Taking Control - A Modified Epidemiological Model Investigating Strategies for Curbing Criminal Behaviour

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Abstract

Crime and its suppression and prevention have become rapidly growing challenges worldwide and are important considerations in many national policies on public safety and security. This model is based on the consideration and treatment of criminal behaviour as a "socially infectious disease"and consequently a public health concern. A modified epidemiological model is used to investigate the effects of two time dependent strategies to curb this behaviour. The first control strategy considered is preventative - aimed at encouraging potential criminals away from a life of crime, while the second one is aimed at rehabilitation of criminals. Local stability analysis around the equilibrium points is performed and Pontryagin's maximum principle is used to derive optimal control strategies for minimizing criminal behaviour. To confirm some of these findings, numerical simulations are performed. Based on the computational results obtained, a strategy using more of developmental crime prevention and early intervention programs may be the most practical to implement. This perspective has implications for the design and development of programs and targets which may be used as guides by policy-makers.

Keywords: Optimal control, Criminal behaviour, Compartmental model, Pontryagin's maximum principle

2010 MSC: 49K15, 34H05, 93C15

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Preprint submitted to xxxxxx *November 17, 2023*

17 Nov 2023 16:57:42 PST
230725-Sooknanan Version 2 - Submitted to Rocky Mountain J. Math.

1. Introduction

A major challenge facing many emerging and developing countries nowadays is that of criminal behaviour [1]. Apart from creating a general feeling of fear and insecurity, this behaviour may result in a decline in tourism, increased emigration in conjunction with a resulting brain drain as well as a loss of investor confidence - factors which contribute to reduced economic growth and development. With limited resources (financial, infrastructural) available to tackling this problem, mitigation strategies need to be implemented in a well-planned manner that is both cost-effective and time efficient.

In 1996, the WHO declared violence as a major, growing public health problem worldwide [2]. Violent young people tend to commit a range of crimes [2]. An interdisciplinary approach to tackling crime that is rapidly gaining popularity is to consider violent behaviour as a "contagious brain process"[3] and then use a public health approach to mitigate its spread [4, 3]. In this approach, after defining the problem, risk and protective factors are identified and used to develop and then test mitigation strategies. If these strategies are deemed effective, they are then implemented. In the absence of the wherewithal to adopt this approach fully, but still in keeping with the general precepts, mathematical models can be used to bridge the gap by "testing"different prevention and intervention strategies to determine the most effective ones.

This is not a new concept - modeling of behaviour using an infectious disease approach has been done for juvenile delinquency [5] fanatical and violent ideology [6, 7, 8, 9], violent crime and burglary [10, 11] and gang membership [12, 13]. By considering criminal behaviour as a socially infectious disease spread by peer influence, a similar approach can be used to model criminal behaviour and to explore treatment and prevention control strategies for its mitigation.

Mathematical models are commonly used to explore possible "what if"scenarios and compare the relative merits of different control strategies, both in preparation for and during an outbreak [14]. These may include combinations of prevention strategies (such as vaccination and quarantine) and treatment strategies such as isolation, which can lead to disease elimination when administered at the right time and in the right amount [15, 16]. By applying these concepts from the Public Health Approach to violence prevention, models can assist in determining "what proportion of individuals to treat at each point in time subject to the cost of treatment, the value of susceptible and infected individuals to social welfare, as well as the transmission dynamics"[17].

The majority of modelling work incorporating control strategies is of two types. In the first type, the control strategies are represented by a parameter and the goal is to understand how changing the value of the parameter changes the dynamics of the system. In the second category, the controls are allowed to vary with time and the goal is to minimise the cost of infection or the cost of implementing the control, or both. This technique is known as optimal control theory and can provide valuable information about the optimal use of prevention and treatment resources especially when resources are limited [18]. Here, the aim of applying optimal control is to control the spreading of the disease while considering cost of an activity or program to society.

Optimal control theory has been used to identify strategies for the treatment of many diseases - COVID-19 [19], Ebola [20], Dengue [21] and Pandemic Influenza [18]. It has also been applied to social problems [22], crime to determine optimal intervention strategies aimed at reducing property crime while considering the effects of unemployment [23], gangs and financial crime [24, 25].

This paper applies optimal control methods specifically Pontryagin's maximum principle, to a dynamic model where criminal behaviour is treated as an infectious disease. Our aim is to find the best strategy that will minimize the total number of potential criminals. This paper is organised as follows: Section 2 describes the model. Section 3 analyses the model with constant controls in terms of basic reproduction number, equilibria and stability. The formulation of the optimal control problem is presented in Section 4 and the results of numerical simulations and the discussion are described in Sections 5 and 6 with conclusions summarised in Section 7.

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2. Formulation of the Model

Criminal behaviour and violence behave as contagious diseases with similar characteristics to an infectious disease such as clustering, spread, and transmission [26]. From this perspective, an epidemiological model may therefore be modified to describe the transmission of criminal behaviour in a population while including strategies to both mitigate spread as well as to to reduce the likelihood of developing "the disease"analogous to public health interventions to managing an infectious disease outbreak. Primary prevention strategies aim to prevent the onset of criminal behaviour. This is based on the use of developmental crime prevention and early intervention strategies and programs designed to build protective factors and provide support and guidance to at risk individuals. Strategies include educational, skill and competency building, and mentoring programs [27] as well as practices such as the use of youth street workers to find and assist at risk individuals [28]. In contrast, tertiary prevention focuses on the aftermath of this behaviour with rehabilitation and reintegration treatment programs designed to reduce recidivism. This includes strategies such as educational and vocational programs, treatment center placement, and mental health counseling.

