Note on the Polynomials $L_n^{\alpha,\beta}(x)$

Vinod Kumar Jatav[a,b], A.K. Shukla[b]
vinodkumarjatav@vitbhopal.ac.in, kumar17vinod@gmail.com, ajayshukla2@rediffmail.com, aks@amhd.svnit.ac.in
[a]Division of Mathematics, School of Advanced Sciences & Languages,
VIT Bhopal University, Kothrikalan, Sehore-466 114, Madhya Pradesh, India.
[b]Department of Mathematics, Sardar Vallabhbhai National Institute of Technology,
Surat-395 007, Gujarat, India.

Abstract

In this paper, we discuss some properties for the polynomials $L_n^{\alpha,\beta}(x)$ including integral representations, orthogonal condition, Rodrigues formula, recurrence relation and fractional differential equation.

Keywords: Integral representation, Rodrigues formula, Recurrence function, Fractional differential equation, Orthogonality.

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1 Introduction and Preliminaries

Prabhakar and Suman [1] introduced the polynomials $L_n^{\alpha,\beta}(x)$ and defined as,

$$L_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{\Gamma(\alpha k+\beta+1)k!}, \tag{1.1}$$

where $(\cdot)_k$ denotes the Pochhammer symbol and $n \in \mathbf{N}\cup\{\mathbf{0}\}$, $\alpha,\beta \in \mathbf{C}$, $\Re(\alpha)>0, \Re(\beta)>-1$. For $\alpha=1$, this reduces to the generalized Laguerre polynomial (Rainville [2]),

$$L_n^{\beta}(x) = \sum_{k=0}^{n}\frac{(-1)^k(1+\beta)_n x^k}{k!(n-k)!(1+\beta)_k}. \tag{1.2}$$

Jatav and Shukla [3] represented (1.1) as,

$$L_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha n+\beta+1)}{n!}\,{}_1R_0\left[\begin{matrix}-n\\ -\end{matrix}\middle|\,\alpha,\beta+1;x\right]. \tag{1.3}$$

and also obtained the following generating function of (1.1) as,

$$\sum_{n=0}^{\infty}\frac{(\beta+1)_n}{\Gamma(\alpha n+\beta+1)}L_n^{\alpha,\beta}(x)t^n = (1-t)^{-(\beta+1)}\;{}_1R_0\left[\begin{matrix}\beta+1\\ -\end{matrix}\middle|\,\alpha,\beta+1;\frac{xt}{t-1}\right]. \tag{1.4}$$

Desai and Shukla [4, 5] defined the ${}_pR_q(\tau_1,\tau_2;z)$ function as,

$$\begin{aligned}{}_pR_q(\tau_1,\tau_2;z) &= {}_pR_q\left[\begin{matrix}\xi_1,\xi_2,\ldots,\xi_p\\ \zeta_1,\zeta_2,\ldots,\zeta_q\end{matrix}\middle|\,\tau_1,\tau_2;z\right]\\ &= \sum_{m=0}^{\infty}\frac{1}{\Gamma(\tau_1 m+\tau_2)}\frac{(\xi_1)_k\ldots(\xi_p)_m}{(\zeta_1)_k\ldots(\zeta_q)_m}\frac{z^m}{m!},\end{aligned} \tag{1.5}$$

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where $p, q \in \mathbf{N} \cup \{\mathbf{0}\}$ and $\tau_1, \tau_2 \in \mathbf{C}$, $\Re(\tau_1), \Re(\tau_2), \Re(\xi_i), \Re(\zeta_j) > 0$; for any $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$. When no ζ_j $(j = 1, 2, \cdots, q)$ is zero or a negative integer, the series (1.5) is defined. The series (1.5) terminates to polynomial in z if any numerator parameter ξ_i $(i = 1, 2, \ldots, p)$ is a zero or negative integer.

The series (1.5) having the following convergence conditions:

(i). If $\Re(\tau_1) \geq p - q$, the series converges for all finite values of z.

(ii). If $\Re(\tau_1) = p - q - 1$, the series converges for all $|z| < 1$ and diverges for $|z| > 1$.

(iii). When $\Re(\tau_1) = p - q - 1$ and $|z| = 1$, the series can converges on condition depending on the parameters. If $\Re(\tau_1) = p - q - 1$, the series is absolutely convergent on the circle $|z| = 1$ if

$$\Re\left(\tau_1 + \sum_{j=1}^{q} \zeta_j - \sum_{i=1}^{p} \xi_i\right) > 0.$$

The confluent hypergeometric function is defined (Rainville [2]) as,

$$ {}_1F_1(\xi; \zeta; z) = \sum_{m=0}^{\infty} \frac{(\xi)_n}{(\zeta)_n} \frac{z^m}{m!}, \tag{1.6}$$

in which $\zeta \neq 0$ or a negative integer and the series is convergent for all finite z.
The Pochhammer symbol is denoted by $(\wp)_m$ and defined (Rainville [2]) for $\wp \in \mathbf{C}$ as,

$$(\wp)_m = \frac{\Gamma(\wp + m)}{\Gamma(\wp)} = \begin{cases} \wp(\wp+1)(\wp+2)\ldots(\wp+m-1) & (m \in \mathbf{N}) \\ 1 & (m = 0,\ \wp \neq 0). \end{cases} \tag{1.7}$$

The Beta function is defined (Rainville [2]) as,

$$\mathbf{B}(\sigma, \upsilon) = \frac{\Gamma(\sigma)\Gamma(\upsilon)}{\Gamma(\sigma + \upsilon)} = \int_0^1 \omega^{\sigma-1}(1-\omega)^{\upsilon-1} d\omega, \quad (\Re(\sigma) > 0, \Re(\upsilon) > 0). \tag{1.8}$$

In this investigation, we need to recall the following result due to Lavoie and Trottier [6],

$$\int_0^1 \vartheta^{\chi-1}(1-\vartheta)^{2\eta-1}\left(1 - \frac{\vartheta}{3}\right)^{2\chi-1}\left(1 - \frac{\vartheta}{4}\right)^{\eta-1} d\vartheta = \left(\frac{2}{3}\right)^{2\chi} \frac{\Gamma(\chi)\Gamma(\eta)}{\Gamma(\chi + \eta)}, \tag{1.9}$$

where $\chi, \eta \in \mathbf{C}$ with $\Re(\chi) > 0$ and $\Re(\eta) > 0$.

