GRADED BETTI NUMBERS OF SOME SPLIT HYPERGRAPHS

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ABSTRACT. In this paper, we deduce a combinatorial formula for computing the $\mathbb{N}$- graded Betti numbers of edge ideals of a class of hypergraphs, known as d-uniform complete split hypergraphs. As a consequence, we obtain graded Betti numbers of $d-$uniform complete hypergraphs, complete split graphs, its projective dimension and depth. Apart from this, by looking at the edge ideals with linear resolutions as a Stanley-Reisner ideal of the independence complex of associated graph, being called a fat forest, we compute the graded Betti numbers of some families of 2-uniform split hypergraphs by computing it's Hilbert series. Also, we obtain some combinatorial identities by comparing the formulae obtained for graded Betti numbers of 2-uniform split hypergraphs with known formulae.

1. INTRODUCTION

Let $\mathcal{V}$ be a finite set and $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ be a finite collection of non-empty subsets of $\mathcal{V}$. Then we call a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, a *hypergraph*. The elements of $\mathcal{E}$ are called edges and the elements of $\mathcal{V}$ are called vertices. We call a hypergraph $\mathcal{H}$, a *simple hypergraph*, if $|E_i| \geq 2; 1 \leq i \leq m$ and if $E_j \subseteq E_i$ then $i = j$. If the cardinality of vertex set $\mathcal{V}$ is n, we use the set $[n] = \{1, 2, \dots, n\}$ for $\mathcal{V}$. If cardinality of each edge of a hypergraph $\mathcal{H}$ is 2, then the hypergraph $\mathcal{H}$ is a simple graph. Thus, one may say that a *hypergraph* is the generalisation of a simple graph, in which an edge can have any number of vertices.

Let $\mathcal{R} = k[x_1, x_2, \dots, x_n]$ denotes the polynomial ring in n variables over a field k. Consider the monomial $x^E = \prod_{j \in E} x_j \in \mathcal{R}$ for an edge E of a hypergraph $\mathcal{H}$. The ideal $\langle x^E : E \in \mathcal{E}(\mathcal{H}) \rangle \subseteq \mathcal{R}$ generated by monomials x^E, for each edge E of $\mathcal{H}$, is called an edge ideal $\mathcal{I}(\mathcal{H})$ of $\mathcal{H}$ and the quotient ring $\mathcal{R}/I(\mathcal{H})$ is called the edge ring of $\mathcal{H}$.

Edge ideals were first introduced and studied by Villarreal [17] to investigate the relationship between algebraic properties of edge ideals and combinatorial properties of the corresponding graphs. For an edge ideal $\mathcal{I}(\mathcal{H})$ in $\mathcal{R} = k[x_1, x_2, \dots, x_n]$, there exists an $\mathbb{N}$-graded minimal free resolution

$$\mathcal{F} : 0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to \mathcal{R}/\mathcal{I}(\mathcal{H}) \to 0,$$

where $p \leq n$, $F_i = \oplus_j \mathcal{R}(-j)^{\beta_{i,j}}$ and $\mathcal{R}(-j)$ is the graded free $\mathcal{R}$-module obtained by shifting the graded components of $\mathcal{R}$ by j. The numbers $\beta_{i,j}$ are called the i^{th} *graded Betti numbers* of $\mathcal{R}/\mathcal{I}(\mathcal{H})$ in degree j and we write $\beta_{i,j}(\mathcal{H})$ for $\beta_{i,j}(\mathcal{R}/\mathcal{I}(\mathcal{H}))$. The edge ring $\mathcal{R}/\mathcal{I}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ has a t-linear resolution if $\mathcal{I}(\mathcal{H})$ is generated by homogeneous elements of degree t, and all higher

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syzygies are linear. Thus, $\mathcal{R}/\mathcal{I}(\mathcal{H})$ has a t-linear resolution if and only if $Tor_{i,j}(\mathcal{R}/\mathcal{I}(\mathcal{H}), k) = 0$ if $j \neq i + t - 1$ [18, Page 136, 139].

The length p of the resolution is called the *projective dimension* of $\mathcal{R}/\mathcal{I}(\mathcal{H})$ and is denoted by $\text{pd}(\mathcal{R}/\mathcal{I}(\mathcal{H}))$ (we write $\text{pd}(\mathcal{H})$ for $\text{pd}(\mathcal{R}/\mathcal{I}(\mathcal{H}))$), i.e.,

$$\text{pd}(\mathcal{H}) = \max\{i \mid \beta_{i,j}(\mathcal{H}) \neq 0 \text{ for some j}\}.$$

Also, the *Castelnuovo–Mumford regularity* or simply *regularity* of $\mathcal{R}/\mathcal{I}(\mathcal{H})$ denoted by $reg(\mathcal{R}/\mathcal{I}(\mathcal{H}))$ (we write $reg(\mathcal{H})$ for $reg(\mathcal{R}/\mathcal{I}(\mathcal{H}))$ is defined as

$$reg(\mathcal{H}) = \max\{j - i \mid \beta_{i,j}(\mathcal{H}) \neq 0\}.$$

The graded Betti numbers, Castelnuovo-Mumford regularity and projective dimension are among the most important homological invariants of a monomial ideal encoded in its minimal graded free resolution. These homological invariants have been investigated by various authors (see [1, 2, 4, 8–14, 16] and references therein).

During the last few decades, there has been a lot of activity to find the relationship between the homological properties of edge ideals of hypergraphs and combinatorics associated to them by different methods. The notion of Stanley-Reisner ideals where edge ideals of hypergraphs are seen as ideals generated by monomials associated to non-faces of a certain simplicial complex is one and the set of facets of a simplicial complex as a hypergraph is another adopted by Faridi in [6]. For details, we refer to [4, 6, 18] and references therein. In [4], the author deduced nice combinatorial formulae for computing the Betti numbers of many families of $d-$uniform complete hypergraphs. Singh and Verma [16] studied the edge ideals of split graphs and obtained combinatorial formulae for computing the graded Betti numbers of some families of split graphs, namely, complete split graphs and nearly complete split graphs by using the well known Hochster's formula.

In this paper, we introduce the notion of $d-$uniform split hypergraphs, which is a natural generalization of split graphs investigated in [16], and derive a combinatorial formula for graded Betti numbers of edge rings of $d-$uniform complete split hypergraphs. As a consequence, we obtain combinatorial formulae for graded Betti numbers of $d-$uniform complete hypergraphs. Also, we compute the depth and projective dimension of edge rings of these hypergraphs. The edge rings of $d-$uniform complete split hypergraphs has linear resolutions. The problem of finding the classes of square-free monomial ideals that have linear resolution is of great interest. One of the reason is that the Alexander dual of a square-free monomial ideal with linear resolution is Cohen-Macaulay. Our paper is structured as follows:

In Section 2, we recall some definitions and well known results that will be used throughout this article. In Section 3, we deduce a combinatorial formula for computing the graded Betti numbers of d-uniform complete split hypergraphs $(CS_m^n)^d$ by using Hochster's formula. As a consequence, we obtain the formula for the graded Betti numbers of d-uniform complete hypergraph $\mathcal{K}_{n+1}^d$ (already computed in [4]) as follows.