Using this framework for our model, the population is divided into four disjoint compartments/ classes based on status with respect to criminal behavior:

P : Members of the population who are at risk/ susceptible to criminal behaviour - Potential Criminals.

C : Members of the population who are engaged in criminal behaviour - Criminals.

J : Members of the population who are incarcerated.

R : Members of the population who are in an prevention or rehabilitation program and are no longer susceptible to criminal behaviour - Reformed/ Recovered people.

2.1. Model Equations

Figure 1 illustrates the flow of individuals among these compartments. By considering the contagiousness of criminal behaviour, vulnerable individuals P through interaction with criminals C may engage in criminal behaviour at the rate $\frac{\beta CP}{N}$. However, prevention programs represented by $u_1(t)$ target these vulnerable individuals to to lower the likelihood of future violence and criminality so that they are no longer susceptible to criminal behaviour. Though criminals may be removed via incarceration at a rate Φ, on release, they may return to the same environment that initially created the opportunities for criminal behavior and some may re-join the criminal class at a rate $(1-f)\gamma$, where γ^{-1} represents the time spent in incarceration. Yet there are alternatives to incarceration for certain types of offences (depending on their seriousness/severity) - these can take the form of rehabilitation programs - $u_2(t)$ - with the aim of preventing future crime by altering a criminal's behavior. The rate of entry and exit into the system is proportional to the population size and given by μ.

A description of model parameters is given in Table 1. Since the model monitors changes in the human population, the variables and the parameters are assumed to be positive for all $t \geq 0$. The following system of nonlinear differential equations describes the dynamics of the system:

$$P' = \mu N - \frac{\beta CP}{N} - u_1(t)P - \mu P \tag{1}$$

$$C' = \frac{\beta CP}{N} + (1-f)\gamma J - u_2(t)C - \phi C - \mu C \tag{2}$$

$$J' = \phi C - \gamma J - \mu J \tag{3}$$

$$R' = u_1(t)P + u_2(t)C + f\gamma J - \mu R \tag{4}$$

$$N = P + C + J + R \tag{5}$$

Assuming a constant population size N, and re-scaling, so that $p = \frac{P}{N}$; $c = \frac{C}{N}$; $j = \frac{J}{N}$ and $r = \frac{R}{N}$ gives the following system of five equations:

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Table 1: Description of the Parameters used in the Model

Parameter	Description
μ	Entry rate and Death rate
$1-f$	Proportion of incarcerated people who return to life of crime
γ	(average length of incarceration)$^{-1}$
β	Contact rate which results in becoming a criminal
Φ	Rate of incarceration
u_1	The proportion of people in a prevention program per year
u_2	The proportion of criminals in a rehabilitation program per year

$$p' = \mu - \beta cp - u_1(t)p - \mu p \tag{6}$$

$$c' = \beta cp + (1-f)\gamma j - u_2(t)c - \phi c - \mu c \tag{7}$$

$$j' = \phi c - \gamma j - \mu j \tag{8}$$

$$r' = u_1(t)p + u_2(t)c + f\gamma j - \mu r \tag{9}$$

$$p + c + j + r = 1 \tag{10}$$

3. Analysis of the Model with Constant Controls

The first step of the analysis is to check that the model is well-posed. Since $p' \geq 0$ if $p = 0$, $c' \geq 0$ if $c = 0$, $j' \geq 0$ if $j = 0$ and $r' \geq 0$ if $r = 0$, we have $p \geq 0$, $c \geq 0$, $j \geq 0$,$r \geq 0$ for $t \geq 0$. Also, since $p' \leq 0$ if $p = \frac{\mu}{u_1+\mu}$, we have $p \leq \frac{\mu}{u_1+\mu}$ for $t \geq 0$. Thus the solution always remains in the biologically realistic region $0 \leq p \leq \frac{\mu}{u_1+\mu}$, $c \geq 0$, $j \geq 0$,$r \geq 0$, where $0 \leq p, c, j, r \leq 1$. We now examine the existence and stability behaviour of the system at equilibrium points.

3.1. Equilibrium points

Equilibrium states (steady states of the system) provide insight into the long-term behavior of a system and can be used to determine if the system has

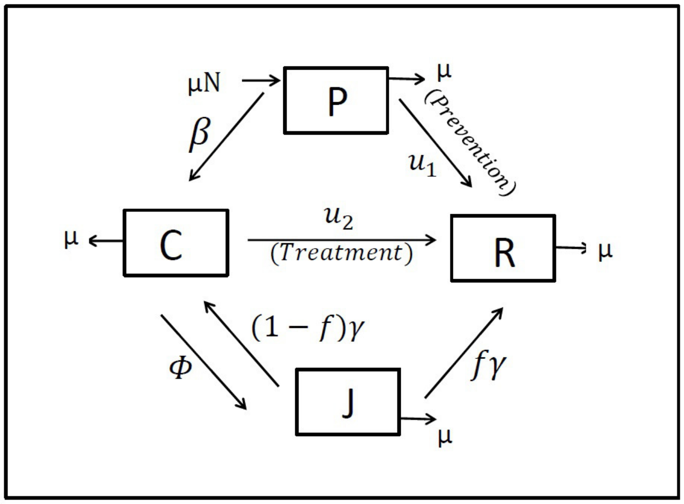

Figure 1: Model Diagram

periodic behavior or not, i.e., whether there are oscillations. After setting the system of equations equal to zero, and solving the resultant equations, there are two possible equilibrium states - the criminal-free equilibrium where criminal behaviour is not present in the population and the coexistence equilibrium. At the criminal-free equilibrium , $E_0 : (p, c, j, r) = (\frac{\mu}{u_1+\mu}, 0, 0, \frac{u_1}{u_1+\mu})$ and at the co-existence equilibrium E_1: $p = \frac{u_2+\phi+\mu-(1-f)\gamma\lambda}{\beta}$, $c = \frac{\mu}{\beta p} - \frac{(u_1+\mu)}{\beta}$, $j = \frac{\phi}{\gamma+\mu} c$ and $r = 1 - p - c - j$. In order to determine the conditions necessary to mitigate criminal behaviour so that the system is in a criminal-free equilibrium, the basic reproduction number R_0 is determined.

