and (k_1,k_2) be as in Theorem 5.4. Then for all $q \in \mathbb{R}\setminus\{0\}$, the GCP $(F_{\mu_1}*F_{\mu_2})_q^{(k_1,k_2)}$ is in the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$.*

Proof. Define a set function $\varphi_q^{(k_1,k_2)}:\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)\to\mathbb{C}$ by the formula

$$
\varphi_q^{(k_1,k_2)}(V)=\iint_V\exp\bigg\{-\frac{i}{4q}\sum_{j=1}^n\|u_j\alpha_jk_1-v_j\alpha_jk_2\|_2^2\bigg\}d\mu_1(\vec{u})d\mu_2(\vec{v}) \tag{5.4}
$$

for $V\in\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$. Then $\varphi_q^{(k_1,k_2)}$ is a complex measure with finite total variation on $\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$. Next, let $\phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be the continuous function given by $\phi(\vec{u},\vec{v})=(\vec{u}+\vec{v})/\sqrt{2}$. Then ϕ is $\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$–$\mathscr{B}(\mathbb{R}^n)$ measurable. Finally, let the set function $\Phi_{c,q}^{(k_1,k_2)}:\mathscr{B}(\mathbb{R}^n)\longrightarrow\mathbb{C}$ be given by

$$
\Phi_{c,q}^{(k_1,k_2)}=\varphi_q^{(k_1,k_2)}\circ\phi^{-1}. \tag{5.5}
$$

Then $\Phi_{c,q}^{(k_1,k_2)}$ satisfies the countable additivity obviously. Based on these structure above, it follows that

$$
\begin{aligned}
\|\Phi_{c,q}^{(k_1,k_2)}\| &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bigg|\exp\bigg\{-\frac{i}{4q}\sum_{j=1}^n\|u_j\alpha_jk_1-v_j\alpha_jk_2\|_2^2\bigg\}\bigg|d|\mu_1|(\vec{u})d|\mu_2|(\vec{v})\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}d|\mu_1|(\vec{u})d|\mu_2|(\vec{v})\\
&=\bigg(\int_{\mathbb{R}^n}d|\mu_1|(\vec{u})\bigg)\bigg(\int_{\mathbb{R}^n}d|\mu_2|(\vec{v})\bigg)\\
&=\|\mu_1\|\|\mu_2\|<+\infty.
\end{aligned}
$$

Thus the set function $\Phi_{c,q}^{(k_1,k_2)}$ is a complex measure with finite total variation on $\mathscr{B}(\mathbb{R}^n)$. Using equation (5.3) together with (5.4) and (5.5), it follows that

$$
\begin{aligned}
(F_{\mu_1}*F_{\mu_2})_q^{(k_1,k_2)}(y) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\bigg\{i\bigg(\langle\vec{\alpha},y\rangle,\frac{1}{\sqrt{2}}(\vec{u}+\vec{v})\bigg)_{\mathbb{R}^n}\bigg\}d\varphi_q^{(k_1,k_2)}(\vec{u},\vec{v})\\
&=\int_{\mathbb{R}^n}\exp\Big\{i\big(\langle\vec{\alpha},y\rangle,\vec{r}\big)_{\mathbb{R}^n}\Big\}d\Phi_{c,q}^{(k_1,k_2)}(\vec{r})\\
&=F_{\Phi_{c,q}^{(k_1,k_2)}}(y)
\end{aligned} \tag{5.6}
$$

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for SI-a.e. $y \in C_0[0,T]$, where $(\cdot,\cdot)_{\mathbb{R}^n}$ denote the standard inner product on $\mathbb{R}^n$. Hence the GCP $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$ belongs to the class $\widehat{\mathfrak{T}}_{\mathscr{A}}$. □