Theorem 1.1. *(Theorem 3.5) The graded Betti numbers $\beta_{i,j}(\mathcal{CS}_m^n)^d$ of the d-complete split hypergraph $(\mathcal{CS}_m^n)^d$ are given by*

$$\beta_{i,j}(\mathcal{CS}_m^n)^d = \begin{cases} \binom{n+m}{j}\binom{j-1}{d-1} - \sum\limits_{j_1=0}^{j} \binom{m}{j_1}\binom{n}{j-j_1}\binom{j_1-1}{d-1} & \text{if } j = i+(d-1) \\ 0 & \text{if } j \neq i+(d-1). \end{cases}$$

Corollary 1.2. *(Corollary 3.6) The graded Betti numbers of edge ideals of d-uniform complete hypergraph $\mathcal{K}_{n+1}^d$ are given by*

$$\beta_{i,j}(\mathcal{K}_{n+1}^d) = \begin{cases} \binom{n+1}{j}\binom{j-1}{d-1} & \text{if } j = i+(d-1) \\ 0 & \text{if } j \neq i+(d-1). \end{cases}$$

Apart from this, Fröberg [15] computed the Betti numbers of edge rings with 2-linear resolutions by looking at the edge ring as the Stanley Reisner ring of the independence complex of associated graph. The associated complexes of the Stanley Reisner rings with 2-linear resolution are called fat forests. In [15], the author gives an alternate technique to determine the Betti numbers of fat forests by computing it's Hilbert series. Singh and Verma [16] use the well-known Hochster's formula to compute the Betti numbers of one extremity of split graphs known as complete split graphs $\mathcal{CS}_m^n$, for the intermediate case, i.e., for nearly complete split graphs $\mathcal{NCS}_m^n$ and the other extremity is the split graph $\mathcal{S}_m^n$, (that is, a split graph with no edge between a vertex of the stable set and a vertex of the clique). However, Hochster's formula is somewhat daunting to use for computing the Betti numbers when the structure of associated independence complex becomes complicated.

In Section 4, we consider the family of 2-uniform hypergraphs. Here, by using the technique of Hilbert series for fat forest given in [15], we compute the Betti numbers of edge ideals of $\mathcal{S}_m^n$. Also, we provide an alternate combinatorial formula for the Betti numbers of edge rings of complete split graphs $\mathcal{CS}_m^n$ and nearly complete split graphs $\mathcal{NCS}_m^n$. Then, by comparing the formulae for Betti numbers so obtained with the formulae obtained in [16], we get certain combinatorial identities also.

For $d = 2$, the main result Theorem 3.5 gives the formula for the graded Betti numbers of edge ring of complete split graph $\mathcal{CS}_m^n$. Using the notion of fat forest given by Fröberg [15], we obtain the following formula for graded Betti numbers of $\mathcal{CS}_m^n$.

Theorem 1.3. *(Theorem 4.4) The edge ring of $\mathcal{CS}_m^n$ has a 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{CS}_m^n) = n\binom{m+n}{i+1} - n\binom{m+n-1}{i+1} - \binom{n}{i+1}.$$

On comparing the formula for graded Betti numbers of complete split graph $\mathcal{CS}_m^n$ obtained in [16, Theorem 3.2] with a special case of Theorem 3.5 for $d = 2$, we obtain the following combinatorial identities.

Identity 1.4. *(see equation 4.1)*

$$i\binom{n}{i+1}+\sum_{\substack{r+s=i+1\\ r,s\geq 1}} r\binom{n}{r}\binom{m}{s} \;=\; n\binom{m+n}{i+1}-n\binom{m+n-1}{i+1}-\binom{n}{i+1}.$$

and

$$i\binom{m+n}{i+1}-\sum_{j=0}^{i+1}\binom{m}{j}\binom{n}{i+1-j}(j-1)= \; n\binom{m+n}{i+1}-n\binom{m+n-1}{i+1}-\binom{n}{i+1}.$$

Using the notion of fat forests, we deduce the following formula for graded Betti numbers of nearly complete split graphs $\mathcal{NCS}_m^n$.

Theorem 1.5. *(Theorem 4.8) The edge ring of $\mathcal{NCS}_m^n$ has a 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{NCS}_m^n)=\begin{cases} n\binom{m+n-1}{i+1}-n\binom{m+n-2}{i+1}-\binom{n}{i+1} & \text{if } m\geq n\\ (n-m)\binom{m+n}{i+1}-(n-2m)\binom{m+n-1}{i+1}-m\binom{m+n-2}{i+1}-\binom{n}{i+1} & \text{if } m<n. \end{cases}$$

On comparing the above formula for graded Betti numbers of nearly complete split graphs $\mathcal{NCS}_m^n$ with the formula obtained in [16, Theorem 3.7], we obtain the following combinatorial identities.

Identity 1.6. *(see equations 4.3 and 4.4)*

$$i\binom{n}{i+1}+\sum_{p=1}^{i}p\binom{n}{p}\binom{m-p}{i+1-p}+\sum_{t=1}^{n}\binom{n}{t}\sum_{p=1}^{i+1-2t}p\binom{n-t}{p}\binom{m-t-p}{i+1-2t-p}=n\binom{m+n-1}{i+1}-n\binom{m+n-2}{i+1}-\binom{n}{i+1}.$$

and

$$i\binom{n}{i+1}+\sum_{p=1}^{i}p\binom{n}{p}\binom{m-p}{i+1-p}+\sum_{t=1}^{m}\binom{m}{t}\sum_{p=1}^{i+1-2t}p\binom{n-t}{p}\binom{m-t-p}{i+1-2t-p}=(n-m)\binom{m+n}{i+1}-(n-2m)\binom{m+n-1}{i+1}-m\binom{m+n-2}{i+1}-\binom{n}{i+1}.$$

Also, we obtain a formula for graded Betti numbers of split graph $\mathcal{S}_m^n$ using the notion of fat forests as follows.

Theorem 1.7. *(Theorem 4.10) The edge ring of $\mathcal{S}_m^n$ has 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{S}_m^n)=(n-1)\binom{n}{i+1}-n\binom{n-1}{i+1}.$$

2. Preliminaries

In this section, we recall some preliminaries and a well-known result that will be used in this paper.

2.1. **Hypergraphs and Independence Complexes.** In this article, we consider only simple hypergraphs.