3.2. Calculation of R_0

In the early stages of a crime outbreak, R_0 is the key quantity of interest, and the goal is to identify mitigation strategies to reduce it below the threshold $R_0 = 1$. When this occurs, the system will be in the desirable criminal-free equilibrium state. R_0 is calculated using the next generation operator method [29]. In applying this method, c and j are considered the infective compartments and s and r represent non-infective classes. We obtain

$$R_0 = \frac{\beta P*}{(u_2 + \phi + \mu) - \frac{(1-f)\gamma\Phi}{\gamma+\mu}},$$

where $P* = \frac{\mu}{u_1+\mu}$. For our model, R_0 represents the fraction of individuals leaving p and j who progress to c. Re-writing the co-existence equilibrium in terms of R_0 i.e. $j = \frac{\phi}{\gamma+\mu}, c$, $p = \frac{\mu}{u_1+\mu}\frac{1}{R_0}$ and $c = \frac{\eta}{\beta}(R_0 - 1)$, shows its relation to R_0 where it exists when $R_0 > 1$. This means that to control the spread of criminal behaviour, we need to reduce R_0 to values less than one.

3.3. Stability

The only steady state solutions that are observable in a physical system are the stable equilibria. In this section, the local stability of the criminal-free equilibrium and the co-existence equilibrium will be investigated. In order to determine the conditions necessary for the stability of each state, the Jacobian

$$J = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix}$$

of the system is determined such that

$$A_1 = -\beta c - u_1 - \mu$$

$$A_2 = -\beta p$$

$$A_3 = 0$$

$$A_4 = \beta c$$

$$A_5 = \beta p - u_2 - \phi - \mu = \beta p - \psi$$

$$A_6 = (1 - f)\gamma$$

$$A_7 = 0$$

$$A_8 = \phi$$

$$A_9 = -\gamma - \mu$$

where $\psi = u_2 + \phi + \mu$, and where for ease of analysis the system is reduced to three differential equations since $r = 1 - p - c - j$.

3.4. Stability of the criminal-free equilibrium

The criminal-free equilibrium values are substituted into J. The algebraic expressions in the Jacobian matrix are simple enough to allow for a straightforward determination of the eigenvalues.

The eigenvalues are:

1. $-u_1 - \mu < 0$,
2. $\frac{1}{2}\left[\beta\frac{\mu}{u_1+\mu} - \psi - \gamma - \mu\right] \pm \frac{1}{2}\sqrt{\begin{array}{c}\left(\beta\frac{\mu}{u_1+\mu} - \psi - \gamma - \mu\right)^2 + \\ 4(\gamma+\mu)\left(\beta\frac{\mu}{u_1+\mu} - \psi\right) + 4(1-f)\gamma\phi\end{array}}$

These eigenvalues will be negative when $(\gamma+\mu)\left(\beta\frac{\mu}{u_1+\mu} - \psi\right) + (1-f)\gamma\phi < 0$. Hence $R_0 = \frac{\beta\frac{\mu}{u_1+\mu}}{\left(\psi - \frac{(1-f)\gamma\phi}{\gamma+\mu}\right)} < 1$ and $\beta\frac{\mu}{u_1+\mu} - \psi - \gamma - \mu < 0$. It follows that the criminal-free equilibrium is locally (asymptotically) stable if $R_0 < 1$.

3.5. Stability of the co-existence equilibrium

$$p' = \mu - \beta cp - u_1(t)p - \mu p \tag{11}$$

$$c' = \beta cp + (1-f)\gamma j - u_2(t)c - \phi c - \mu c \tag{12}$$

$$j' = \phi c - \gamma j - \mu j \tag{13}$$

$$r' = u_1(t)p + u_2(t)c + f\gamma j - \mu r \tag{14}$$

$$p + c + j + r = 1 \tag{15}$$

After substituting the co-existence equilibrium expressions in J and obtaining the characteristic equation $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, the Routh-Hurwitz criteria $a_1 > 0$; $a_3 > 0$ and $a_1a_2 - a3 > 0$ are used to obtain conditions for stability.