6. Relationships between the GFFT and the GCP

In order to obtain our main results in this paper, we define the following conventions. Note that for any nonzero functions h_1 and h_2 in $L_2[0,T]$, there exists a function s in $L_2[0,T]$ such that

$$s^2(t) = h_1^2(t) + h_2^2(t) \tag{6.1}$$

for m_L-a.e. $t \in [0,T]$. It is clear that the function s satisfying (6.1) is not unique. Thus we will use the symbol $s(h_1,h_2)$ for any functions s which satisfy (6.1). Given functions h_1 and h_2 in $L_2[0,T] \setminus \{0\}$, a lot of functions, $s(h_1,h_2)$, exist in $L_2[0,T]$. Thus $s(h_1,h_2)$ can be considered as an equivalence class of the equivalence relation $\sim$ on $L_2[0,T]$ given by

$$s_1 \sim s_2 \Longleftrightarrow s_1^2 = s_2^2 \quad m_L\text{-a.e.}.$$

But we see that for every function s in the equivalence class $s(h_1,h_2)$, the Gaussian random variable $\langle s,x\rangle$ has the normal distribution $N(0, \|h_1\|_2^2 + \|h_2\|_2^2)$. If the functions h_1 and h_2 are in $L_\infty[0,T]$, then we can take $s(h_1,h_2)$ to be in $L_\infty[0,T]$.

Theorem 6.1. *Let F_{μ_1} and F_{μ_2} be functionals in $\widehat{\mathfrak{T}}_{\mathscr{A}}$, let h be a function in $\mathscr{O}_\infty(\mathscr{A})$, and let (k_1,k_2) be in the class $\mathscr{P}_\infty(\mathscr{A})$. Assume that*

$$h^2(t) = k_1(t)k_2(t)$$

for m_L-a.e. $t \in [0,T]$. Then for each $p \in [1,2]$ and all real $q \in \mathbb{R} \setminus \{0\}$, it follows that

$$T_{q,h}^{(p)}\big((F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}\big)(y) = T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F_{\mu_1})\left(\frac{y}{\sqrt{2}}\right) T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(F_{\mu_2})\left(\frac{y}{\sqrt{2}}\right) \tag{6.2}$$

for SI-a.e. $y \in C_0[0,T]$, where $s(h,k_1)$ and $s(h,k_2)$ are the functions which satisfy the relation

$$s(h,k_1)^2(t) = h^2(t) + k_1^2(t) \tag{6.3}$$

and

$$s(h,k_2)^2(t) = h^2(t) + k_2^2(t) \tag{6.4}$$

for m_L-a.e. $t \in [0,T]$, respectively.

Proof. In view of Theorem 4.2, it suffices to show that

$$T_{q,h}^{(1)}\big((F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}\big)(y) = T_{q,s(h,k_1)/\sqrt{2}}^{(1)}(F_{\mu_1})\left(\frac{y}{\sqrt{2}}\right) T_{q,s(h,k_2)/\sqrt{2}}^{(1)}(F_{\mu_2})\left(\frac{y}{\sqrt{2}}\right)$$

for SI-a.e. $y \in C_0[0,T]$.

Since the GCP $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$ of F_{μ_1} and F_{μ_2} is an element of $\widehat{\mathfrak{T}}_{\mathscr{A}}$ by Theorem 5.5, the GFFT of $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$, $T_{q,h}^{(1)}((F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)})$ exists for all $q \in \mathbb{R} \setminus \{0\}$ by Theorem 4.1. Next using (4.1) with F_μ replaced with $(F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}$, and (5.6), it follows that

$$T_{q,h}^{(1)}\big((F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}\big)(y) = \int_{\mathbb{R}^n} \exp\left\{ i \sum_{j=1}^n \langle \alpha_j, y\rangle r_j \right\} d(\Phi_{c,q}^{(k_1,k_2)})_{t,q}^h(\vec{r}) \tag{6.5}$$

where $(\Phi_{c,q}^{(k_1,k_2)})_{t,q}^h$ is the complex measure in $\mathscr{M}(\mathbb{R}^n)$ given by

$$(\Phi_{c,q}^{(k_1,k_2)})_{t,q}^h(U) = \int_U \exp\bigg\{-\frac{i}{2q}\sum_{j=1}^n \|\alpha_j h\|_2^2 r_j^2\bigg\} d\Phi_{c,q}^{(k_1,k_2)}(\vec{r}) \tag{6.6}$$

for $U \in \mathscr{B}(\mathbb{R}^n)$, and where $\Phi_{c,q}^{(k_1,k_2)}$ is given by (5.5). Now the assumption yields the equality that given any $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$,