- Let $\mathcal{H}$ be a hypergraph, we call a hypergraph $\mathcal{K}$ to be a *subhypergraph* of $\mathcal{H}$ if $\mathcal{V}(\mathcal{K}) \subseteq \mathcal{V}(\mathcal{H})$ and $\mathcal{E}(\mathcal{K}) \subseteq \mathcal{E}(\mathcal{H})$.
- If $\mathcal{W} \subseteq \mathcal{V}$, then the *induced hypergraph* on $\mathcal{W}, \mathcal{H}_{\mathcal{W}}$ is the subgraph with $\mathcal{V}(\mathcal{H}_{\mathcal{W}}) = \mathcal{W}$ and edge set of $\mathcal{H}_{\mathcal{W}}$ is the set of edges that lies entirely in $\mathcal{W}$.
- If $|E_i| = d$ for every edge $E_i \in \mathcal{E}(\mathcal{H})$, then the hypergraph $\mathcal{H}$ is called *d-uniform hypergraph.* One can note that a 2-*uniform hypergraph* is a simple graph.
- A d-uniform hypergraph on n vertices with edge set $\mathcal{E}$ as the set of all subsets of $[n]$ having d elements is called *d-complete hypergraph* and denoted as $\mathcal{K}_n^d$.
- A hypergraph $\mathcal{K}^d_{(n_1,\ldots,n_t)}$ is called *d-uniform multipartite hypergraph* on vertex set $[n_1] \sqcup [n_2] \cdots \sqcup [n_t]$, and its edge set consists of all d-edges except those of the form $\{l_{i_1}, \ldots, l_{i_d}\}$, where $l_{i_j} \in [n_i]$; for some i and each $j = 1, 2, \ldots, d$.

A *simplicial complex* on a vertex set $V = \{x_1, x_2, \ldots, x_n\}$ is a set Δ whose elements are subset of V such that (a) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$, and (b) for each $i = 1, 2, \ldots, n$, $\{x_i\} \in \Delta$. Note that the set $\emptyset \in \Delta$. An element $F \in \Delta$ is called a *face*. The maximal elements of Δ, with respect to inclusion, are called the *facets* of Δ. If $\{F_1, F_2, \ldots, F_t\}$ is a complete list of the facets of Δ, we will sometimes write $\Delta = \langle F_1, \ldots, F_t\rangle$. The *dimension* of a face $F \in \Delta$, denoted by $\dim F$, is given by $\dim F = |F| - 1$, where we make the convention that $\dim \emptyset = -1$. The *dimension* of Δ, denoted by $\dim \Delta$, is defined to be $\dim \Delta = max_{F\in\Delta}\{\dim F\}$. A simplicial complex is called *pure* if all its facets have the same dimension. A simplicial complex having exactly one facet is called a *simplex*. For each subset $W \subset V$, the collection

$$\Delta_W = \{F \in \Delta \mid F \subset W\}$$

is a simplicial complex on the vertex set W, known as *induced subcomplex* of Δ on W.

Given a simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$, we can associate it with a monomial ideal I_Δ in the polynomial ring $\mathcal{R} = k[x_1, x_2, \ldots, x_n]$ (for a field k) in the following way: For every subset F of V, we define a monomial $x_F := \prod_{x_i\in F} x_i$ in $\mathcal{R}$. Then the ideal $I_\Delta := \langle x_F | \, F \notin \Delta\rangle$ is called the *Stanley-Reisner* ideal of Δ and the quotient ring $k[\Delta] = \mathcal{R}/I_\Delta$ is called the *Stanley-Reisner* ring.

For a hypergraph $\mathcal{H} = ([n], \mathcal{E})$, consider an edge ideal $\mathcal{I}(\mathcal{H}) \subseteq \mathcal{R}$. One can note that the *Stanley-Reisner ring* of the simplicial complex

$$\Delta(\mathcal{H}) = \{F \in \ [n] \mid E \nsubseteq F, \forall E \in \mathcal{E}(\mathcal{H})\}$$

is $\mathcal{R}/\mathcal{I}(\mathcal{H})$. This is the independence complex of $\mathcal{H}$. One can easily note that the edges of $\mathcal{H}$ are precisely the minimal non-faces of $\Delta(\mathcal{H})$.

In this article, for basics of simplicial homology, we refer to *Algebraic Topology* by E. H. Spanier [5].

15 Aug 2023 03:31:22 PDT
230719-Singh Version 2 - Submitted to Rocky Mountain J. Math.

2.2. **Hochster's formula.** Now, we recall the Hochster's formula which we shall use for computing the Betti numbers of edge ideals of complete split hypergraphs.

Proposition 2.1. [7, Hochster's formula] *For a simplicial complex Δ on vertex set $[n]$, let $k[\Delta] = \mathcal{R}/I_\Delta$ denote its Stanley-Reisner ring. Then for $i \geq 0$, the Betti numbers $\beta_{i,j}$ of $\mathcal{R}/I_\Delta$ are given by*

$$\beta_{i,j}(k[\Delta]) = \sum_{\substack{W \subset [n] \\ |W|=j}} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k).$$

Here Δ_W denotes the simplicial subcomplex of Δ induced on the vertex set W.

3. Split Hypergraph and the Betti numbers

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a *d-uniform hypergraph.* A subset $\mathcal{C} \subseteq \mathcal{V}$ is called a *clique* if every d-subset of $\mathcal{C}$ is an edge of $\mathcal{H}$. A subset $\mathcal{S} \subseteq \mathcal{V}$ is called a *stable set* if no subset of $\mathcal{S}$ is an edge of $\mathcal{H}$. A d-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called a *d-split hypergraph* if vertex set $\mathcal{V}$ of $\mathcal{H}$ can be written as disjoint union $\mathcal{V} = \mathcal{C} \sqcup \mathcal{S}$, where $\mathcal{C}$ is the clique and $\mathcal{S}$ is the stable set.

Definition 3.1. A d-split hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called a *d-complete split hypergraph* if each subset of the form $\{x_{i_1}, x_{i_2}, \dots, x_{i_t}, y_{j_1}, y_{j_2}, \dots, y_{j_{d-t}}\} \subseteq \mathcal{V}$, $(1 \leq t \leq d-1)$ is also an edge of $\mathcal{H}$, where $\{x_{i_1}, x_{i_2}, \dots, x_{i_t}\}$ is a subset of the clique $\mathcal{C}$ and $\{y_{j_1}, y_{j_2}, \dots, y_{j_{d-t}}\}$ is a subset of the stable set $\mathcal{S}$. If $\mathcal{H}$ is a complete split hypergraph with $|\mathcal{C}| = n$ and $|\mathcal{S}| = m$, where $\mathcal{C}$ is the clique and $\mathcal{S}$ is the stable set, then we shall denote the d-complete split hypergraph $\mathcal{H}$ by $(CS^n_m)^d$. In such a case, we shall write $\mathcal{V}(\mathcal{CS}^n_m)^d = C \sqcup S$, where $\mathcal{C} = \{x_1, x_2, \dots, x_n\}$ is the clique and $\mathcal{S} = \{y_1, y_2, \dots, y_m\}$ is the stable set, and edge set $\mathcal{E}(\mathcal{CS}^n_m)^d = \{\{x_{i_1}, x_{i_2}, \dots, x_{i_d}\} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\} \cup \{\{x_{i_1}, x_{i_2}, \dots, x_{i_r}, y_{j_1}, y_{j_2}, \dots, y_{j_s}\} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n, 1 \leq j_1 < j_2 < \cdots < j_s \leq m;\ r+s = d;\ r, s \geq 1\}$.