Here

$$a_1 = (\gamma+\mu) - (\beta p - \psi) + (\beta c + u_1 + \mu).$$

From eq (7)

$$\beta p - \psi = -(1-f)\gamma j/c = -(1-f)\gamma\frac{\phi}{\gamma+\mu}$$

which gives

$$a_1 = (\gamma+\mu) + (1-f)\gamma\frac{\phi}{\gamma+\mu} + (\beta c + u_1 + \mu) > 0.$$

Also,

$$a_3 = -\phi(1-f)\gamma\left(\beta c + u_1 + \mu\right) + \left(\gamma+\mu\right)\beta c\left(\beta p\right) - \left(\gamma+\mu\right)\left(\beta p - \psi\right)\left(\beta c + u_1 + \mu\right).$$

Using eq (7),

$$a_3 = \left(\gamma+\mu\right)\beta c\left(\beta p\right) > 0,$$

$$a_2 = \left(\gamma+\mu\right)\left(\beta c + u_1 + \mu\right) + \beta c \beta p + \left(\beta c + u_1 + \mu\right)(1-f)\gamma\frac{\phi}{\gamma+\mu},$$

$$a_1 a_2 - a_3 = \begin{pmatrix} \gamma+\mu \\ +\frac{(1-f)\gamma\phi}{\gamma+\mu} \\ +\beta c + u_1 + \mu \end{pmatrix} \begin{pmatrix} \left(\gamma+\mu\right)\left(\beta c + u_1 + \mu\right) \\ +\beta c \beta p \\ +\frac{(\beta c + u_1 + \mu)(1-f)\gamma\phi}{\gamma+\mu} \end{pmatrix} - \left(\left(\gamma+\mu\right)\beta c\left(\beta p\right)\right)$$

$$= \begin{pmatrix} \gamma+\mu \\ +\frac{(1-f)\gamma\phi}{\gamma+\mu} \\ +\eta(R_0-1) \\ +u_1+\mu \end{pmatrix} \begin{pmatrix} (\gamma+\mu)\begin{pmatrix} \eta(R_0-1) \\ +u_1+\mu \end{pmatrix} + \\ \beta\mu(1-\frac{1}{R_0})+ \\ \frac{(\eta(R_0-1)+u_1+\mu)(1-f)\gamma\phi}{\gamma+\mu} \end{pmatrix} - \left(\beta\mu\left(\gamma+\mu\right)(1-\frac{1}{R_0})\right)$$

and we can conclude that $a_1a_2 - a_3 > 0$ when $R_0 > 1$. Thus, the co-existence equilibrium is locally (asymptotically) stable for $R_0 > 1$.

4. Formulation and Analysis of the Optimal Control Problem

So far, we have considered prevention and treatment programs that must be maintained at constant levels at all times. However, this may be costly over time and it may be more practical to consider time dependent controls. This section describes the circumstances under which criminal behaviour can be controlled or curtailed using optimal control theory. Though there are seven parameters in the model, u_1 and u_2 are the measures that are designated for control in this model.

A straightforward computation shows their relation to R_0 and the partial derivatives

$$\frac{\partial R_0}{\partial u_1} = -\frac{\beta\mu}{\left((u_1+\mu)^2\,(u_2+\phi+\mu) - \frac{(1-f)\gamma\Phi}{\gamma+\mu}\right)}$$

$$\frac{\partial R_0}{\partial u_2} = -\frac{\beta P*}{\left((u_2+\phi+\mu) - \frac{(1-f)\gamma\Phi}{\gamma+\mu}\right)^2}.$$

Since $\frac{\partial R_0}{\partial u_1} < 0$ and $\frac{\partial R_0}{\partial u_2} < 0$, increasing the prevention and treatment rate decreases R_0. This means that by controlling u_1 and u_2 the transmission of criminal behaviour may be reduced.

An important consideration when designing any public health policy is the costs associated with its implementation. For this model, such costs include not only the cost of prevention and rehabilitation measures but the other costs mentioned in the introduction. Our aim is to minimize the number of criminals while at the same time minimizing the cost of controls $u_1(t)$, $u_2(t)$ on $[0;T]$ where T is the time period over which the mitigation strategies will be applied. Thus, we are seeking an optimal control pair $(u_1^*(t), u_2^*(t))$ so that

$$J(u_1^*, u_2^*) = min\,\{J(u_1,u_2) : (u_1,u_2) \in U\} \tag{16}$$

where the Lebesgue measurable control set U is defined as

$$U = \{u_1(t), u_2(t) : 0 \le u_1 \le 1, 0 \le u_2 \le 1, t \in [0,T]\}$$

subject to

$$p' = \mu - \beta cp - u_1 p - \mu p \tag{17}$$

$$c' = \beta cp + (1-f)\gamma j - u_2 c - \phi c - \mu c \tag{18}$$

$$j' = \phi c - \gamma j - \mu j \tag{19}$$

$$r' = u_1 p + u_2 c + f\gamma j - \mu r \tag{20}$$

$$p + c + j + r = 1. \tag{21}$$

The control scheme is optimal if it minimizes the objective function

$$J(u_1, u_2) = \int_0^T \left[Kc(t) + \frac{B_1}{2} u_1^2 + \frac{B_2}{2} u_2^2 \right] dt \tag{22}$$

where B_1 and B_2 are the relative weights attached to the cost or effort required to implement each of the control measures and reflect the importance of one type of measure over the other. Here, the weight factor K balances out the relative importance of the $c(t)$ in the objkective functional. It is a relative measure of the importance of reducing the number of criminals. Therefore, the term $Kc(t)$ represents the cost due to the number of criminals. The first term on the right hand side in eq.22 represents the goal of the control measures - to reduce the number of the criminals, while the other terms denote the systemic cost of the control measures.

The squares on the control variables are used to capture the effects of non-linear costs potentially arising from implementation of the controls. This form was chosen as implementation of any public health intervention does not have a linear cost, but instead there are increasing costs with reaching higher fractions of the population [30]. The quadratic cost on the control is the simplest and most widely used nonlinear representation of control cost [31, 32].

The necessary conditions that an optimal control must satisfy come from Pontryagin's Maximum Principle. The existence of the optimal controls u_1^*, u_2^* and the corresponding optimal solutions $p^*(t), c^*(t), j^*(t), r^*(t)$ using Pontryagin's maximum principle are given in the Appendix. The state system of differential equations and its adjoint system together with the control characterization form the optimality system, which is solved numerically in the next section.

