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DELAYED FRACTIONAL DISCRETE SINE AND COSINE MATRIX FUNCTIONS AND THEIR APPLICATIONS TO LINEAR FRACTIONAL DELAYED DIFFERENCE OSCILLATING SYSTEMS

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ABSTRACT. The fractional discrete retarded cosine and sine matrix functions are defined for the first time in the current paper, and some their relations are discussed. The variation of constants technique is exploited to obtain an exact analytical form of a general solution to the Cauchy type problem for the linear Riemann Liouville fractional discrete retarded difference system of order $1 < 2\alpha \leq 2$ with the noncommutative coefficient matrices. Novel special cases are theoretically presented. In addition, numerical and simulated examples are given to illustrate all of the obtained results.

1. Introduction

In the recent decades, it has been noticed that fractional differential systems are more appropriate to represent the real-world problems in many of areas like mathematical physics, biophysics, electrochemistry, engineering; see [1]-[7] and the references therein.

Retarded differential equations which depend simultaneously on the present and past states enable to model various systems which have memory like automatic steering, control, stabilization; see [8]-[15].

Even though there are so many works about fractional retarded differential systems in continuous time and almost all of their aspects such as different kinds of stabilities and distinct sorts of controllability, etc are investigated, ones about fractional delayed difference systems of order $1 < \alpha \leq 2$ in the discrete-time are not that rich even if there is enough work in case of order $0 < \alpha \leq 1$; see [27]-[38]. This paper is an effort to make up for the deficiency in this regard. Everyone knows that the sine and the cosine functions as two different trigonometric functions are the solutions of the second-order differential systems. In the study[16], the delayed sine and cosine matrices are proposed to acquire a solution formula to the Cauchy problem for a second-order linear delayed system. These trigonometric functions are exploited in the works of the controllability and stability of the second ordered differential system with time-delay in the continuous time[17]-[25]. To the best of our knowledge, there are no studies in the discrete time to correspond to the ones in the just above-counted works. So, motivated by the works [16], [20], and [25] we dedicate this paper to the exploration of the following nonlinear Riemann-Liouville fractional retarded difference system of order $1 < 2\alpha \leq 2$ with the noncommutative

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coefficient matrices,

$$
\begin{cases} \left({}^{R}\nabla^{2\alpha}_{-r}n\right)(e) + Tn(e) + Sn(e-r) = \daleth(e), \quad e \in \mathbb{N}_1, \\ n(e) = \phi(e), \quad \left({}^{R}\nabla^{\alpha}_{-r}n\right)(e) = \left({}^{R}\nabla^{\alpha}_{-r}\phi\right)(e), \quad e \in \mathbb{N}^{0}_{-r+1}, \end{cases} \tag{1}
$$

where ${}^{R}\nabla^{\alpha}_{-r}$ symbolises the Riemann-Liouville fractional difference of order $\frac{1}{2} < \alpha \leq 1$, $n : \mathbb{N}_1 \to \mathbb{R}^n$, $\daleth : \mathbb{N}_1 \to \mathbb{R}^n$ is a function, $r \in \mathbb{N}_1$ is a retardation, $T, S \in \mathbb{R}^{n\times n}$ are constant coefficient matrices, $\phi : \mathbb{N}^{0}_{-r+1} \to \mathbb{R}^n$ is an initial function. This current paper is organized as follows:

- we give the auxiliary available tools in the literature for the present paper to use(see Section 2);
- we newly propose the discrete delayed perturbation of the nabla sine and cosine matrix function, and establish a couple of its relations with each other(see Section 3);
- we look for a representation of solutions to the nabla semi-linear Riemann Liouville fractional delayed difference system with noncommutative coefficient matrices step by step(see Section 4);

  A solution to homogeneous part with and without the initial circumstances(see Section 4.1)

  A solution to nonhomogeneous part with the zero condition and with the initial circumstances(see Section 4.2)
- we offer a few valuable new special cases(see Section 5);
- we present a few practical examples with simulations to exemplify our theoretical results(see Section 6);
- we express a couple of open problems together with the conclusion(see Section 6).

2. Preliminaries

In this section, we offer the auxiliary tools to be used in the coming sections, which is available in the literature.

$\mathbb{N}_a = \{a, a+1, a+2, \dots\}$, $\mathbb{N}^a = \{\dots, a-2, a-1, a\}$, $\mathbb{N}^b_a = \{a, a+1, a+2, \dots, b\}$ where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_1$.

Definition 1. [39, Definition 3.4] *The generalized rising function is defined by*

$$
e^{\overline{r}} = \frac{\Gamma(e+r)}{\Gamma(e)},
$$

whenever the right-hand side of this equation is sensible for those values of e and r.

Definition 2. [39, Definition 3.56] *Assume that $\alpha \notin \mathbb{N}^{-1}$. The α-th order (nabla) fractional Taylor monomial $\mathscr{H}_\alpha(e,a)$ is defined by*

$$
\mathscr{H}_\alpha(e,a) = \frac{(e-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)},
$$

where the right-hand side of the above equation is sensible.

$\mathscr{H}_\alpha(e,a)$ has the following properties.

Lemma 3. [39, Definition 3.56] *The following hold:*

(a): $\mathscr{H}_\alpha(a,a) = 0$,

(b): *For* $k \in \mathbb{N}_1$, $e \in \mathbb{N}_a$, $\mathscr{H}_{-k}(e,a) = 0$,
(c): $\mathscr{H}_{-1}(e,e-1) = 1$,
(d): $\nabla \mathscr{H}_\alpha(e,a) = \mathscr{H}_{\alpha-1}(e,a)$.

We are ready to offer two main definitions and one useful lemma about the (nabla) fractional sum with respect to the (nabla) fractional Taylor monomial and the (nabla) Riemann-Liouville fractional difference.

Definition 4. [39, Theorem 3.93] *Let* $n : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $\alpha > 0$. *Then the (nabla) fractional sum is given by*

$$\nabla_a^{-\alpha} n(e) = \int_a^e \mathscr{H}_{\alpha-1}(e,\rho(s))n(s)\nabla s = \sum_{s=a+1}^{e} \mathscr{H}_{\alpha-1}(e,\rho(s))n(s), \ \ e \in \mathbb{N}_a,$$

where $\rho(s) = s - 1$.

Definition 5. [39, Definition 3.61, Theorem 3.62] *Let* $n : \mathbb{N}_{a+1} \to \mathbb{R}$. *Then the (nabla) Riemann-Liouville fractional difference of order* $0 < \alpha < 1$ *is given by*

$$^{R}\nabla_a^{\alpha} n(e) = \int_a^e \mathscr{H}_{-\alpha-1}(e,\rho(s))n(s)\nabla s = \sum_{s=a+1}^{e} \mathscr{H}_{-\alpha-1}(e,\rho(s))n(s),$$

for $e \in \mathbb{N}_a$.

Lemma 6. [39, Theorem 3.107] *Let* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that* $\beta - \alpha$, $\beta - 1$ *and* $\alpha + \beta - 1$ *are nonnegative integers. Then*

$$\nabla_a^{-\alpha} \mathscr{H}_{\beta-1}(e,a) = \mathscr{H}_{\alpha+\beta-1}(e,a), \ \ e \in \mathbb{N}_a,$$

and

$$^{R}\nabla_a^{\alpha} \mathscr{H}_{\beta-1}(e,a) = \mathscr{H}_{\beta-\alpha-1}(e,a), \ \ e \in \mathbb{N}_a,$$

and

$$^{R}\nabla_a^{\alpha} (e-a)^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(e-a)^{\overline{\beta-\alpha}},$$

hold true.