Remark 3.2. (1) The d-complete split hypergraph $(CS^n_m)^d$ has $m+n$ vertices and $\binom{n}{d} + \sum\limits_{\substack{r+s=d \\ r,s \geq 1}} \binom{n}{r}\binom{m}{s}$ edges. Observe that the d-uniform complete split hypergraph $(CS^n_1)^d$ is the d-uniform hypergraph $\mathcal{K}^d_{n+1}$ on $(n+1)$-vertices which have been studied in [4].

(2) The independence complex $\Delta\mathcal{K}^d_n$ is the $(d-2)$ skeleton of Δ_n, while observe that $\Delta(\mathcal{CS}^n_m)^d$ is the union of $(d-2)$-skeleton of simplices $\{\langle x_1, x_2, \dots, x_n, y_{k_1}, \dots, y_{k_{d-1}}\rangle; 1 \leq k_1 < k_2 < \dots k_{d-1} \leq m\}$ with the simplexes $\{\langle y_{j_1}, y_{j_2}, \dots, y_{j_d}\rangle; 1 \leq j_1 < j_2 < \dots j_d \leq m\}$.

Example 3.3. Let $\mathcal{H}$ be 3-complete split hypergraph $(\mathcal{CS}^4_3)^3$. Then $\mathcal{V}(\mathcal{CS}^4_3)^3 = C \sqcup S$, where $\mathcal{C} = \{x_1, x_2, x_3, x_4\}$ is the clique and $\mathcal{S} = \{y_1, y_2, y_3\}$ is the stable set, and the edge set is $\mathcal{E}(\mathcal{CS}^4_3)^3 =$ $\{(1,2,3), (1,3,4), (2,3,4), (1,2,4), (1,2,1'), (1,3,1'), (1,4,1'), (2,3,1'), (2,4,1'), (3,4,1'), (1,2,2'),$ $(1,3,2'), (1,4,2'), (2,3,2'), (2,4,2'), (3,4,2'), (1,2,3'), (1,3,3'), (1,4,3'), (2,3,3'), (2,4,3'), (3,4,3'),$ $(1,1',2'), (2,1',2'), (3,1',2'), (4,1',2'), (1,1',3'), (2,1',3'), (3,1',3'), (4,1',3'), (1,2',3'), (2,2',3'),$

$(3, 2', 3'), (4, 2', 3')\}$.
where by the edge (i, j, k) we mean $\{x_i, x_j, x_k\}$, by (i, j, k') we mean $\{x_i, x_j, y_k\}$ and by (i, j', k') we mean $\{x_i, y_j, y_k\}$.

Theorem 3.4. *The Stanley-Reisner ring $\mathcal{R}/\mathcal{I}(CS_m^n)^d$ of d-uniform complete split hypergraph has d-linear resolution, and $\beta_{i,j}(CS_m^n)^d \neq 0$ only if $j = i + (d-1)$.*

Proof. Consider the edge ideal $\mathcal{I}(CS_m^n)^d$ of d-complete split hypergraph. It is generated by all possible d-edges except those which lies entirely in the stable set, i.e., edges of the form $\{\{y_{j_1}, y_{j_2}, \ldots, y_{j_d}\} \mid 1 \leq j_1 < j_2 < \cdots < j_d \leq m\}$. Thus, the independence complex $\Delta(CS_m^n)^d$ of d-uniform complete split hypergraph must contains the $(d-1)-$faces $\{y_{j_1}, y_{j_2}, \ldots, y_{j_d}\}$.

Since the generators of $\mathcal{I}(CS_m^n)^d$ have degree d, $\beta_{i,j}(CS_m^n)^d = 0$ if $j < i + (d-1)$. Suppose $\beta_{i,j}(CS_m^n)^d \neq 0$ for some $j > i + (d-1)$. Then by *Hochster's formula* there must exist non-zero homology group $\widetilde{H}_t(\Delta(CS_m^n)_W^d; k)$, for some $W \subseteq [n] \sqcup [m]$ and $t \geq d-1$.

But cycles in such a degree t lies entirely in the simplex Δ_m on the vertices of the stable set $\{y_1, y_2, \ldots, y_m\}$ (which follows from the structure of $\Delta(CS_m^n)^d$). Also, the homology of a simplex is zero. Thus, every cycle in such a degree t is a boundary and thus $\beta_{i,j}(CS_m^n)^d = 0$ for all $j > i+(d-1)$ also. Hence, the conclusion holds.

□

Theorem 3.5. *With the same notations as in Definition 3.1, the graded Betti numbers of $\mathcal{R}/\mathcal{I}(CS_m^n)^d$ are given by*

$$\beta_{i,j}(\mathcal{CS}_m^n)^d = \begin{cases} \binom{n+m}{j}\binom{j-1}{d-1} - \sum\limits_{j_1=0}^{j} \binom{m}{j_1}\binom{n}{j-j_1}\binom{j_1-1}{d-1} & \text{if } j = i + (d-1) \\ 0 & \text{if } j \neq i + (d-1). \end{cases} \tag{3.1}$$

Proof. By *Hochster's formula*, we have

$$\beta_{i,j}(\mathcal{CS}_m^n)^d = \sum_{\substack{W \subset [n] \sqcup [m] \\ |W| = j}} \dim_k \widetilde{H}_{|W|-i-1}(\Delta(\mathcal{CS}_m^n)_W^d; k).$$

As $\Delta\mathcal{K}_n^d$ is the $(d-2)$ skeleton of Δ_n while $\Delta(\mathcal{CS}_m^n)^d$ is the union of $(d-2)$-skeleton of simplices $\{\langle x_1, x_2, \ldots, x_n, y_{k_1}, \ldots, y_{k_{d-1}}\rangle; 1 \leq k_1 < k_2 < \ldots k_{d-1} \leq m\}$ and the simplexes $\{\langle y_{j_1}, y_{j_2}, \ldots, y_{j_d}\rangle; 1 \leq j_1 < j_2 < \ldots j_d \leq m\}$. Thus, their might be the faces $F \in \Delta(\mathcal{CS}_m^n)^d$ such that $|F| \geq d$, which would result in non-zero boundary group $B_{d-2}(\Delta(\mathcal{CS}_m^n)^d)$ in the chain complex of $\Delta(\mathcal{CS}_m^n)^d$.

Therefore, while comparing the terms $\dim_k \widetilde{H}_{|W|-i-1}(\Delta(\mathcal{CS}_m^n)_W^d; k)$ occurring in Hochster's formula for d- complete split hypergraphs with the corresponding terms $\dim_k \widetilde{H}_{|W|-i-1}(\Delta(\mathcal{K}_{n+m}^d)_W; k)$ of complete hypergraph on $n+m$ vertices, which we have encountered in the [4, Theorem 3.1] we have,

$$\dim_k \widetilde{H}_{|W|-i-1}(\Delta(\mathcal{CS}_m^n)_W^d; k) \leq \dim_k \widetilde{H}_{|W|-i-1}(\Delta(\mathcal{K}_{n+m}^d)_W; k)$$

for every set $W \subset [n] \sqcup [m]$.