5. Numerical Results

Prevention and treatment programs are essential components of any crime reduction program. A central concern among policy makers and planners is allocation of controls i.e. the division of control efforts between treatment and prevention schemes. In this section, the control strategies are implemented

Table 2: Values of the Parameters used in the Model

Parameter	Value
μ	0.0503+0.00258=0.05288 yr^{-1}
$1-f$	0.56
γ	0.2
β	0.71
u_1,u_2,T	$0 \leq u_1, u_2 \leq 1$, $t \in [0, 5]$
Φ	0.115
p(0), c(0) j(0)	$0.8278, 0.04, 0.0322$

and their effect in reducing or eliminating potential criminals and criminals is investigated. This is done by solving the state equations using a numerical approach known as the backward-forward sweep method in Matlab [33, 21]. The values for the initial conditions and model parameters are given in Table 2. Though parameters were a challenge to determine, the values were estimated using data from Trinidad and Tobago with full details provided in [12].

The age at which individuals can be held liable for criminal offences varies around the world - ranging generally from from the age of 10 years in England and Wales to 14 years in Trinidad and Tobago to 16 years in Argentina [34]. Bearing this in mind, we consider a population $N(t)$ of individuals aged more that 14 years who are “at risk”for engaging in or have engaged in criminal behaviour. Based on the ages of the population considered, the term delinquent may be used interchangebly in this model as delinquency is a wrongful act committed by a juvenile, whereas a “crime”is generally attributed to an adult.

Three scenarios are investigated for the control measures which are applied over a five year period:

Scenario 1 : The situation where control $u_1(t)$ is more costly than $u_2(t)$ (i.e. the weight factors are $B_1 = 10$, $B_2 = 1$) so that prevention strategies require a bigger cost or effort than rehabilitation ones, and $K = 10$.

Scenario 2 : The situation where control control $u_2(t)$ is more costly than $u_1(t)$ (i.e. the weight constants are $B_1 = 1$, $B_2 = 10$) so that rehabilitation strategies require a bigger cost or effort than intervention ones, and $K = 10$.

Scenario 3 : The situation where the cost or effort of implementing both controls are weighed equally (i.e. the weight constants are $B_1 = 1$, $B_2 = 1$), and $K = 10$.

A relatively high value of $K = 10$ is chosen since we want to emphasize that in our optimization attempts, the resulting size of the criminal group should be as small as possible. Note also that for optimality, it is preferable to use less of the control with the bigger weight and more of the control with the lesser weight.

As shown in Figure 2, Scenario 1 resulted in the lowest proportion of recovered people, while Scenario 2 generated the highest proportion of individuals in the recovered category, having renounced a life of crime. It is also interesting to observe that the largest proportion of detained criminals corresponds to Scenario 2, though this is only approximately a 2% more than the others.

Numerical results from Figure 3 show that, in general, the use of prevention and rehabilitation controls results in a marked reduction from the onset in the number of the criminals. This is true for all three scenarios under consideration. In the control profile for scenario 1, the optimal treatment control u_2 remains at the upper bound until time t =2 years, before steadily decreasing to the lower bound. Note that in scenario 1, more of the rehabilitation control $u_2(t)$ is used, while the prevention control $u_1(t)$ does not play a significant role. From a practical viewpoint, this scenario is far from ideal. In most treatment programs, challenges such as recruitment, attendance and compliance [35] are common - which hampers the overall effectiveness of the strategy.

In the case where both weights are equal $B_1 = B_2 = 1$ (i.e. Scenario 3), although there is a high proportion of recovered people (see Figure 2), we must apply more of the rehabilitation control u_2 - which remains at the upper bound for almost two years, while the prevention control u_1 starts off at the upper bound but rapidly decreases to zero therafter. As previously explained, keeping u_2 at the maximum value for this long may be difficult and impractical to sustain. This Scenario also requires significantly more of the treatment control u_2 than of the prevention control u_1 at any given time. For these reasons, we do not consider this scenario to be an efficient stategy for the reduction of criminal behaviour.

In contrast, for Scenario 2 with $B_1 = 1$, $B_2 = 10$, the main emphasis is on prevention of criminal activity. Research has demonstrated the success of developmental crime prevention and early intervention programs [36]. Despite this scenario having the highest proportion of incarcerated people (see Figure 2), a significant reduction of criminals is still achieved (Figure 3 indicates an overall decrease from 30% to 10%). The control profile in Figure 3 indicates that the optimal treatment control u_2 is at the upper bound for $t = \frac{1}{2}$ years before rapidly decreasing to the lower bound. In contrast, the prevention control u_1 starts off at approximately 80%, and takes just over 2 years to decrease to the same level (just under 30%) as the treatment control u_2. The rapid decrease in the level of u_2 is of great significance, given that this is the most difficult type of control strategy to implement successfully [35]. This strategy where we "hit them hard "at the beginning and then taper all efforts is undoubtedly the most practical of the three scenarios examined.

It is interesting to note that none of the scenarios examined reduces the proportion of criminals to zero over the time period of implementation. Since the emphasis is on prevention and rehabilitation, it is also not surprising that there is no significant change in the number of people incarcerated for any of the scenarios under consideration (see Figure 2).