Lemma 7. [39, Theorem 3.108] *For* $k \in \mathbb{N}_0$, $\mu > 0$, *and* $N \in \mathbb{N}$ *such that* $N - 1 < \mu \leq N$, *we have*

$$\nabla^k \, {}^{R}\nabla_a^{\mu} f(e) = {}^{R}\nabla_a^{k+\mu} f(e), \quad e \in \mathbb{N}_{a+k}.$$

Lemma 8. [39, Theorem 3.41] *Assume that* $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. *Then*

$$\nabla \left(\int_a^e f(e,s)\nabla s \right) = \int_a^{e-1} \nabla_e f(e,s)\nabla s + f(e,e), \ \ e \in \mathbb{N}_{a+1}.$$

From the following section on, we present all of our new original contributions.

3. Discrete delayed perturbation of sine and cosine matrix functions

Here, the fractional discrete delayed perturbation of sine and cosine matrix functions composing of the building stones of a representation of solution formulas of the homogeneous and nonhomogeneous Riemann-Liouville fractional retarded difference system are newly defined. A couple of their relations to be used in the upcoming findings will be debated.

We introduce the determining matrix equation $Q(i,j)$ as the following recursive form

$$Q(i+1,j) = -TQ(i,j) - SQ(i,j-1), \tag{2}$$

and

$$Q(-1,j) = Q(i,-1) = \Theta,\ Q(0,0) = I, \tag{3}$$

where $i,j \in \mathbb{N}_0$, Θ is the zero matrix, and I is the identity matrix.

Remark 9. *It should be stated that the determining function $Q(i,j)$ has been employed in the works* [26], [40] *to describe delayed perturbation of discrete matrix exponential, delayed perturbation of Mittag-Leffler matrix function, and discrete delay perturbation of Mittag-Leffer matrix function , respectively. By using the recursive equation (2), one can easily acquire the explicit expansion in the following table:*

	$k=0$	$k=1$	$k=2$	$k=3$	$\cdots$	$k=p$
$Q(0,k)$	I	Θ	Θ	Θ	$\cdots$	Θ
$Q(1,k)$	$-T$	$-S$	Θ	Θ	$\cdots$	
$Q(2,k)$	T^2	$TS+ST$	S^2	Θ	$\cdots$	Θ
$Q(3,k)$	$-T^3$	$-T(TS+ST)-ST^2$	$-TS^2-S(TS+ST)$	$-S^3$	$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	Θ
$Q(p,k)$	$(-1)^pT^p$	Θ	Θ	Θ	$\cdots$	$(-1)^pS^p$

Everyone can smoothly gain the below remark from the above table.

Remark 10. *It is clear that $Q(i,j) = \Theta$ for $i < j$.*

Remark 11. *Let $Q(i,j)$ be defined as in the equations (2) and (3).*

- *If $T = \Theta$, then we have $Q(i,j) = (-1)^j S^j$.*
- *If $S = \Theta$, then we have $Q(i,j) = (-1)^i T^i$.*

Definition 12. *The (fractional) discrete delayed perturbation of cosine and sine matrix functions ${}_CX^{T,S}_{r,\alpha,\beta}$ and ${}_SX^{T,S}_{r,\alpha,\beta}$ generated by T,S is defined, respectively, as follows:*

$$ {}_CX^{T,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e \in \mathbb{N}^{-r-1}, \\ \sum\limits_{i=0}^{\infty}(-1)^i T^i \dfrac{(e+r)^{\overline{2i\alpha+\beta-1}}}{\Gamma(2i\alpha+\beta)} + \sum\limits_{i=1}^{\infty} Q(i,1) \dfrac{e^{\overline{2i\alpha+\beta-1}}}{\Gamma(2i\alpha+\beta)} & \\ +\ldots+\sum\limits_{i=p}^{\infty} Q(i,p) \dfrac{(e-(p-1)r)^{\overline{2i\alpha+\beta-1}}}{\Gamma(2i\alpha+\beta)}, & e \in \mathbb{N}^{pr}_{(p-1)r}, \end{cases} \tag{4}$$

and

$$(5)\qquad {}_{S}X^{T,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e\in\mathbb{N}^{-r-1},\\ \sum\limits_{i=0}^{\infty}(-1)^iT^i\dfrac{(e+r)^{\overline{(2i+1)\alpha+\beta-1}}}{\Gamma((2i+1)\alpha+\beta)}+\sum\limits_{i=1}^{\infty}Q(i,1)\dfrac{(e+r)^{\overline{(2i+1)\alpha+\beta-1}}}{\Gamma((2i+1)\alpha+\beta)} & \\ +\ldots+\sum\limits_{i=p}^{\infty}Q(i,p)\dfrac{(e-(p-1)r)^{\overline{(2i+1)\alpha+\beta-1}}}{\Gamma((2i+1)\alpha+\beta)}, & e\in\mathbb{N}^{pr}_{(p-1)r}.\end{cases}$$

Remark 13. *It is clear that the series for cosine (4) and sine (5) converge absolutely for fixed e, provided that $\|T\|<1$, see [34], [35] Lemma 2.3. If $T=\Theta$, then we have a finite sum instead of series, see Proposition 24.*

Remark 14. *The exclusive matrix equation embedded in the (fractional) discrete delayed perturbation of cosine and sine matrix functions provides the non-commutativity of the coefficient constant matrices T and S.*

For more convenience, we could restate the (fractional) discrete delayed perturbation of cosine and sine matrix functions ${}_{C}X^{T,S}_{r,\alpha,\beta}$ and ${}_{S}X^{T,S}_{r,\alpha,\beta}$ in terms of the (nabla) fractional Taylor monomial as follows $\mathscr{H}_\alpha(e,a)$.

$$(6)\qquad {}_{C}X^{T,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e\in\mathbb{N}^{-r-1},\\ \sum\limits_{i=0}^{\infty}(-1)^iT^i\mathscr{H}_{2i\alpha+\beta-1}(e,-r)+\sum\limits_{i=1}^{\infty}Q(i,1)\mathscr{H}_{2i\alpha+\beta-1}(e,0), & \\ +\ldots+\sum\limits_{i=p}^{\infty}Q(i,p)\mathscr{H}_{2i\alpha+\beta-1}(e,(p-1)r), & e\in\mathbb{N}^{pr}_{(p-1)r},\end{cases}$$

and

$$(7)\qquad {}_{S}X^{T,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e\in\mathbb{N}^{-r-1},\\ \sum\limits_{i=0}^{\infty}(-1)^iT^i\mathscr{H}_{(2i+1)\alpha+\beta-1}(e,-r)+\sum\limits_{i=1}^{\infty}Q(i,1)\mathscr{H}_{(2i+1)\alpha+\beta-1}(e,-r), & \\ +\ldots+\sum\limits_{i=p}^{\infty}Q(i,p)\mathscr{H}_{(2i+1)\alpha+\beta-1}(e,(p-1)r), & e\in\mathbb{N}^{pr}_{(p-1)r}.\end{cases}$$

Remark 15. *It is clear that for $e\in\mathbb{N}^0_{-r}$*

$$\begin{aligned} {}_{C}X^{T,S}_{r,\alpha,1}(e) &= \sum_{i=0}^{\infty}(-1)^iT^i\mathscr{H}_{2i\alpha}(e,-r)\\ &= I\mathscr{H}_0(e,-r)+\sum_{i=1}^{\infty}(-1)^iT^i\mathscr{H}_{2i\alpha}(e,-r)\\ &= I+\sum_{i=1}^{\infty}(-1)^iT^i\mathscr{H}_{2i\alpha}(e,-r).\end{aligned}$$

By Lemma 3(a) we have

$${}_{C}X^{T,S}_{r,\alpha,1}(-r) = I+\sum_{i=1}^{\infty}(-1)^iT^i\mathscr{H}_{2i\alpha}(-r,-r)=I.$$

Similarly, for $e \in \mathbb{N}_{-r}^{0}$

$$
\begin{aligned}
{}_{S}X_{r,\alpha,\beta}^{T,S}(e) &= \sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{(2i+1)\alpha+\beta-1}(e,-r),\\
{}_{S}X_{r,\alpha,\beta}^{T,S}(-r) &= \Theta.
\end{aligned}
$$

In the following lemma, we debate what the matrix equation $Q(i,j)$ happens in the case of the commutativity of T and S.