$$\begin{aligned}
&(u_j+v_j)^2\|\alpha_j h\|_2^2 + \|u_j\alpha_j k_1 - v_j\alpha_j k_2\|_2^2 \\
&= (u_j+v_j)^2\int_0^T \alpha_j^2(t)h^2(t)dt + \int_0^T \big(u_j\alpha_j(t)k_1(t) - v_j\alpha_j(t)k_2(t)\big)^2 dt \\
&= u_j^2\int_0^T \alpha_j^2(t)\big(h^2(t)+k_1^2(t)\big)dt + v_j^2\int_0^T \alpha_j^2(t)\big(h^2(t)+k_2^2(t)\big)dt \\
&= u_j^2\|\alpha_j s(h,k_1)\|_2^2 + v_j^2\|\alpha_j s(h,k_2)\|_2^2
\end{aligned} \tag{6.7}$$

for each $j \in \{1, \dots, n\}$. Thus, using (6.5), (6.6), (5.5), (5.4), (6.7), the Fubini theorem, (4.1), (6.3), and (6.4), it follows that

$$\begin{aligned}
&T_{q,h}^{(1)}\big((F_{\mu_1} * F_{\mu_2})_q^{(k_1,k_2)}\big)(y) \\
&= \int_{\mathbb{R}^n} \exp\bigg\{i\sum_{j=1}^n \langle\alpha_j, y\rangle r_j - \frac{i}{2q}\sum_{j=1}^n \|\alpha_j h\|_2^2 r_j^2\bigg\} d\Phi_{c,q}^{(k_1,k_2)}(\vec{r}) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\bigg\{i\sum_{j=1}^n \langle\alpha_j, y\rangle \frac{(u_j+v_j)}{\sqrt{2}} - \frac{i}{4q}\sum_{j=1}^n \|\alpha_j h\|_2^2 (u_j+v_j)^2 \\
&\qquad\qquad - \frac{i}{4q}\sum_{j=1}^n \|u_j\alpha_j k_1 - v_j\alpha_j k_2\|_2^2\bigg\} d\mu_1(\vec{u})d\mu_2(\vec{v}) \\
&= \int_{\mathbb{R}^n} \exp\bigg\{i\sum_{j=1}^n \Big\langle \alpha_j, \frac{y}{\sqrt{2}}\Big\rangle u_j - \frac{i}{2q}\sum_{j=1}^n \Big\|\alpha_j\frac{s(h,k_1)}{\sqrt{2}}\Big\|_2^2 u_j^2\bigg\} d\mu_1(\vec{u}) \\
&\quad \times \int_{\mathbb{R}^n} \exp\bigg\{i\sum_{j=1}^n \Big\langle \alpha_j, \frac{y}{\sqrt{2}}\Big\rangle v_j - \frac{i}{2q}\sum_{j=1}^n \Big\|\alpha_j\frac{s(h,k_2)}{\sqrt{2}}\Big\|_2^2 v_j^2\bigg\} d\mu_2(\vec{v}) \\
&= T_{q,s(h,k_1)/\sqrt{2}}^{(1)}(F_{\mu_1})\Big(\frac{y}{\sqrt{2}}\Big) T_{q,s(h,k_2)/\sqrt{2}}^{(1)}(F_{\mu_2})\Big(\frac{y}{\sqrt{2}}\Big)
\end{aligned}$$

for SI-a.e. $y \in C_0[0,T]$. □

Setting $k_1 = k_2 = h$ in equation (6.2), we have the following corollary which agrees with the results in [3, 11].

Corollary 6.2. *Let F_{μ_1} and F_{μ_2} be functionals in $\widehat{\mathfrak{T}}_{\mathscr{A}}$ and let h be a function in $\mathscr{O}_\infty(\mathscr{A})$. Then for each $p \in [1,2]$ and all real $q \in \mathbb{R}\setminus\{0\}$, it follows that*

$$T_{q,h}^{(p)}\big((F_{\mu_1} * F_{\mu_2})_q^{(h,h)}\big)(y) = T_{q,h}^{(p)}(F_{\mu_1})\Big(\frac{y}{\sqrt{2}}\Big) T_{q,h}^{(p)}(F_{\mu_2})\Big(\frac{y}{\sqrt{2}}\Big)$$

for SI-a.e. $y \in C_0[0,T]$.