Also we have,

$$\dim_k \widetilde{H}_{d-2}(\Delta(\mathcal{CS}^n_m)^d_W; k) = \dim_k Z_{d-2}(\Delta(\mathcal{CS}^n_m)^d_W; k) - \dim_k B_{d-2}(\Delta(\mathcal{CS}^n_m)^d_W; k)$$

Because of the $(d-2)$-skeleton structure of the independence complexes, the cycle group $Z_{d-2}(\Delta(\mathcal{CS}^n_m)^d_W)$ and $Z_{d-2}(\Delta(\mathcal{K}^d_{n+m})_W)$ clearly coincides and $B_{d-2}(\Delta(\mathcal{K}^d_{n+m})_W) = 0$, the only thing we have to compute is the dimension of $B_{d-2}(\Delta(\mathcal{CS}^n_m)^d_W)$. Let $W = W_1 \sqcup W_2$, where $W_1 \subset [n]$ and $W_2 \subset [m]$, Now we have,

$$B_{d-2}(\Delta(\mathcal{CS}^n_m)^d_W) = B_{d-2}(\Delta(\mathcal{CS}^n_m)^d_{W_2}).$$

This is because the potential $(d-1)$-faces of $\Delta(\mathcal{CS}^n_m)^d$ lies in $\Delta(\mathcal{CS}^n_m)^d_{[m]}$. Now, we only have to compute the dimension of $\Delta(\mathcal{CS}^n_m)^d_{[m]}$. From [4, Theorem 3.1] we have

$$dim_k B_{d-2}(\Delta(\mathcal{CS}^n_m)^d_{W_2}) = \binom{|W_2|-1}{d-1} \tag{3.2}$$

Take $|W_2| = j_1$ and taking sum over all possible $V \subseteq [n] \sqcup [m]$, we have,

$$\beta_{i,j}(\mathcal{CS}^n_m)^d = \begin{cases} \binom{n+m}{j}\binom{j-1}{d-1} - \sum_{j_1=0}^{j} \binom{m}{j_1}\binom{n}{j-j_1}\binom{j_1-1}{d-1} & \text{if } j = i + (d-1) \\ 0 & \text{if } j \neq i + (d-1). \end{cases} \tag{3.3}$$

□

Now we shall obtain a formula for computing the graded Betti numbers of edge ideals of d-complete hypergraph $\mathcal{K}^d_{n+1}$ on $(n+1)$ vertices, computed in [4], as a special case of Theorem 3.5.

Corollary 3.6. *The graded Betti numbers of edge ideals of d-complete hypergraph $\mathcal{K}^d_{n+1}$ are given by*

$$\beta_{i,j}(\mathcal{K}^d_{n+1}) = \begin{cases} \binom{n+1}{j}\binom{j-1}{d-1} & \textit{if } j = i + (d-1) \\ 0 & \textit{if } j \neq i + (d-1). \end{cases}$$

Proof. We have $(\mathcal{CS}^n_1)^d = \mathcal{K}^d_{n+1}$. Thus, in view of equation 3.3, we get as required.

□

Remark 3.7. *For $d = 2$, Corollary 3.6 provides the formula for the graded Betti numbers of edge ring of complete graph $\mathcal{K}_{n+1}$ on $n+1$ vertices as*

$$\beta_{i,j}(\mathcal{K}_{n+1}) = \begin{cases} i\binom{n+1}{i+1} & \textit{if } j = i+1 \\ 0 & \textit{if } j \neq i+1. \end{cases}$$

Now, we derive a formula for graded Betti numbers of complete split graph $\mathcal{CS}^n_m$(see [16]) as a special case of Theorem 3.5 as follows.

Corollary 3.8. *The Betti numbers of edge ring of a complete split graph $\mathcal{CS}^n_m$ are given by*

$$\beta_{i,j}(\mathcal{CS}^n_m) = \begin{cases} \binom{n+m}{j}(j-1) - \sum_{j_1=0}^{j} \binom{m}{j_1}\binom{n}{j-j_1}(j_1-1) & \textit{if } j = i+1 \\ 0 & \textit{if } j \neq i+1. \end{cases}$$

15 Aug 2023 03:31:22 PDT
230719-Singh Version 2 - Submitted to Rocky Mountain J. Math.

Proof. We see that $(\mathcal{CS}^n_m)^2 = \mathcal{CS}^n_m$. Thus, in view of equation 3.3, we get as required.

$\square$

Example 3.9. The non-zero graded Betti numbers of $\mathcal{R}/\mathcal{I}(\mathcal{CS}^4_3)^3$, in view of Theorem 3.5, are given as follows:

$$\begin{aligned}
\beta_{1,3}(\mathcal{CS}^4_3)^3 &= \binom{4+3}{3}\binom{3-1}{2} - \left[\binom{3}{0}\binom{4}{3}\binom{0-1}{3-1} + \binom{3}{1}\binom{4}{2}\binom{1-1}{3-1} + \binom{3}{2}\binom{4}{1}\binom{2-1}{3-1}\right. \\
&\quad \left. + \binom{3}{3}\binom{4}{0}\binom{3-1}{3-1}\right] = 34, \\
\beta_{2,4}(\mathcal{CS}^4_3)^3 &= \binom{4+3}{4}\binom{4-1}{2} - \left[\binom{3}{0}\binom{4}{4}\binom{0-1}{3-1} + \binom{3}{1}\binom{4}{3}\binom{1-1}{3-1} + \binom{3}{2}\binom{4}{2}\binom{2-1}{3-1}\right. \\
&\quad \left. + \binom{3}{3}\binom{4}{1}\binom{3-1}{3-1} + \binom{3}{4}\binom{4}{0}\binom{4-1}{3-1}\right] = 101, \\
\beta_{3,5}(\mathcal{CS}^4_3)^3 &= \binom{4+3}{5}\binom{5-1}{2} - \left[\binom{3}{0}\binom{4}{5}\binom{0-1}{3-1} + \binom{3}{1}\binom{4}{4}\binom{1-1}{3-1} + \binom{3}{2}\binom{4}{3}\binom{2-1}{3-1}\right. \\
&\quad \left. + \binom{3}{3}\binom{4}{2}\binom{3-1}{3-1} + \binom{3}{4}\binom{4}{1}\binom{4-1}{3-1} + \binom{3}{5}\binom{4}{0}\binom{5-1}{3-1}\right] = 120, \\
\beta_{4,6}(\mathcal{CS}^4_3)^3 &= \binom{4+3}{6}\binom{6-1}{2} - \left[\binom{3}{0}\binom{4}{6}\binom{0-1}{3-1} + \binom{3}{1}\binom{4}{5}\binom{1-1}{3-1} + \binom{3}{2}\binom{4}{4}\binom{2-1}{3-1}\right. \\
&\quad \left. + \binom{3}{3}\binom{4}{3}\binom{3-1}{3-1} + \binom{3}{4}\binom{4}{2}\binom{4-1}{3-1} + \binom{3}{5}\binom{4}{1}\binom{5-1}{3-1} + \binom{3}{6}\binom{4}{0}\binom{6-1}{3-1}\right] = 66, \\
\beta_{5,7}(\mathcal{CS}^4_3)^3 &= \binom{4+3}{7}\binom{7-1}{2} - \left[\binom{3}{0}\binom{4}{7}\binom{0-1}{3-1} + \binom{3}{1}\binom{4}{6}\binom{1-1}{3-1} + \binom{3}{2}\binom{4}{5}\binom{2-1}{3-1}\right. \\
&\quad + \binom{3}{3}\binom{4}{4}\binom{3-1}{3-1} + \binom{3}{4}\binom{4}{3}\binom{4-1}{3-1} + \binom{3}{5}\binom{4}{2}\binom{5-1}{3-1} + \binom{3}{6}\binom{4}{1}\binom{6-1}{3-1} \\
&\quad \left. + \binom{3}{7}\binom{4}{0}\binom{7-1}{3-1}\right] = 14.
\end{aligned}$$