6. Cost Effectiveness Analysis

Cost-effectiveness analysis is a method used to compare the cost benefits of implementing the control strategies implemented. In this section we will consider two approaches, the average cost-effectiveness ratio (ACER) and the

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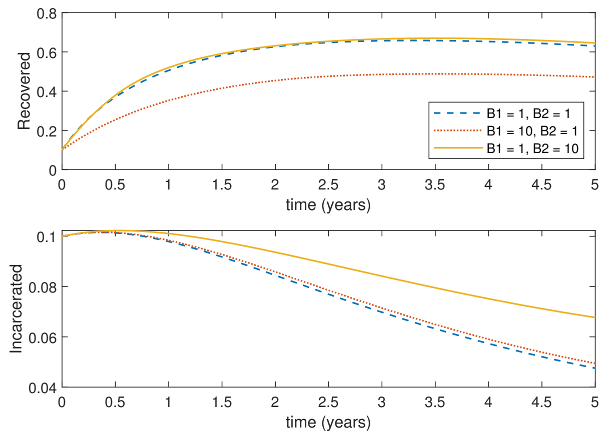

Figure 2: Comparison of the effect of all the different control strategies on the number of recovered individuals and those incarcerated

incremental cost-effectiveness ratio (ICER).

Average Cost-Effectiveness Ratio (ACER) deals with a single intervention approach. The ACER is calculated as

$$ACER = \frac{\text{Total cost for the implementation of the strategy}}{\text{Total number of infections reduced by the strategy}}.$$

Based on this cost analysis, the most cost-effective strategy is the one with the smallest ACER value.

In view of the objective functional , the total cost produced by an intervention is expressed mathematically as

$$\int_0^T \left[\frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2\right] dt.$$

The number of infections reduced due to the particular strategy is estimated as the difference between the total number of infected individuals without control and the total number of infected individuals with control in the simulation period.

Incremental Cost-Effectiveness Ratio (ICER) is used to compare the differences between the costs and health outcomes of two optional approaches. To

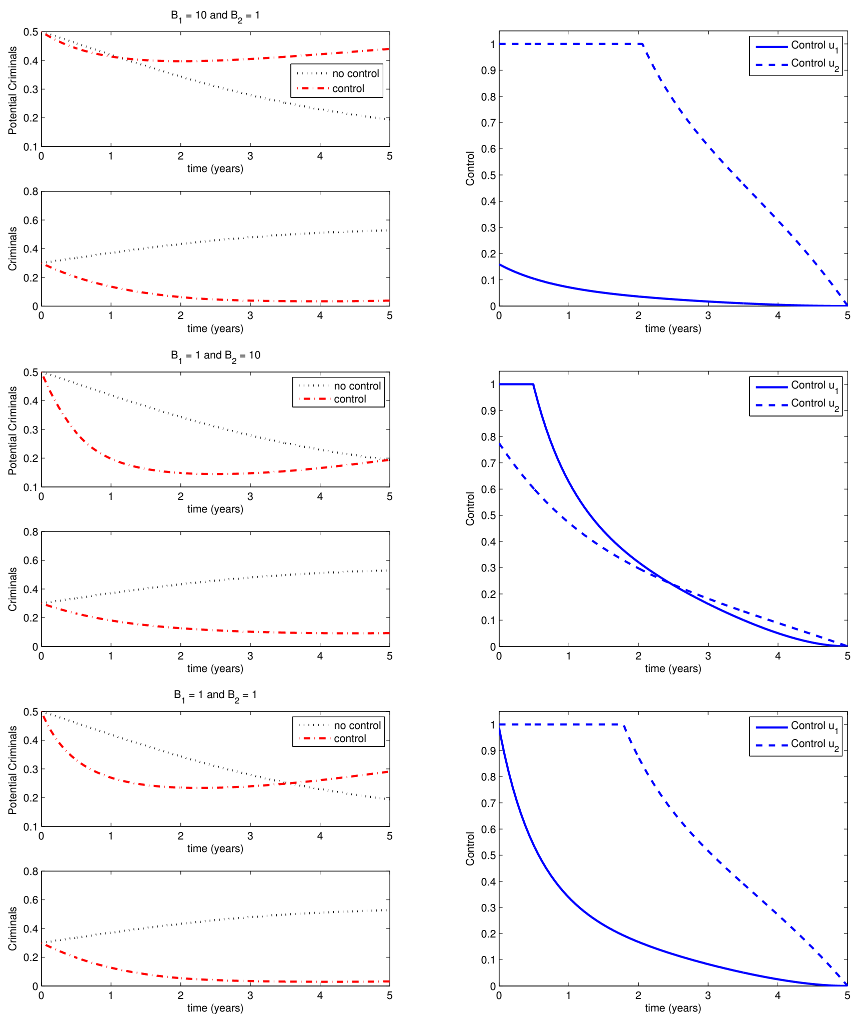

Figure 3: Comparison of Potential Criminals and Criminals with respect to different control options with the top graphs showing scenario 1, the middle ones scenario 2 and the lowest scenario 3

Table 3: Total infection reduction, total cost and ACER

Strategy	Infection Reduction	Total Cost	ACER
Strategy 1: $B_1 = 10, B_2 = 0$	0.4895	1.5266	3.1186
Strategy 2: $B_1 = 0, B_2 = 50$	0.8019	3.2919	4.1054
Strategy 3: $B_1 = 10, B_2 = 50$	0.9730	3.5519	3.6505

compare two or more competing intervention strategies incrementally, one intervention is compared with the next-less-effective alternative. Thus, the ICER is calculated as

$$ICER = \frac{\text{Total Cost (Strategy A)} - \text{Total Cost (Strategy B)}}{\text{\# Infections Reduced (Strategy A)} - \text{\# Infections Reduced (Strategy B)}}$$

The strategy with the least ACER is the most cost effective.