Lemma 16. *Under the commutativity of the constant coefficient matrices T ad S, we have*

$$
Q(i,j) = (-1)^{i}\binom{i}{j}T^{i-j}S^{j}, \quad i,j \in \mathbb{N}_{0}.
$$

Proof. We apply the mathematical induction on $j \in \mathbb{N}_0$ to the recursive equation to prove this lemma. Let us see that it is true for $j = 0$.

$$
\begin{aligned}
Q(i,0) &= -TQ(i-1,0) - SQ(i-1,-1)\\
&= -TQ(i-1,0) = (-1)^{i}T^{i} = (-1)^{i}\binom{i}{0}T^{i-0}S^{0}.
\end{aligned}
$$

Assume that it is true for $j = n$, that is,

$$
Q(i,n) = (-1)^{i}\binom{i}{n}T^{i-n}S^{n}.
$$

Let us check its validity for $j = n+1$.

$$
\begin{aligned}
Q(i,n+1) &= -TQ(i-1,n+1) - SQ(i-1,n)\\
&= -T(-1)^{i-1}\binom{i-1}{n+1}T^{i-1-n-1}S^{n+1} - S(-1)^{i-1}\binom{i-1}{n}T^{i-1-n}S^{n}\\
&= (-1)^{i}\left[\binom{i-1}{n+1} + \binom{i-1}{n}\right]T^{i-1-n}S^{n+1}\\
&= (-1)^{i}\binom{i}{n+1}T^{i-(n+1)}S^{n+1}.
\end{aligned}
$$

□

The following lemma expresses the relations we can determine between discrete delayed sine and cosine matrix function.

Lemma 17. *The nabla Riemann-Liouville fractional difference of the fractional discrete delayed sine ${}_{S}X_{r,\alpha,\beta}^{T,S}$ and cosine ${}_{C}X_{r,\alpha,\beta}^{T,S}$ are as follows:*

$$
{}^{R}\nabla_{-r}^{\alpha}\,{}_{S}X_{r,\alpha,\alpha}^{T,S}(e) = {}_{C}X_{r,\alpha,\alpha}^{T,S}(e), \tag{8}
$$

and

$$
{}^{R}\nabla_{-r}^{\alpha}\,{}_{C}X_{r,\alpha,\alpha}^{T,S}(e) = -T\,{}_{S}X_{r,\alpha,\alpha}^{T,S}(e) - S\,{}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r). \tag{9}
$$

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Proof. For $e \in \mathbb{N}^{-r-1}$, equation (8) is hold because ${}_{C}X^{T,S}_{r,\alpha,\alpha}(e) = {}_{S}X^{T,S}_{r,\alpha,\alpha}(e) = \Theta$. For $e \in \mathbb{N}^{0}_{-r}$, by using Lemma 6 we have

$$\begin{aligned}{}^{R}\nabla^{\alpha}_{-r}{}_{S}X^{T,S}_{r,\alpha,\alpha}(e) &= \sum_{i=0}^{\infty}(-1)^{i}T^{i}\,{}^{R}\nabla^{\alpha}_{-r}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e,-r)\\ &= \sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{2i\alpha+\alpha-1}(e,-r)\\ &= {}_{C}X^{T,S}_{r,\alpha,\alpha}(e).\end{aligned}$$

For $e \in \mathbb{N}^{pr}_{(p-1)r+1}$, $p \in \mathbb{N}_{1}$, by considering the subintervals and using Lemma 6 we have

$$\begin{aligned}{}^{R}\nabla^{\alpha}_{-r}{}_{S}X^{T,S}_{r,\alpha,\alpha}(e) &= \sum_{i=0}^{\infty}\sum_{j=0}^{p}Q(i,j)\,{}^{R}\nabla^{\alpha}_{(j-1)r}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e,(j-1)r)\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{p}Q(i,j)\,\mathscr{H}_{2i\alpha+\alpha-1}(e,(j-1)r)\\ &= {}_{C}X^{T,S}_{r,\alpha,\alpha}(e).\end{aligned}$$

For the proof of equality (9), assume that $e \in \mathbb{N}^{-r-1}$. Then equation (9) is hold because ${}_{C}X^{T,S}_{r,\alpha,\alpha}(e) = {}_{S}X^{T,S}_{r,\alpha,\alpha}(e) = {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r) = \Theta$. For $e \in \mathbb{N}^{0}_{-r}$, ${}_{C}X^{T,S}_{r,\alpha,\alpha}(e) = \sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{2i\alpha+\alpha-1}(e,-r)$ and ${}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r) = \Theta$, we get

$$\begin{aligned}{}^{R}\nabla^{\alpha}_{-r}{}_{C}X^{T,S}_{r,\alpha,\alpha}(e) &= \sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{2i\alpha-1}(e,-r)\\ &= \sum_{i=1}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{2i\alpha-1}(e,-r)\\ &= -T\,{}_{S}X^{T,S}_{r,\alpha,\alpha}(e) - S\,{}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r).\end{aligned}$$

For $e \in \mathbb{N}^{pr}_{(p-1)r+1}$, $p \in \mathbb{N}_{1}$, by considering the subintervals and using Lemma 6 we get

$${}^{R}\nabla^{\alpha}_{-r}{}_{C}X^{T,S}_{r,\alpha,\alpha}(e) = \sum_{i=0}^{\infty}\sum_{j=0}^{p}Q(i,j)\,\mathscr{H}_{2i\alpha-1}(e,(j-1)r).$$

Applying (2) and (3) to the above equation, one can see

$$\begin{aligned}{}^{R}\nabla^{\alpha}_{-r\,C}X^{T,S}_{r,\alpha,\alpha}(e) &= -T\sum_{i=1}^{\infty}\sum_{j=0}^{p}Q(i-1,j)\,\mathscr{H}_{2i\alpha-1}(e,(j-1)r) \\ &\quad -S\sum_{i=1}^{\infty}\sum_{j=1}^{p}Q(i-1,j-1)\,\mathscr{H}_{2i\alpha-1}(e,(j-1)r) \\ &= -T\sum_{i=0}^{\infty}\sum_{j=0}^{p}Q(i,j)\,\mathscr{H}_{2i\alpha+2\alpha-1}(e,(j-1)r) \\ &\quad -S\sum_{i=0}^{\infty}\sum_{j=0}^{p-1}Q(i,j)\,\mathscr{H}_{2i\alpha+2\alpha-1}(e,jr) \\ &= -T\,{}_{S}X^{T,S}_{r,\alpha,\alpha}(e) - S\,{}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r),\end{aligned}$$

which is the foregone conclusion. This completely ends the proof. □

4. The explicit solution of RL fractional retarded difference system

In this section, our aim is to investigate an explicit solution to the nonlinear Riemann Liouville fractional retarded difference system (1) by dividing into three subsections.