Theorem 6.3. *Let F_{μ_1}, F_{μ_2}, h, and (k_1,k_2) be as in Theorem 6.1 under the same assumption. Then for each $p \in [1,2]$ and all $q \in \mathbb{R}\setminus\{0\}$, it follows that*

$$(6.8)\qquad \Big(T^{(p)}_{q,s(h,k_1)/\sqrt{2}}(F_{\mu_1}) * T^{(p)}_{q,s(h,k_2)/\sqrt{2}}(F_{\mu_2})\Big)^{(k_1,k_2)}_{-q}(y) = T^{(p)}_{q,h}\bigg(F_{\mu_1}\Big(\frac{\cdot}{\sqrt{2}}\Big)F_{\mu_2}\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y)$$

for SI-a.e. $y \in C_0[0,T]$, *where* $s(h,k_1)$ *and* $s(h,k_2)$ *are the functions which satisfy the relations* (6.3) *and* (6.4) *respectively.*

Proof. Applying (4.3), (6.2) with F_{μ_1}, F_{μ_2}, and q replaced with $T^{(p)}_{q,s(h,k_1)/\sqrt{2}}(F_{\mu_1})$, $T^{(p)}_{q,s(h,k_2)/\sqrt{2}}(F_{\mu_2})$ and $-q$ respectively, and (4.3) again, it follows that for SI-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&\Big(T^{(p)}_{q,s(h,k_1)/\sqrt{2}}(F_{\mu_1}) * T^{(p)}_{q,s(h,k_2)/\sqrt{2}}(F_{\mu_2})\Big)^{(k_1,k_2)}_{-q}(y)\\
&= T^{(p)}_{q,h}\bigg(T^{(p)}_{-q,h}\bigg(\Big(T^{(p)}_{q,s(h,k_1)/\sqrt{2}}(F_{\mu_1}) * T^{(p)}_{q,s(h,k_2)/\sqrt{2}}(F_{\mu_2})\Big)^{(k_1,k_2)}_{-q}\bigg)\bigg)(y)\\
&= T^{(p)}_{q,h}\bigg(T^{(p)}_{-q,s(h,k_1)/\sqrt{2}}\Big(T^{(p)}_{q,s(h,k_1)/\sqrt{2}}(F_{\mu_1})\Big)\Big(\frac{\cdot}{\sqrt{2}}\Big)T^{(p)}_{-q,s(h,k_2)/\sqrt{2}}\Big(T^{(p)}_{q,s(h,k_2)/\sqrt{2}}(F_{\mu_2})\Big)\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y)\\
&= T^{(p)}_{q,h}\bigg(F_{\mu_1}\Big(\frac{\cdot}{\sqrt{2}}\Big)F_{\mu_2}\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y)
\end{aligned}$$

as desired. □

Setting $k_1 = k_2 = h$ in equation (6.8), we also have the following corollary.

Corollary 6.4. *Let F_{μ_1} and F_{μ_2} be functionals in $\widehat{\mathfrak{T}}_{\mathscr{A}}$ and let h be a function in $\mathscr{O}_\infty(\mathscr{A})$. Then for each $p \in [1,2]$ and all real $q \in \mathbb{R}\setminus\{0\}$, it follows that*

$$\Big(T^{(p)}_{q,h}(F_{\mu_1}) * T^{(p)}_{q,h}(F_{\mu_2})\Big)^{(h,h)}_{-q}(y) = T^{(p)}_{q,h}\bigg(F_{\mu_1}\Big(\frac{\cdot}{\sqrt{2}}\Big)F_{\mu_2}\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(y)$$

for SI-a.e. $y \in C_0[0,T]$.

Note that if $h = k_1 = k_2 \equiv 1$ on $[0,T]$, then the definitions of the L_p analytic GFFT $T^{(p)}_{q,h} \equiv T^{(p)}_{q,1}$ and the GCP $(\cdot * \cdot)^{(k_1,k_2)}_q \equiv (\cdot * \cdot)^{(1,1)}_q$ agree with the previous definitions [8, 9, 10, 17] of the analytic FFT $T^{(p)}_q$ and the CP $(\cdot * \cdot)_q$, because $Z_1(x,\cdot) = x$ for all $x \in C_0[0,T]$. Thus, setting $k_1 = k_2 = h = 1$ in (6.2) and (6.8), we also have the following corollary.

Corollary 6.5. *Let F_{μ_1} and F_{μ_2} be functionals in $\widehat{\mathfrak{T}}_{\mathscr{A}}$. Then for each $p \in [1,2]$ and all real $q \in \mathbb{R}\setminus\{0\}$, equations* (1.1) *and* (1.2) *with F and G replaced with F_{μ_1} and F_{μ_2} respectively hold true for SI-a.e.* $y \in C_0[0,T]$.

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