The ACER values for three control strategies are calculated in Table 3. These are

Strategy 1 : Prevention as the control. In this case, only u_1 is taken as a control variable.

Strategy 2 : Rehabilitation as the control. In this case, only u_2 as the control variable.

Strategy 3 : Combination of preventative and intervention strategies. In this case u_1 and u_2 are defined as control variables.

It is assumed that the cost of rehabilitation is significantly greater than the cost of prevention, we assume the weight constants $B_1 = 10$ and $B_2 = 50$.

From Table 3, it follows that Strategy 2 is more costly and less effective than Strategy 1 and Strategy 3 . We conclude that the Strategy 1 is the most cost effective of all for this particular study. Thus, according to ACER the control strategies from the most to the least are listed as Strategy 1, Strategy 3, and Strategy 2.

To implement the ICER, the interventions strategies are then ranked according to their increasing order of total number of infections reduced. The strategy to be discarded from the list of alternative interventions at each step is that corresponding to the highest ICER value. We first compared the ICER value for Strategies 1 and 2. Their ICER values are computed as follows:

$$\text{ICER}(1) = \frac{1.5266}{0.4895} = 3.1186$$

$$\text{ICER}(2) = \frac{3.2919 - 1.5266}{0.8019 - 0.4895} = 5.6508$$

From ICER(1) and ICER(2)), it is observed that the ICER for Strategy 2 is greater than the ICER for Strategy 1. This implies that Strategy 2 strongly dominates Strategy 1, indicating that Strategy 1 is less costly and more effective in comparison with Strategy 2. As a result, Strategy 2 is eliminated from subsequent ICER computations.

Strategies 1 and 3 are now compared.

$$\text{ICER}(1) = \frac{1.5266}{0.4895} = 3.1186$$

$$\text{ICER}(3) = \frac{3.5519 - 1.5266}{0.9730 - 0.4895} = 54.1888$$

From ICER(1) and ICER(3), it is observed that the ICER for Strategy 3 is greater than the ICER for Strategy 1. This implies that Strategy 3 strongly dominates Strategy 1, indicating that Strategy 1 is less costly and more effective in comparison with Strategy 3. Therefore, Strategy 1 (the Strategy that implements prevention controls) has the least ICER and is thus most cost-effective of all the control strategies. This agrees with the results obtained before using the ACER method, that Strategy 1 is the most cost-effective strategy.

7. Discussion

Violence and criminality affects not only their victims but ultimately exacts both a human and an economic toll on countries. Model results confirm that that like any disease, prevention is better than cure and it is more cost effective to prevent than it is to treat. Research has shown that rapid, early intervention and prevention programs are more effective in reducing delinquent behavior [37] than rehabilitation programs. Figure 2 showed that when using both strategies with an emphasis on prevention, the prevention control needs only to be applied at a maximum value for a short time initially before gradually decreasing in contrast to when there is an emphasis on rehabilitation. This means that changing behaviour with family, school, and community interventions can make a significant difference so that protective factors outweigh risk factors.

The recent COVID-19 pandemic has highlighted the importance of mathematical models in informing policy decisions. In the design and development of crime mitigation initiatives, cost and strategy are important considerations for effective delivery of programs. This method has implications for designing strategies to inform policy responses. It allows us to conduct "social experiments"without the ethics and costs attached to experimenting on human beings, and may provide valuable insights into the effectiveness of time-dependent control efforts. Using this method, controls $u_1(t)$ and $u_2(t)$ may be gradually relaxed from maximum effort $u_1 = 1$ and $u_2 = 1$ in an approximate time frame, as indicated by the model, and the resources diverted elsewhere. While parameters were estimated, the model has the potential to assist in setting targets for the amount and length of of control strategies. Future research can investigate whether these strategies are more effective than improving areas such as deterrence (related to contact rates), incarceration rate and rehabilitation/recividism in the criminal justice system.

8. Conclusion

Implementing programs for reducing the number of individuals involved in criminal behaviour is usually a challenge for policy makers and planners. Mathematical models may be used to determine "dynamically varying policies"subject to the cost of controls and the burden of criminals to society. With constant contols, achieving the goal of reducing criminals may be costly, as the intervention and prevention programs will need to keep running for infinite time at those levels. However, when time dependent controls via optimal control is applied, new insights on different intervention and rehabilitation policies may arise. Based on the computational results obtained, a strategy using more of developmental crime prevention and early intervention programs may be the most practical and least costly to implement.

Appendix A. The optimal control: Existence and characterization

The necessary conditions that an optimal control must satisfy come from Pontryagin's Maximum Principle. We first show the existence of solutions of the system and then we will prove the existence of optimal control.

Appendix A.1. Existence of an Optimal Control Pair

From the definition of the controls u_1 and u_2 and the restrictions on the nonnegativeness of the state variables we see that a solution of the system exists [38].

The existence of the optimal control can be obtained using a result by [39]. To use this result, we must check the following properties:

(1) The set of controls and corresponding state variables are nonempty.

(2) The control set U is convex and closed.

(3) The right-hand side of the state system is bounded by a linear function in the state and control variables.

(4) The integrand of the objective functional is convex on U.

(5) There exist constants $c_1, c_2 > 0$ and $\rho > 1$ such that the integrand $L(T, u_1, u_2)$ of the objective functional satisfies

$$L(T, u_1, u_2) \geq -c_2 + c_1 \left\|(u_1, u_2)\right\|^{\rho}.$$

In order to verify these conditions, we use a result by Lukes [10, Th 9.2.1, p. 182] [40] to give the existence of solutions of the system. Thus property (1) is satisfied. By definition of convex set, the control set U is convex and closed, hence, the second property is also satisfied. Since our state system is bilinear in u_1, u_2, the RHS of eq.17-eq.20 satisfies condition 3, using the boundedness of the solutions.