We share a couple of main theorems to achieve our objective. When it comes to most of their proofs, we use the method of variations' technique for them since we easily find some solutions to the homogeneous part of the Riemann Liouville fractional delayed difference system.

4.1. ***Homogeneous case:*** In this subsection, we will make an effort to find a solution to the below homogeneous retarded Riemann Liouville fractional difference system,

$$\begin{cases} \left({}^{R}\nabla^{2\alpha}_{-r}n\right)(e) + Tn(e) + Sn(e-r) = 0, & e\in\mathbb{N}_1, \\ n(e) = \phi(e), \ \left({}^{R}\nabla^{\alpha}_{-r}n\right)(e) = \left({}^{R}\nabla^{\alpha}_{-r}\phi\right)(e), & e\in\mathbb{N}^{0}_{-r+1}, \end{cases} \tag{10}$$

where ${}^{R}\nabla^{\alpha}_{-r}$ symbolises the Riemann-Liouville fractional difference of order $\frac{1}{2}<\alpha\le 1$, $n:\mathbb{N}_1\to\mathbb{R}^n$, $\daleth:\mathbb{N}_1\to\mathbb{R}^n$ is a function, $r\in\mathbb{N}_1$ is a retardation, $T,S\in\mathbb{R}^{n\times n}$ are constant coefficient matrices, $\phi:\mathbb{N}^{0}_{-r+1}\to\mathbb{R}^n$ is an initial function.

We firstly investigate solutions to homogeneous Riemann Liouville fractional delayed difference system (1) regardless of the initial conditions.

Theorem 18. *The fractional discrete delayed sine and cosine matrix functions ${}_{S}X^{T,S}_{r,\alpha,\alpha}$ and ${}_{C}X^{T,S}_{r,\alpha,\alpha}$ satisfy homogeneous system (10) without any initial conditions in the case of $\alpha=\beta$, that is,*

$$\left({}^{R}\nabla^{2\alpha}_{-r\,S}X^{T,S}_{r,\alpha,\alpha}\right)(e) = -T\,{}_{S}X^{T,S}_{r,\alpha,\alpha}(e) - S\,{}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r),$$

and

$$\left({}^{R}\nabla^{2\alpha}_{-r\,C}X^{T,S}_{r,\alpha,\alpha}\right)(e) = -T\,{}_{C}X^{T,S}_{r,\alpha,\alpha}(e) - S\,{}_{C}X^{T,S}_{r,\alpha,\alpha}(e-r).$$

Proof. We omit the proofs since they are the similar to that of Lemma 17. □

Now, we will share a practical lemma to be used in the forthcoming proofs of the following theorems stated by homogeneous system (10) with the initial conditions and non-homogeneous system (11) with the zero initial condition.

Lemma 19. *The following equalities*

$$ {}^{R}\nabla_{-r}^{\alpha} \int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k = \int_{-r}^{e} {}_{C}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k, \quad e \in \mathbb{N}_{-r+1}, $$

and

$$ \begin{aligned} &{}^{R}\nabla_{-r}^{2\alpha} \int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k \\ &\qquad = \eta(e) - T\int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k, \qquad e \in \mathbb{N}_{-r+1}^{0}, \end{aligned} $$

and

$$ \begin{aligned} {}^{R}\nabla_{-r}^{2\alpha} \int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k &= \eta(e) - T\int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k \\ &\quad - S\int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-2r-k)\,\eta(k)\nabla k, \qquad e \in \mathbb{N}_{-r+1}, \end{aligned} $$

are hold true for $\eta : \mathbb{N}_{-r+1} \to \mathbb{R}^{n}$.

Proof. We prove lemma only for $e \in \mathbb{N}^{0}_{-r+1}$, since for $e \in \mathbb{N}^{pr}_{(p-1)r}$, $p \in \mathbb{N}_1$ the proof is very similar. We use Lemma 6 to acquire the following:

$$\begin{aligned}
&{}^{R}\nabla^{\alpha}_{-r} \int_{-r}^{e} {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,\eta(k)\nabla k \\
&= \sum_{s=-r+1}^{e} \mathscr{H}_{-\alpha-1}(e,\rho(s)) \left(\int_{-r}^{s} {}_{S}X^{T,S}_{r,\alpha,\alpha}(s-r-k)\,\eta(k)\nabla k \right) \\
&= \sum_{s=-r+1}^{e} \sum_{k=-r+1}^{s} \mathscr{H}_{-\alpha-1}(e,\rho(s))\, {}_{S}X^{T,S}_{r,\alpha,\alpha}(s-r-k)\,\eta(k) \\
&= \sum_{k=-r+1}^{e} \sum_{s=k}^{e} \left(\mathscr{H}_{-\alpha-1}(e,\rho(s))\ {}_{S}X^{T,S}_{r,\alpha,\alpha}(s-r-k) \right) \eta(k) \\
&= \sum_{k=-r+1}^{e} \sum_{s=k+1}^{e} \left(\mathscr{H}_{-\alpha-1}(e,\rho(s))\ {}_{S}X^{T,S}_{r,\alpha,\alpha}(s-r-k) \right) \eta(k) \\
&= \sum_{k=-r+1}^{e} {}^{R}\nabla^{\alpha}_{k}\ {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,\eta(k) \\
&= \int_{-r}^{e} {}^{R}\nabla^{\alpha}_{k}\ {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,\eta(k)\nabla k \\
&= \int_{-r}^{e} {}^{R}\nabla^{\alpha}_{k} \sum_{i=0}^{\infty} (-1)^{i} T^{i} \mathscr{H}_{(2i+1)\alpha+\alpha-1}(e-r-k,-r)\eta(k)\nabla k \\
&= \int_{-r}^{e} \sum_{i=0}^{\infty} (-1)^{i} T^{i}\ {}^{R}\nabla^{\alpha}_{k}\ \mathscr{H}_{(2i+1)\alpha+\alpha-1}(e-r-k,-r)\eta(k)\nabla k \\
&= \int_{-r}^{e} \sum_{i=0}^{\infty} (-1)^{i} T^{i}\ {}^{R}\nabla^{\alpha}_{k} \frac{(e-k)^{\overline{(2i+1)\alpha+\alpha-1}}}{\Gamma((2i+1)\,\alpha+\alpha)} \eta(k)\nabla k \\
&= \int_{-r}^{e} \sum_{i=0}^{\infty} (-1)^{i} T^{i} \frac{\Gamma((2i+1)\,\alpha+\alpha)(e-k)^{\overline{2i\alpha+\alpha-1}}}{\Gamma(2i\alpha+\alpha)\Gamma((2i+1)\,\alpha+\alpha)} \eta(k)\nabla k \\
&= \int_{-r}^{e} {}_{C}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,\eta(k)\nabla k.
\end{aligned}$$