So the minimal graded free resolution of $\mathcal{R}/\mathcal{I}(\mathcal{CS}^4_3)^3$ is of the form

$$0 \to \mathcal{R}[-5]^{14} \to \mathcal{R}[-4]^{66} \to \mathcal{R}[-3]^{120} \to \mathcal{R}[-2]^{101} \to \mathcal{R}[-1]^{34} \to \mathcal{R} \to \mathcal{R}/\mathcal{I}(\mathcal{CS}^4_3)^3 \to 0.$$

Observe that the graded Betti numbers $\beta_{i,i+2}(\mathcal{CS}^4_3)^3$ computed using Theorem 3.5 are same as given in the Betti table of the minimal graded free resolution of $\mathcal{R}/\mathcal{I}(\mathcal{CS}^4_3)^3$ computed using Singular 2.0 [3] as follows:

	0	1	2	3	4	5
0:	1	-	-	-	-	-
1:	-	-	-	-	-	-
2:	-	34	101	120	66	14
total:	1	34	101	120	66	14

where the number at jth row and ith column is the graded Betti number $\beta_{i,i+j}(\mathcal{CS}^4_3)^3$.

15 Aug 2023 03:31:22 PDT
230719-Singh Version 2 - Submitted to Rocky Mountain J. Math.

Theorem 3.10. *The projective dimension of edge ring of d-uniform complete split hypergraph $(\mathcal{CS}_m^n)^d$ with $m+n \geq d$ is given by*

$$pd(\mathcal{CS}_m^n)^d = m + n - (d-1).$$

Proof. From Theorem 3.5 we have,

$$\beta_{i,j}(\mathcal{CS}_m^n)^d = \begin{cases} \binom{n+m}{j}\binom{j-1}{d-1} - \sum\limits_{j_1=0}^{j} \binom{m}{j_1}\binom{n}{j-j_1}\binom{j_1-1}{d-1} & \text{if } j = i + (d-1) \\ 0 & \text{if } j \neq i + (d-1). \end{cases}$$

Put $j = n + m$ in above expression we get,

$$\binom{n+m-1}{d-1} - \binom{m-1}{d-1}$$

Now, one can easily see that $\binom{a}{c} > \binom{b}{c}$, when $a > b \geq c$. Thus, $\binom{n+m-1}{d-1} - \binom{m-1}{d-1} > 0$ and the result follows.

□

Corollary 3.11. *The depth of edge ring of d-uniform complete split hypergraph $(\mathcal{CS}_m^n)^d$ is given by*

$$depth_{\mathcal{R}}(\mathcal{CS}_m^n)^d = d - 1.$$

Proof. The Auslander-Buchsbaum formula [18, Theorem 3.5.13] tells us that

$$pd(\mathcal{CS}_m^n)^d + depth_{\mathcal{R}}(\mathcal{CS}_m^n)^d = m + n$$

and the result follows in view of Theorem 3.10.

□

4. 2-uniform split hypergraphs and Betti numbers

In this section, we consider the family of 2-uniform hypergraphs. R. Fröberg [15] determined the Betti numbers of certain edge rings with 2- linear resolution by looking at the edge ring as a Stanley-Reisner ring and the associated complex being called a fat forest.

Fat forests are recursively defined as follows. A ℓ-simplex F_1 of dimension ≥ 0 (i.e. with $\ell+1$ vertices) is a fat forest. If $F_i, i = 1, \ldots k$, are simplices and $G_{k-1} = F_1 \cup \ldots \cup F_{k-1}$ is a fat forest, then $G_{k-1} \cup F_k$ is a fat forest if $H = G_{k-1} \cap F_k$ is a simplex, $\dim H \geq -1$. (If $\dim H = -1$, then G_{k-1} and F_k are disjoint.)

If $\mathcal{R}/\mathcal{I}$ has a 2-linear resolution it looks like this:

$$0 \longrightarrow \mathcal{R}[-p-1]^{b_p} \longrightarrow \ldots \longrightarrow \mathcal{R}[-3]^{b_2} \longrightarrow \mathcal{R}[-2]^{b_1} \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{I} \longrightarrow 0$$

where $\mathcal{R}[-i]$ means that we have shifted degrees of $\mathcal{R}$ i steps. Using that the alternating sum of the $k-$dimensions in each degree is 0, we get that the Hilbert series of $k[\Delta]$ with 2-linear resolution equals $\frac{1-\beta_{1,2}t^2+\beta_{2,3}t^3-\ldots(-1)^p\beta_{p,p+1}t^{p+1}}{(1-t)^n}$ where $\beta_{i,j}$ are the graded Betti numbers $dim_k\ Tor_{i,j}^{\mathcal{R}}(k[\Delta], k)$, and n is the number of vertices in Δ. The Betti numbers $\beta_{i,j} = dim_k\ Tor^{\mathcal{R}}_{i,j}(\mathcal{R}/\mathcal{I}, k)$ of Stanley-Reisner rings of fat forests and the Hilbert series contains the same information as the set of Betti numbers.

The following theorem computes the Hilbert series of the Stanley-Reisner ring $k[\Delta]$ with 2-linear resolution.

Theorem 4.1. [15, Theorem 1] *Let $F = F_1 \cup \ldots \cup F_k$ be a fat forest with F_i a simplex of dimension d_i and $(F_1 \cup \ldots \cup F_{j-1}) \cap F_j$ a simplex of dimension r_j. Then the Hilbert series of $k[F]$ is $\sum_{i=1}^{k} \frac{1}{(1-t)^{d_i+1}} - \sum_{i=2}^{k} \frac{1}{(1-t)^{r_i+1}}$. The projective dimension is $\sum_{i=1}^{k} d_i - \sum_{i=2}^{k} r_i + 1 - min\{r_i\} - 2$. The depth of $k[F]$ is $min\{r_i\} + 2$, and F is CM(Cohen- Macaulay) if and only if there is a d such that $d_i = d$ for all i and $r_i = d - 1$ for all i.*

Using this technique of Hilbert series, we compute the graded Betti numbers of some families of 2-uniform split hypergraphs. Throughout this section, we are dealing with 2-uniform split hypergraphs, so for the sake of brevity, we drop the exponent $d = 2$ in notations.

Let G be a 2-uniform hypergraph (i.e. an ordinary simple graph) with vertex set V and edge set E then we call G a *split graph* if the vertex set V can be written as disjoint union $V = \mathcal{C} \sqcup \mathcal{S}$, where $\mathcal{C}$ is a clique and $\mathcal{S}$ is stable set. If there is no edge between $\mathcal{C}$ and $\mathcal{S}$ and $|\mathcal{C}| = n$ and $|\mathcal{S}| = m$, then we shall denote such a split graph by $\mathcal{S}^n_m$.