The integrand of our objective functional is convex. In addition, we can easily see that there exist a constant $\rho > 1$ and positive numbers c_1 and c_2 satisfying $L(T, u_1, u_2) \geq c_1 \left(|u1|^2 + |u2|^2\right)^{\beta/2} - c_2$ because the state variables are bounded. Hence, the existence of optimal control follows from the existence results by Fleming and Rishel [39].

Appendix A.2. Characterisation of an Optimal Control

We characterize the optimal controls u_1^*, u_2^* which give the optimal levels for the various control measures and the corresponding states (p^*, c^*, j^*). The necessary conditions that optimal solutions must satisfy are derived from Pontryagin's Maximum Principle [41]. This principle converts the system of equations 17 - 21 and eq.22 into a problem of minimizing pointwise a Hamiltonian H, with respect to $(u_1,\, u_2)$

$$H = Kc(t) + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 + \lambda_1(\mu - \beta cp - u_1 p - \mu p)$$

$$+\lambda_2(\beta cp + (1-f)\gamma j - u_2 c - \phi c - \mu c) + \lambda_3(\phi c - \gamma j - \mu j)$$

where λ_i, for $i = 1, 2, 3$ are the adjoint functions associated with states p, c, and j respectively. By differentiating the Hamiltonian (H) with respect to each state variable, we find the differential equation for the associated adjoint. Hence, the adjoint system is,

$$\lambda_1^{'} = -\frac{\partial H}{\partial p} = \lambda_1(\beta c + u_1 + \mu) - \lambda_2 \beta c$$

$$\lambda_2^{'} = -\frac{\partial H}{\partial c} = -K + \beta p \lambda_1 + \lambda_2(-\beta p + u_2 + \phi + \mu) - \lambda_3 \phi$$

$$\lambda_3^{'} = -\frac{\partial H}{\partial j} = -\lambda_2(1-f)\gamma + \lambda_3(\gamma + \mu)$$

and the transversality conditions $\lambda_i(T) = 0$ for $i = 1, 2, 3$.

By considering the optimality conditions, $\frac{\partial H}{\partial u_1} = 0$, $\frac{\partial H}{\partial u_2} = 0$ and solving for u_1^*, u_2^*, subject to the constraints, the characterizations can be derived.

$$\frac{\partial H}{\partial u_1} = B_1 u_1 - \lambda_1 p = 0.$$

thus $u_1^* = \frac{\lambda_1 p}{B}$. Also

$$\frac{\partial H}{\partial u_2} = Bu_2 - \lambda_2 c = 0,$$

so that $u_2^* = \frac{\lambda_2 c}{B_2}$. Taking into account the bounds on u_1^*, we obtain the characterization of u_1^*,$u_1^* = min(max(0, \frac{\lambda_1 p}{B_1}), 1)$ and $u_2^* = min(max(0, \frac{\lambda_2 c}{B_2}), 1)$

Appendix A.3. Optimality System

Therefore the system is described as:

State equations:

$$p' = \mu - \beta cp - u_1 p - \mu p \tag{A.1}$$

$$c' = \beta cp + (1-f)\gamma j - u_2 c - \phi c - \mu c \tag{A.2}$$

$$j' = \phi c - \gamma j - \mu j \tag{A.3}$$

with initial conditions, $p(0) = p_0 \geq 0, c(0) = c_0 \geq 0, j(0) = j_0 \geq 0$. Adjoint equations:

$$\lambda_1^{'} = \lambda_1(\beta c + u_1 + \mu) - \lambda_2 \beta c$$

$$\lambda_2^{'} = -K + \beta p \lambda_1 + \lambda_2(-\beta p + u_2 + \phi + \mu) - \lambda_3 \phi$$

$$\lambda_3^{'} = -\lambda_2(1-f)\gamma + \lambda_3(\gamma + \mu)$$

Transversality equations (for $i = 1, 2, 3$):

$$\lambda_i(T) = 0$$

Characterization of the optimal control u_1^*, u_2^*:

$$u_1^* = \begin{cases} 0 & if \quad \frac{\lambda_1 p}{B_1} < 0 \\ \frac{\lambda_1 p}{B_1} & if \quad 0 \leq \frac{\lambda_1 p}{B_1} \leq 1 \\ 1 & if \quad \frac{\lambda_1 p}{B_1} > 1 \end{cases}$$

$$u_2^* = \begin{cases} 0 & if \quad \frac{\lambda_2 c}{B_2} < 0 \\ \frac{\lambda_2 c}{B_2} & if \quad 0 \leq \frac{\lambda_2 c}{B_2} \leq 1 \\ 1 & if \quad \frac{\lambda_2 c}{B_{22}} > 1 \end{cases}$$

In compact notion we can write

$$u_1^* = min(max(0, \frac{\lambda_1 p}{B_1}), 1)$$

$$u_2^* = min(max(0, \frac{\lambda_2 c}{B_2}), 1).$$

Due to the a priori boundedness of the state and adjoint functions and the resulting Lipschitz structure of the ODEs, we obtain the uniqueness of the optimal control for small T. The uniqueness of the optimal control follows from the uniqueness of the optimality system.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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17 Nov 2023 16:57:42 PST
230725-Sooknanan Version 2 - Submitted to Rocky Mountain J. Math.