Next, we apply the nabla Riemann-Liouville difference ${}^{R}\nabla_{-r}^{2\alpha}$ and use Lemmas 3, 6 and 7 to get

$$\begin{aligned}
&{}^{R}\nabla_{-r}^{2\alpha}\int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k \\
&= \nabla\,{}^{R}\nabla_{-r}^{2\alpha-1}\int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k \\
&= \nabla\int_{-r}^{e}\mathscr{H}_{-2\alpha}(e,\rho(s))\left(\int_{-r}^{s} {}_{S}X_{r,\alpha,\alpha}^{T,S}(s-r-k)\,\eta(k)\nabla k\right)\nabla s \\
&= \nabla\int_{-r}^{e}\left(\int_{k}^{e}\mathscr{H}_{-2\alpha}(e,\rho(s))\,{}_{S}X_{r,\alpha,\alpha}^{T,S}(s-r-k)\nabla s\right)\eta(k)\nabla k \\
&= \nabla\int_{-r}^{e} {}^{R}\nabla_{k}^{2\alpha-1}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e-r-k,-r)\eta(k)\nabla k \\
&= \nabla\int_{-r}^{e} {}^{R}\nabla_{k}^{2\alpha-1}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e,k)\eta(k)\nabla k \\
&= \nabla\int_{-r}^{e}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\,{}^{R}\nabla_{k}^{2\alpha-1}\,\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e,k)\eta(k)\nabla k \\
&= \nabla\int_{-r}^{e}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{2i\alpha}(e,k)\eta(k)\nabla k \\
&= \nabla\int_{-r}^{e}\mathscr{H}_{0}(e,k)\eta(k)\nabla k+\nabla\int_{-r}^{e}\sum_{i=1}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{2i\alpha}(e,k)\eta(k)\nabla k \\
&= \int_{-r}^{e-1}\nabla\mathscr{H}_{0}(e,k)\eta(k)\nabla k+\int_{-r}^{e-1}\sum_{i=1}^{\infty}(-1)^{i}T^{i}\nabla\mathscr{H}_{2i\alpha}(e,k)\eta(k)\nabla k \\
&= \int_{-r}^{e-1}\mathscr{H}_{-1}(e,k)\eta(k)\nabla k-T\int_{-r}^{e-1}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e,k)\eta(k)\nabla k \\
&= \eta(e)-T\int_{-r}^{e}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e,k)\eta(k)\nabla k \\
&= \eta(e)-T\int_{-r}^{e}\sum_{i=0}^{\infty}(-1)^{i}T^{i}\mathscr{H}_{(2i+1)\alpha+\alpha-1}(e-r-k,-r)\eta(k)\nabla k \\
&= \eta(e)-T\int_{-r}^{e} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\,\eta(k)\nabla k.
\end{aligned}$$

This is the expected result. □

Theorem 20. *The following function*

$$\begin{aligned}
n(e) &= \int_{-r}^{0} {}_{S}X_{r,\alpha,\alpha}^{T,S}(e-r-k)\left[\left({}^{R}\nabla_{-r}^{2\alpha}\phi\right)(k)+T\phi(k)\right]\nabla k \\
&\quad + {}_{C}X_{r,\alpha,1}^{T,S}(e)\,\phi(-r)+{}_{S}X_{r,\alpha,1}^{T,S}(e)\left({}^{R}\nabla_{-r}^{\alpha}\phi\right)(-r),
\end{aligned}$$

satisfies homogeneous system (10).

Proof. We use the variation of parameters method to find a general solution formula to system (10). Therefore we are looking for a solution of the form

$$n(e) = {}_{C}X^{T,S}_{r,\alpha,1}(e)\lambda_1 + {}_{S}X^{T,S}_{r,\alpha,1}(e)\lambda_2 + \int_{-r}^{0} {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\lambda(k)\nabla k,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^n$ are unknown constants, $\lambda(k)$ is an unknown function. Taking the nabla Riemann-Liouville differences and using the Lemma 19, we get

$$n(e) = {}_{C}X^{T,S}_{r,\alpha,1}(e)\lambda_1 + {}_{S}X^{T,S}_{r,\alpha,1}(e)\lambda_2 + \int_{-r}^{0} {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\lambda(k)\nabla k = \phi(e), \quad e \in \mathbb{N}^0_{-r},$$

$$\begin{aligned}\left({}^{R}\nabla^{\alpha}_{-r}n\right)(e) &= -T\ {}_{S}X^{T,S}_{r,\alpha,1}(e)\lambda_1 + {}_{C}X^{T,S}_{r,\alpha,1}(e)\lambda_2 \\ &\quad + \int_{-r}^{0} {}_{C}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\lambda(k)\nabla k = \left({}^{R}\nabla^{\alpha}_{-r}\phi\right)(e), \quad e \in \mathbb{N}^0_{-r},\end{aligned}$$

$$\begin{aligned}\left({}^{R}\nabla^{2\alpha}_{-r}\phi\right)(e) &= -T\ {}_{C}X^{T,S}_{r,\alpha,1}(e)\lambda_1 - T\ {}_{S}X^{T,S}_{r,\alpha,1}(e)\lambda_2 \\ &\quad - T\int_{-r}^{0} {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\lambda(k)\nabla k + \lambda(e) \\ &= -Tn(e) + \lambda(e) = -T\phi(e) + \lambda(e).\end{aligned}$$

Now using the initial conditions properties

$$\int_{-r}^{0} {}_{S}X^{T,S}_{r,\alpha,\alpha}(-r-r-k)\lambda(k)\nabla k = 0, \quad \int_{-r}^{0} {}_{C}X^{T,S}_{r,\alpha,\alpha}(r-r-k)\lambda(k)\nabla k = 0,$$

and Remark 15, we get

$$\begin{aligned}n(-r) &= \lambda_1 = \phi(-r), \\ \left({}^{R}\nabla^{\alpha}_{-r}n\right)(-r) &= \lambda_2 = \left({}^{R}\nabla^{\alpha}_{-r}\phi\right)(-r), \\ \lambda(e) &= \left({}^{R}\nabla^{2\alpha}_{-r}\phi\right)(e) + T\phi(e).\end{aligned}$$

This is the end of the proof of this theorem. □

4.2. *Nonhomogeneous case:* In this subsection, we try to research for a solution to the nonhomogeneous Riemann Liouville fractional delayed difference system with the zero initial condition.

$$\left\{\begin{array}{l}\left({}^{R}\nabla^{2\alpha}_{-r}n\right)(e) + Tn(e) + Sn(e-r) = \daleth(e), \quad e \in \mathbb{N}_1, \\ n(e) = 0, \quad \left({}^{R}\nabla^{\alpha}_{-r}n\right)(e) = 0, \quad e \in \mathbb{N}^0_{-r+1}.\end{array}\right. \tag{11}$$

Theorem 21. *The following integral expression*

$$n(e) = \int_{0}^{e} {}_{S}X^{T,S}_{r,\alpha,\alpha}(e-r-k)\daleth(k)\nabla k,$$

fulfills nonhomogeneous system (11) with the zero initial condition.