Definition 4.2. *A complete split graph is a graph in which each vertex of the stable set of G is adjacent to every vertex of the clique of G. Let G be a complete split graph with $|\mathcal{C}| = n$ and $|\mathcal{S}| = m$, where $\mathcal{C}$ is the clique and $\mathcal{S}$ is the stable set, then we shall denote the complete split graph G by $\mathcal{CS}^n_m$ (for sake of brevity, we use $\mathcal{CS}^n_m$ for $(\mathcal{CS}^n_m)^2$). Let $\mathcal{C} = \{x_1, x_2, \ldots, x_n\}$ be the clique and $\mathcal{S} = \{y_1, y_2, \ldots, y_m\}$ be the stable set, then we shall write $V(\mathcal{CS}^n_m) = \mathcal{C} \sqcup \mathcal{S}$ and edge set $E(\mathcal{CS}^n_m) = \{\{x_i, x_j\} \mid 1 \leq i < j \leq n\} \cup \{\{x_i, y_j\} \mid 1 \leq i \leq n$ and $1 \leq j \leq m\}$.*

The graded Betti numbers of this family of complete split graphs $\mathcal{CS}^n_m$ were computed by Singh and Verma [16] and is given as

Theorem 4.3. [16, Theorem 3.2] *The edge ring of $\mathcal{CS}^n_m$ has a 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{CS}^n_m) = i\binom{n}{i+1} + \sum_{\substack{r+s=i+1\\ r,s\geq 1}} r\binom{n}{r}\binom{m}{s}.$$

Here, we obtain an alternate formula for computing the Betti numbers for the complete split graph $\mathcal{CS}^n_m$ by looking the simplicial complex $\Delta(\mathcal{CS}^n_m)$ as a fat forest and as a consequence, we obtain the combinatorial identities (see equation 4.1 and 4.2).

Theorem 4.4. *The edge ring of $\mathcal{CS}^n_m$ has a 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{CS}^n_m) = n\binom{m+n}{i+1} - n\binom{m+n-1}{i+1} - \binom{n}{i+1}.$$

Proof. Here $V(\mathcal{CS}_m^n) = \mathcal{C} \sqcup \mathcal{S}$, where $\mathcal{C} = \{x_1, x_2, \dots, x_n\}$ is the clique and $\mathcal{S} = \{y_1, y_2, \dots, y_m\}$ is the stable set and the edge set is $E(\mathcal{CS}_m^n) = \{\{x_i, x_j\} | 1 \leq i < j \leq n\} \cup \{\{x_i, y_j\} | 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$.

The simplicial complex $\Delta(\mathcal{CS}_m^n)$ associated to complete split graph $\mathcal{CS}_m^n$ is the disjoint union of $(m-1)$-simplex $< y_1, y_2, \dots, y_m >$ and the 0-simplices $< x_i > \forall 1 \leq i \leq n$. The fat tree with this Stanley-Reisner ideal $I(\mathcal{CS}_m^n)$ has a maximal face $< y_1, y_2, \dots, y_m >$. Then the other faces $< x_i >; 1 \leq i \leq n$ are attached to it with empty intersections.

Therefore, the Hilbert series of $k[x_1, \dots, x_n, y_1, \dots, y_m]/I$ is

$$\begin{aligned}
&\frac{1}{(1-t)^m} + \underbrace{\frac{1}{(1-t)} + \dots + \frac{1}{(1-t)}}_{n\text{-times}} - n\frac{1}{(1-t)^0} \\
&= \tfrac{1}{(1-t)^m} + \tfrac{n}{(1-t)} - \tfrac{n}{(1-t)^0} \\
&= \tfrac{(1-t)^n + n(1-t)^{m+n-1} - n(1-t)^{m+n}}{(1-t)^{m+n}}
\end{aligned}$$

So, the Betti numbers are

$$\beta_{i,i+1}(\mathcal{CS}_m^n) = n\binom{m+n}{i+1} - n\binom{m+n-1}{i+1} - \binom{n}{i+1}$$

$\square$

Corollary 4.5. *For positive integers n, m and i, we have the following identities.*

$$i\binom{n}{i+1} + \sum_{\substack{r+s=i+1 \\ r,s\geq 1}} r\binom{n}{r}\binom{m}{s} = n\binom{m+n}{i+1} - n\binom{m+n-1}{i+1} - \binom{n}{i+1} \tag{4.1}$$

and

$$i\binom{m+n}{i+1} - \sum_{j_1=0}^{i+1} \binom{m}{j_1}\binom{n}{i+1-j_1}(j_1-1) = n\binom{m+n}{i+1} - n\binom{m+n-1}{i+1} - \binom{n}{i+1} \tag{4.2}$$

Proof. From [16, Theorem 3.2] we have

$$\beta_{i,i+1}(\mathcal{CS}_m^n) = i\binom{n}{i+1} + \sum_{\substack{r+s=i+1 \\ r,s\geq 1}} r\binom{n}{r}\binom{m}{s}$$

By Theorem 4.4, we get

$$\beta_{i,i+1}(\mathcal{CS}_m^n) = n\binom{m+n}{i+1} - n\binom{m+n-1}{i+1} - \binom{n}{i+1}$$

Also from Theorem 3.5, for $d = 2$ we have,

$$\beta_{i,i+1}(\mathcal{CS}_m^n) = i\binom{m+n}{i+1} - \sum_{j_1=0}^{i+1} \binom{m}{j_1}\binom{n}{i+1-j_1}(j_1-1)$$

Thus, the result follows.

□

Next, we define another class of 2-uniform split hypergraphs namely nearly complete split graphs.

Definition 4.6. *We call the graph obtained from $\mathcal{CS}_m^n$ by removing the edges of matching : $\{x_i, y_i\}; 1 \leq i \leq \ell = min\{m, n\}$; the nearly complete split graph and denote it by $\mathcal{NCS}_m^n$. $\mathcal{NCS}_m^n$ is a split graph with the n-clique $\mathcal{C} = \{x_1, x_2, \ldots, x_n\}$ and the stable set $\mathcal{S} = \{y_1, y_2, \ldots, y_m\}$.*

Singh and Verma [16] computed the graded Betti numbers of this family of graphs and provide the following formula

Theorem 4.7. *The graded Betti numbers of $\mathcal{R}/I(\mathcal{NCS}_m^n)$ are given by*

$$\beta_{i,j}(\mathcal{NCS}_m^n) = \begin{cases} i\binom{n}{i+1} + \sum\limits_{p=1}^{i} p\binom{n}{p}\binom{m-p}{i+1-p} + \sum\limits_{t=1}^{\ell}\binom{\ell}{t}\sum\limits_{p=1}^{i+1-2t} p\binom{n-t}{p}\binom{m-t-p}{i+1-2t-p} & \text{if } j = i+1, \\ 0 & \text{if } j \neq i+1. \end{cases}$$

where $\ell = min.\{m, n\}$, with convention $\binom{u}{v} = 0$ if $u < v$.