Proof. Again, we exploit the variation of parameters method in this proof. Suppose that the representation of the solution formula is given by the following integral expression

$$n(e)=\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,O(k)\nabla k,$$

where $O(k)$, $0\le k\le e$ is the unknown function which will be determined. Now, by keeping Lemma 19 and taking the nabla Riemann-Liouville difference ${}^R\nabla^{2\alpha}_{-r}$ from the above equation

$$\begin{aligned}\left({}^R\nabla^{2\alpha}_{-r}n\right)(e)&=O(e)-T\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,O(k)\nabla k\\&-S\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-2r-k)\,O(k)\nabla k.\end{aligned}$$

In order to get $O(e)=\daleth(e)$, we insert the above equation into equation (1)

$$\begin{aligned}&O(e)-T\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,O(k)\nabla k\\&-S\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-2r-k)\,O(k)\nabla k\\&+T\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,O(k)\nabla k\\&+S\int_0^{e-r} {}_S X^{T,S}_{r,\alpha,\alpha}(e-2r-k)\,O(k)\nabla k=\daleth(e).\end{aligned}$$

The fact that $\int_{e-r}^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-2r-k)\,O(k)\nabla k=\Theta$ finishes our discussion for this proof. □

Now, we will examine the below nonhomogeneous retarded Riemann Liouville fractional difference system,

$$(12)\qquad \begin{cases}\left({}^R\nabla^{2\alpha}_{-r}n\right)(e)+Tn(e)+Sn(e-r)=\daleth(e), & e\in\mathbb{N}_1,\\ n(e)=\phi(e),\ \left({}^R\nabla^{\alpha}_{-r}n\right)(e)=\left({}^R\nabla^{\alpha}_{-r}\phi\right)(e), & e\in\mathbb{N}^0_{-r+1},\end{cases}$$

where ${}^R\nabla^{\alpha}_{-r}$ symbolises the Riemann-Liouville fractional difference of order $\frac{1}{2}<\alpha\le 1$, $n:\mathbb{N}_1\to\mathbb{R}^n$, $\daleth:\mathbb{N}_1\to\mathbb{R}^n$ is a function, $r\in\mathbb{N}_1$ is a retardation, $T,S\in\mathbb{R}^{n\times n}$ are constant coefficient matrices, $\phi:\mathbb{N}^0_{-r+1}\to\mathbb{R}^n$ are initial functions.

Gathering Theorem 20 and 21, we acquire our main result.

Theorem 22. *The whole function*

$$\begin{aligned}n(e)&={}_C X^{T,S}_{r,\alpha,1}(e)\,\phi(-r)+{}_S X^{T,S}_{r,\alpha,1}(e)\left({}^R\nabla^{\alpha}_{-r}\phi\right)(-r)\\&+\int_{-r}^0 {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\left[\left({}^R\nabla^{2\alpha}_{-r}\phi\right)(k)+T\phi(k)\right]\nabla k\\&+\int_0^e {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,\daleth(k)\nabla k,\end{aligned}$$

satisfies nonhomogeneous system (12).

Proof. It is easy to prove by means of the principle of superposition technique. So we omit it. □

5. Special cases

In this section we exhibit a couple of special cases which are new in the literature and each of them is worth the subject of an article.

Example 23. *Let us reconsider the nabla fractional delayed difference system (1) when $T=\Theta$. In this context, Theorem 22 may be reformulated as follows.*

Proposition 24. *The exact analytical general solution of nonhomogeneous version of system (1) has the following explicit form:*

$$\begin{aligned} n(e) &= {}_C X^{\Theta,S}_{r,\alpha,\alpha}(e)\,\phi(-r) + {}_S X^{T,S}_{r,\alpha,\alpha}(e)\left({}^R\nabla^{\alpha}_{-r}\phi\right)(-r) \\ &\quad + \int_{-r}^{0} {}_S X^{\Theta,S}_{r,\alpha,\alpha}(e-r-k)\left[\left({}^R\nabla^{2\alpha}_{-r}\phi\right)(k)\right]\nabla k \\ &\quad + \int_{0}^{e} {}_S X^{\Theta,S}_{r,\alpha,\alpha}(e-r-k)\,\urcorner(k)\nabla k, \end{aligned}$$

where

$${}_C X^{\Theta,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e\in\mathbb{N}^{-r-1}, \\ \sum\limits_{j=0}^{p}(-1)^j S^j \dfrac{(e-(j-1)r)^{\overline{2j\alpha+\beta-1}}}{\Gamma(2j\alpha+\beta)}, & e\in\mathbb{N}^{pr}_{(p-1)r}, \end{cases}$$

and

$${}_S X^{\Theta,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e\in\mathbb{N}^{-r-1}, \\ \sum\limits_{j=0}^{p}(-1)^j S^j \dfrac{(e-(j-1)r)^{\overline{(2j+1)\alpha+\beta-1}}}{\Gamma((2j+1)\alpha+\beta)}, & e\in\mathbb{N}^{pr}_{(p-1)r}. \end{cases}$$

Proof. The proof is an immediate result from combining Theorem 20 with Remark 11. ☐

Remark 25. *Proposition 24 is novel for the nabla RL fractional delayed difference system (1) with $T=\Theta$.*

Example 26. *Let us reconsider the nabla fractional delayed difference system (1) in the case of the commutativity of T and S. In this context, Theorem 22 may be reformulated as follows.*

Proposition 27. *The exact analytical general solution of nonhomogeneous version of system (1) has the following explicit form:*

$$\begin{aligned} n(e) &= {}_C X^{T,S}_{r,\alpha,\alpha}(e)\,\phi(-r) + {}_S X^{T,S}_{r,\alpha,\alpha}(e)\left({}^R\nabla^{\alpha}_{-r}\phi\right)(-r) \\ &\quad + \int_{-r}^{0} {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\left[\left({}^R\nabla^{2\alpha}_{-r}\phi\right)(k) + T\phi(k)\right]\nabla k \\ &\quad + \int_{0}^{e} {}_S X^{T,S}_{r,\alpha,\alpha}(e-r-k)\,\urcorner(k)\nabla k, \end{aligned}$$

where

$${}_C X^{T,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e\in\mathbb{N}^{-r-1}, \\ \sum\limits_{i=0}^{\infty}(-1)^i T^i \dfrac{(e+r)^{\overline{2i\alpha+\beta-1}}}{\Gamma(2i\alpha+\beta)} + \sum\limits_{i=1}^{\infty}(-1)^i \binom{i}{1} T^{i-1}S^1 \dfrac{e^{\overline{2i\alpha+\beta-1}}}{\Gamma(2i\alpha+\beta)} & \\ +\ldots+\sum\limits_{i=p}^{\infty}(-1)^i\binom{i}{p}T^{i-p}S^p \dfrac{(e-(p-1)r)^{\overline{2i\alpha+\beta-1}}}{\Gamma(2i\alpha+\beta)}, & e\in\mathbb{N}^{pr}_{(p-1)r}, \end{cases}$$

and

$$ {}_{S}X^{T,S}_{r,\alpha,\beta}(e) := \begin{cases} \Theta, & e \in \mathbb{N}^{-r-1}, \\ \sum\limits_{i=0}^{\infty}(-1)^i T^i \dfrac{(e+r)^{\overline{(2i+1)\alpha+\beta-1}}}{\Gamma((2i+1)\alpha+\beta)} + \sum\limits_{i=1}^{\infty}(-1)^i\binom{i}{1}T^{i-1}S^1\dfrac{(e+r)^{\overline{(2i+1)\alpha+\beta-1}}}{\Gamma((2i+1)\alpha+\beta)} & \\ +\ldots+\sum\limits_{i=p}^{\infty}(-1)^i\binom{i}{p}T^{i-p}S^p\dfrac{(e-(p-1)r)^{\overline{(2i+1)\alpha+\beta-1}}}{\Gamma((2i+1)\alpha+\beta)}, & e \in \mathbb{N}^{pr}_{(p-1)r}, \end{cases} $$

which can be appropriately called fractional discrete delayed cosine and sine matrix functions, respectively.

Proof. The proof is an immediate result from combining Theorem 22 with Lemma 16. □

Remark 28. *Proposition 24 is new for the nabla RL fractional delayed difference system (1) with the commutativity of T and S.*

6. Practical examples with simulations

In this section, we present a few examples to practically illustrate our theoretical findings. **We plan to draw all of the following figures with the help of the Algorithm 1.**

Example 29. *One considers the Riemann-Liouville fractional retarded difference system with the noncommutative coefficient matrices,*

$$ (13) \quad \begin{cases} \left({}^{R}\nabla^{1.6}_{-2}n\right)(e) + \begin{pmatrix} 0.01 & 0.002 \\ 0.03 & 0.04 \end{pmatrix} n(e) + \begin{pmatrix} 0.1 & 0 \\ 0.1 & 1 \end{pmatrix} n(e-2) = \begin{pmatrix} e^2 \\ e \end{pmatrix}, \\ n(e) = \phi(e),\ \left({}^{R}\nabla^{0.8}_{-2}n\right)(e) = \left({}^{R}\nabla^{0.8}_{-2}\phi\right)(e), \quad e \in \mathbb{N}^{0}_{-2+1}. \end{cases} $$

In the light of Theorem 22, every one can acquire the general solution in the compact form is given as follows

$$ \begin{aligned} n(e) &= {}_{C}X^{T,S}_{2,0.8,1}(e)\,\phi(-2) + {}_{S}X^{T,S}_{2,0.8,1}(e)\,{}^{R}\nabla^{0.8}_{-2}\phi(-2) \\ &+ \sum_{k=-1}^{e}\int_{-2}^{0} {}_{S}X^{T,S}_{2,0.8,0.8}(e-2-k)\,\mathscr{H}_{-2.6}(e,k)\begin{pmatrix} 2k^4+1 \\ k \end{pmatrix}\nabla k \\ &+ \int_{-2}^{0} {}_{S}X^{T,S}_{2,0.8,0.8}(e-2-k)\begin{pmatrix} 0.02k^4+0.002k+0.01 \\ 0.06k^4+0.04k+0.03 \end{pmatrix}\nabla k \\ &+ \int_{0}^{e} {}_{S}X^{T,S}_{2,0.8,0.8}(e-2-k)\begin{pmatrix} k^2 \\ k \end{pmatrix}\nabla k, \end{aligned} $$

where $T = \begin{pmatrix} 0.01 & 0.002 \\ 0.03 & 0.04 \end{pmatrix}$, $S = \begin{pmatrix} 0.1 & 0 \\ 0.1 & 1 \end{pmatrix}$, $\phi(e) = \begin{pmatrix} 2e^4+1 \\ e \end{pmatrix}$. *Under the initial conditions, the graphs of the components of the solution* $n(e) = \begin{pmatrix} n_1(e) \\ n_2(e) \end{pmatrix}$ *in Figure 1 and their values in Table 1.*

Now, we will support Propositions 24 and 27 with practical examples.

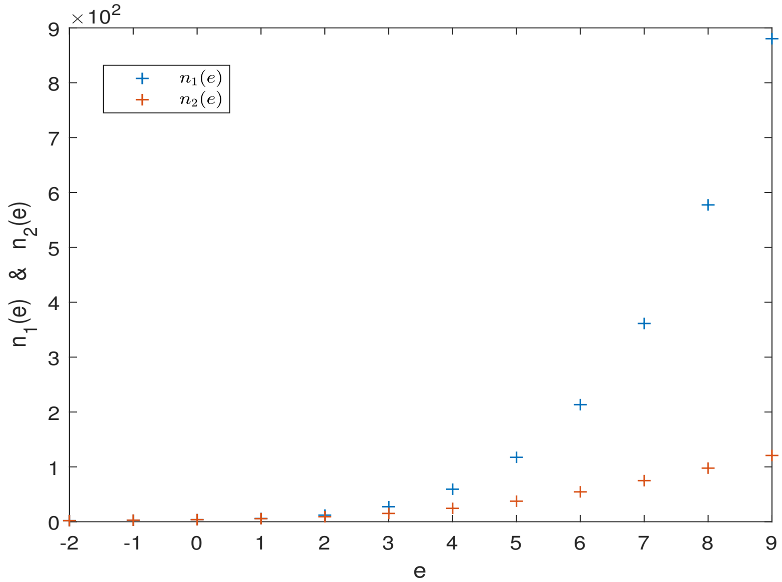

FIGURE 1. Graphs of the component functions $n_i(e),\ i=1,2$ of the solution $n(e)$ to system (13).

e	-2	-1	0	1	2	3
$n_1(e)$	1.99996	2.8499	3.846	5.89184	12.0154	27.3836
$n_2(e)$	1.9976	2.8394	3.8202	5.64006	9.13637	15.15
e	4	5	6	7	8	9
$n_1(e)$	59.3376	117.421	213.406	361.308	577.406	880.257
$n_2(e)$	24.4273	37.4881	54.4648	74.9269	97.7111	120.786

TABLE 1. $n_i(e),\ i=1,2$ for Figure 1.

Example 30. *One takes into consideration the linear Riemann-Liouville fractional retarded difference system (1) with* $T=\Theta$

$$\text{(14)}\qquad \begin{cases} \left({}^{R}\nabla_{-4}^{\frac{4}{5}}n\right)(e)+\frac{3}{100}n(e-4)=e^2, & e\in\mathbb{N}_1, \\ n(e)=\phi(e),\ \left({}^{R}\nabla_{-4}^{\frac{2}{5}}n\right)(e)=\left({}^{R}\nabla_{-4}^{\frac{2}{5}}\phi\right)(e),\ e\in\mathbb{N}_{-4+1}^{0}. \end{cases}$$

Based on Proposition 24, one can easily express the representation of the explicit general solution $n(e)$ in the compact form to system (14) as noted below,

$$
\begin{aligned}
n(e) = {}& {}_{C}X^{0,0.03}_{4,\frac{2}{5},1}(e)\,\phi(-4) + {}_{S}X^{0,0.03}_{4,\frac{2}{5},1}(e)\,{}^{R}\nabla^{\frac{2}{5}}_{-4}\phi(-4) \\
&+ \sum_{k=-3}^{e}\int_{-4}^{0} {}_{S}X^{0,0.03}_{4,\frac{2}{5},\frac{2}{5}}(e-4-k)\,\mathscr{H}_{-1.8}(e,k)(k+1)\nabla k \\
&+ \int_{0}^{e} {}_{S}X^{0,0.03}_{4,\frac{2}{5},\frac{2}{5}}(e-4-k)\,k^{2}\nabla k,
\end{aligned}
$$

where $\phi(e) = e+1$. Under the initial conditions, the graph of the solution $n(e)$ in Figure 2 and its values in Table 2.

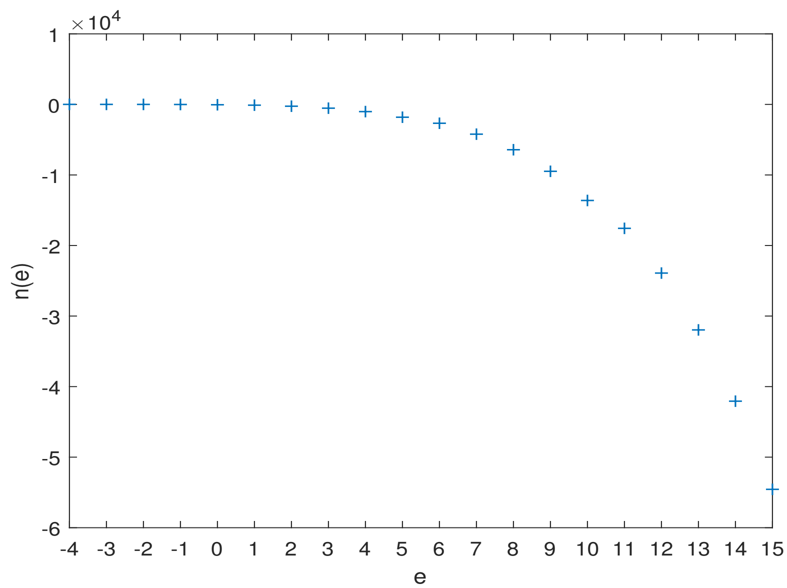

FIGURE 2. Graph of the solution $n(e)$ to system (14).