We obtain an alternate formula for Betti numbers of nearly complete split graph $\mathcal{NCS}_m^n$ given as follows and as a consequence, we obtain two combinatorial identities (see equation 4.3 and 4.4).

Theorem 4.8. *The edge ring of $\mathcal{NCS}_m^n$ has a 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{NCS}_m^n) = \begin{cases} n\binom{m+n-1}{i+1} - n\binom{m+n-2}{i+1} - \binom{n}{i+1} & \text{if } m \geq n \\ (n-m)\binom{m+n}{i+1} - (n-2m)\binom{m+n-1}{i+1} - m\binom{m+n-2}{i+1} - \binom{n}{i+1} & \text{if } m < n. \end{cases}$$

Proof. **Case 1.** If $m \geq n$

$\Delta(\mathcal{NCS}_m^n)$ consists of $(m-1)-$simplex $< y_1, \ldots, y_m >$ and $1-$simplicies $< x_i, y_i >; 1 \leq i \leq n$. Hilbert series in this case will be

$$\begin{aligned} \frac{1}{(1-t)^m} + \underbrace{\left(\frac{1}{(1-t)^2} - \frac{1}{(1-t)}\right) + \ldots + \left(\frac{1}{(1-t)^2} - \frac{1}{(1-t)}\right)}_{n\text{-times}} \\ = \tfrac{1}{(1-t)^m} + \tfrac{n}{(1-t)^2} - \tfrac{n}{(1-t)} \\ = \tfrac{(1-t)^n + n(1-t)^{m+n-2} - n(1-t)^{m+n-1}}{(1-t)^{m+n}} \end{aligned}$$

Therefore,

$$\beta_{i,i+1}(\mathcal{NCS}_m^n) = n\binom{m+n-1}{i+1} - n\binom{m+n-2}{i+1} - \binom{n}{i+1}.$$

Case 2. If $m < n$

15 Aug 2023 03:31:22 PDT
230719-Singh Version 2 - Submitted to Rocky Mountain J. Math.

$\Delta\mathcal{NCS}_m^n$ consists of $(m-1)$ - simplex $< y_1, \dots, y_m >$, 1-simplicies $< x_i, y_i >; 1 \leq i \leq m$ and $0-$ simplicies $< x_i >; m+1 \leq i \leq n$.

Hilbert series in this case will be

$$\tfrac{1}{(1-t)^m} + \underbrace{\left(\frac{1}{(1-t)^2} - \frac{1}{(1-t)}\right) + \ldots + \left(\frac{1}{(1-t)^2} - \frac{1}{(1-t)}\right)}_{m\text{-times}} + \underbrace{\left(\frac{1}{(1-t)} - 1\right) + \ldots + \left(\frac{1}{(1-t)} - 1\right)}_{(n-m)\text{-times}}$$

$$\begin{aligned} &= \tfrac{1}{(1-t)^m} + \tfrac{m}{(1-t)^2} - \tfrac{m}{(1-t)} + \tfrac{n-m}{(1-t)} - (n-m) \\ &= \tfrac{(1-t)^n + m(1-t)^{m+n-2} + (n-2m)(1-t)^{m+n-1} - (n-m)(1-t)^{m+n}}{(1-t)^{m+n}} \end{aligned}$$

Therefore,

$$\beta_{i,i+1}(\mathcal{NCS}_m^n) = (n-m)\binom{m+n}{i+1} - (n-2m)\binom{m+n-1}{i+1} - m\binom{m+n-2}{i+1} - \binom{n}{i+1}.$$

□

Corollary 4.9. *For positive integers m, n and i, we have the following identities*

(4.3)

$$i\tbinom{n}{i+1} + \sum_{p=1}^{i} p\tbinom{n}{p}\tbinom{m-p}{i+1-p} + \sum_{t=1}^{n}\tbinom{n}{t}\sum_{p=1}^{i+1-2t} p\tbinom{n-t}{p}\tbinom{m-t-p}{i+1-2t-p} = n\tbinom{m+n-1}{i+1} - n\tbinom{m+n-2}{i+1} - \tbinom{n}{i+1}.$$

and

(4.4)

$$i\tbinom{n}{i+1} + \sum_{p=1}^{i} p\tbinom{n}{p}\tbinom{m-p}{i+1-p} + \sum_{t=1}^{m}\tbinom{m}{t}\sum_{p=1}^{i+1-2t} p\tbinom{n-t}{p}\tbinom{m-t-p}{i+1-2t-p} = (n-m)\tbinom{m+n}{i+1} - (n-2m)\tbinom{m+n-1}{i+1} - m\tbinom{m+n-2}{i+1} - \tbinom{n}{i+1}.$$

Proof. This follows from Theorem 4.7 and Theorem 4.8.

In [16], authors compute Betti numbers of one extremity of split graphs known as complete split graphs $\mathcal{CS}_m^n$ and for the intermediate case i.e. for nearly complete split graphs $\mathcal{NCS}_m^n$. The other extremity is the split graph $\mathcal{S}_m^n$. Here, by using the technique of Hilbert series for fat forest, we compute Betti numbers of edge ideals of $\mathcal{S}_m^n$.

Theorem 4.10. *The edge ring of $\mathcal{S}_m^n$ has 2-linear resolution and*

$$\beta_{i,i+1}(\mathcal{S}_m^n) = (n-1)\binom{n}{i+1} - n\binom{n-1}{i+1}.$$

Proof. The fat tree with this Stanley-Reisner ideal $I(\mathcal{S}_m^n)$ has a maximal face $< x_i, y_1, \dots, y_m >$ for any $1 \leq i \leq n$. Then the other faces $< x_j, y_1, \dots, y_m >$; for $1 \leq j \neq i \leq n$ are attached to it with intersection as the $(m-1)$-simplex $< y_1, \dots, y_m >$. Therefore, the Hilbert series of

$k[x_1, \dots, x_n, y_1, \dots, y_m]/I$ is

$$\frac{1}{(1-t)^{m+1}} + \underbrace{\left(\frac{1}{(1-t)^{m+1}} - \frac{1}{(1-t)^m}\right) + \dots + \left(\frac{1}{(1-t)^{m+1}} - \frac{1}{(1-t)^m}\right)}_{(n-1)\text{-times}}$$

$$= \frac{1}{(1-t)^{m+1}} + \frac{(n-1)}{(1-t)^{m+1}} - \frac{(n-1)}{(1-t)^m}$$

$$= \frac{n(1-t)^{n-1} - (n-1)(1-t)^n}{(1-t)^{m+n}}$$

So, the Betti numbers are

$$\beta_{i,i+1}(\mathcal{S}_m^n) = (n-1)\binom{n}{i+1} - n\binom{n-1}{i+1}.$$

Remark 4.11. *One can see that $\mathcal{S}_m^n$ is a complete graph on n vertices with m isolated vertices. In view of Theorem 4.10 and Remark 3.7, we have the following identity*

$$(n-1)\binom{n}{i+1} - n\binom{n-1}{i+1} = i\binom{n}{i+1}. \tag{4.5}$$

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