Example 31. *We will examine the Riemann-Liouville fractional retarded difference system (1) with the commutative coefficient matrices,*

$$
(15)\qquad \begin{cases} \left({}^{R}\nabla^{1.8}_{-3}n\right)(e) + 0.2n(e) + 0.5n(e-3) = 2e, & e \in \mathbb{N}_1, \\ n(e) = \phi(e),\ \left({}^{R}\nabla^{0.9}_{-3}n\right)(e) = \left({}^{R}\nabla^{0.9}_{-3}\phi\right)(e), & e \in \mathbb{N}^{0}_{-3+1}, \end{cases}
$$

e	-4	-3	-2	-1	0
$n(e)$	2	1.2	-0.801705	-12.2204	-50.9306
e	1	2	3	4	5
$n(e)$	-108.465	-255.315	-534.099	-1020.48	-1814.97
e	6	7	8	9	10
$n(e)$	-2670.93	-4218.69	-6425.41	-9482.94	-13616.5
e	11	12	13	14	15
$n(e)$	-17567	-23907	-31965.1	-42063.8	-54560.7

TABLE 2. $n(e)$ for Figure 2.

where $\phi(e) = 3e^3$. Based on Proposition 27, one can easily express the representation of the explicit general solution $n(e)$ in the compact form to system (14) as noted below,

$$\begin{aligned} n(e) &= {}_C X^{0.2,0.5}_{3,0.9,1}(e)\,\phi(-3) + {}_S X^{0.2,0.5}_{3,0.9,1}(e)\,{}^R\nabla^{0.9}_{-3}\phi(-3) \\ &+ 3\sum_{k=-2}^{e}\int_{-3}^{0} {}_S X^{0.2,0.5}_{3,0.9,0.9}(e-3-k)\,\mathscr{H}_{-2.8}(e,k)k^3\nabla k \\ &+ 0.6\int_{-3}^{0} {}_S X^{0.2,0.5}_{3,0.9,0.9}(e-3-k)\,k^3\nabla k \\ &+ 2\int_{0}^{e} {}_S X^{0.2,0.5}_{3,0.9,0.9}(e-3-k)\,k\nabla k. \end{aligned}$$

Under the initial conditions, the graph of the solution $n(e)$ in Figure 3 and its values in Table 2.

e	-3	-2	-1	0	1	
$n(e)$	1.66656	-2.19264	-5.49961	-7.14625	-6.01871	
e	2	3	4	5	6	
$n(e)$	0.401262	13.327	32.1537	55.1421	81.5232	
e	7	8	9	10	11	12
$n(e)$	117.946	192.963	385.625	878.788	2054.36	4657.28

TABLE 3. $n(e)$ for Figure 3.

7. Conclusion

We newly define the fractional discrete retarded perturbations of matrix cosine and sine functions and debate about a couple of their ties in order to obtain the representation of an explicit solution to the semilinear Riemann-Liouville fractional retarded difference system.

We would like to remark that it is clear that this fundamental work will become a guiding light for many researchers who are interested in the subject. As open problems, one can investigate

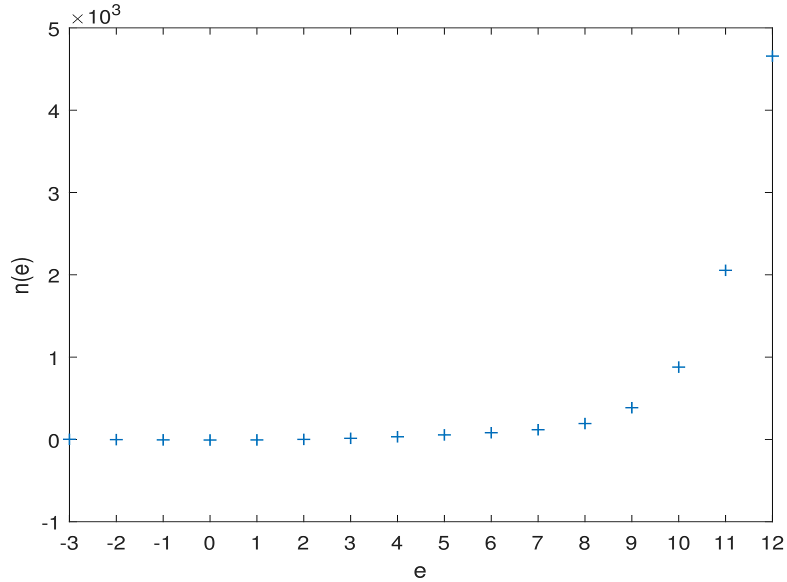

FIGURE 3. Graph of the solution $n(e)$ to system (15).

whether the nabla fractional retarded linear difference system presented in (1) is stable or not in the sense of exponential[47][48], finite-time[52] , asymptotic[49][50][51], and also Lyapunov and is iteratively[45][46] or relatively[41] [42][43][44] controllable or not as in the just cited papers which are examples of discrete or continuous cases.

In addition to the fact that one can consider neutral version[11] and multi-delayed version[53][54] of our problem, one can replace the Caputo fractional difference with the RL fractional one, which causes the transformation of our problem into more challenging problem so that the initial circumstances need to be shifted and the corresponding discrete retarded sine and cosine matrix functions should be redefined again. And then, one can reinvestigate if these newer systems are stable or controllable, etc in the above-mentioned senses.

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Algorithm 1

Require: Problem parameters α, r, T, S, ϕ, $\daleth$.
Ensure: Graph of the solution.
  BEGIN
    Initialize $Q(-1,j)$ and $Q(i,-1)$ as Θ.
    Initialize $Q(0,0)$ as I.
    Construct the matrix $Q(i+1,j) = -TQ(i,j) - SQ(i,j-1)$.
    Define the functions ${}_C X^{T,S}_{r,\alpha,\beta}(e)$ and ${}_S X^{T,S}_{r,\alpha,\beta}(e)$.
    Compute ${}_C X^{T,S}_{r,\alpha,1}(e)\,\phi(-r)$ and ${}_S X^{T,S}_{r,\alpha,1}(e)\left({}^R\nabla^{\alpha}_{-r}\phi\right)(-r)$.
    Define a variable f_1 and set it to 0.
    for $s = -r+1 \to 0$ **do**
      for $k = -r+1 \to s$ **do**
        Add f_1 to ${}_S X^{T,S}_{r,\alpha,\alpha}(e-r-s)\left[T\phi(s) + \dfrac{\Gamma(s-k-2\alpha)}{\Gamma(s-k+1)\Gamma(-2\alpha)}\phi(k)\right]$
      end for
    end for
    Define a variable f_2 and set it to 0.
    for $s = 1 \to e$ **do**
      Add f_2 to ${}_S X^{T,S}_{r,\alpha,\alpha}(e-r-s)\daleth(s)$
    end for
    Compute the solution $n(e) = {}_C X^{T,S}_{r,\alpha,1}(e)\,\phi(-r) + {}_S X^{T,S}_{r,\alpha,1}(e)\left({}^R\nabla^{\alpha}_{-r}\phi\right)(-r) + f_1 + f_2$.
    Plot the solution $n(e)$.
  END

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