2 Main Results

In this section, integral representations, Rodrigues formula, recurrence relation, fractional differential equation and orthogonal condition for the polynomials (1.1) are established.

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2.1 Integral Representations

Theorem 2.1. *Let $\alpha, \beta \in \mathbf{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > -1$ and $\vartheta > 0$. Then the following integral representation formula holds true:*

$$\int_0^1 \vartheta^{\alpha-1}(1-\vartheta)^{2\beta+1}\left(1-\frac{\vartheta}{3}\right)^{2\alpha-1}\left(1-\frac{\vartheta}{4}\right)^{\beta} L_n^{\alpha,\beta}\left(u(1-\vartheta)^{2\alpha}\left(1-\frac{\vartheta}{4}\right)^{\alpha}\right)d\vartheta$$
$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha n+\beta+1)}{n!}\,{}_1R_0\left[\begin{array}{c}-n\\ -\end{array}\middle|\,\alpha,\alpha+\beta+1;u\right]. \tag{2.1}$$

Proof. The left-hand side of (2.1) is denoted by Ξ_1 and using (1.1), further inverting the order of summation and integration, we get

$$\Xi_1 = \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k u^k}{k!\Gamma(\alpha k+\beta+1)}$$
$$\times\left(\int_0^1 \vartheta^{\alpha-1}(1-\vartheta)^{2\alpha k+2\beta+1}\left(1-\frac{\vartheta}{3}\right)^{2\alpha-1}\left(1-\frac{\vartheta}{4}\right)^{\alpha k+\beta}d\vartheta\right),$$

on employing the integral formula (1.9), we obtain the following expression:

$$\Xi_1 = \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k u^k}{k!\Gamma(\alpha k+\alpha+\beta+1)},$$

on solving the above result by using (1.5), this leads to the desired result (2.1). This completes the proof of Theorem 2.1. □

On putting $\alpha = 1$ in (2.1), in view of (1.6), one can easily obtain the following corollary.

Corollary 2.1. *Let $\beta \in \mathbf{C}$ with $\Re(\beta) > -1$ and $\vartheta > 0$. Then the following integral representation formula holds true:*

$$\int_0^1 (1-\vartheta)^{2\beta+1}\left(1-\frac{\vartheta}{3}\right)\left(1-\frac{\vartheta}{4}\right)^{\beta} L_n^{\beta}\left[u(1-\vartheta)^2\left(1-\frac{\vartheta}{4}\right)\right]d\vartheta$$
$$=\left(\frac{2}{3}\right)^{2}\frac{\Gamma(\beta+n+1)}{n!\Gamma(\beta+2)}\,{}_1F_1\left[\begin{array}{c}-n;\\ \beta+2;\end{array}\; u\right]. \tag{2.2}$$

Theorem 2.2. *Let $\alpha, \beta, \varpi \in \mathbf{C}$ such that $\Re(\alpha) > 0$, $\Re(\varpi) > 0$ and $\Re(\beta) > -1$. Then the following integral representation formula holds true:*

$$L_n^{\alpha,\beta+\varpi}(x) = \frac{\Gamma(\alpha n+\beta+\varpi+1)}{n!\Gamma(\varpi)}\int_0^1 \mu^{\beta}(1-\mu)^{\varpi-1}\,{}_1R_0\left[\begin{array}{c}-n\\ -\end{array}\middle|\,\alpha,\beta+1;x\mu^{\alpha}\right]d\mu. \tag{2.3}$$

Proof. The left-hand side of (2.3) is denoted by Ξ_2, using (1.5) and inverting the order of summation and integration, we get

$$\Xi_2 = \frac{\Gamma(\alpha n+\beta+\varpi+1)}{n!\Gamma(\varpi)}\sum_{k=0}^{n}\frac{(-n)_k x^k}{k!\Gamma(\alpha k+\beta+1)}\left(\int_0^1 \mu^{\alpha k+\beta}(1-\mu)^{\varpi-1}d\mu\right)$$
$$= \frac{\Gamma(\alpha n+\beta+\varpi+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{k!\Gamma(\alpha k+\beta+\varpi+1)},$$

on further simplification the above result by using (1.1), this yields the desired result. □

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On putting $\alpha = 1$ in (2.3), in view of (1.6), we obtain the following corollary.

Corollary 2.2. *Let $\beta, \varpi \in \mathbf{C}$ such that $\Re(\varpi) > 0$ and $\Re(\beta) > -1$. Then the following integral representation formula holds true:*

$$L_n^{\beta+\varpi}(x) = \frac{\Gamma(\beta+\varpi+n+1)}{n!\Gamma(\varpi)\Gamma(\beta+1)} \int_0^1 \mu^{\beta}(1-\mu)^{\varpi-1} {}_1F_1\left[\begin{array}{c} -n; \\ \beta+1; \end{array} \mu x\right] d\mu. \tag{2.4}$$

2.2 Rodrigues Formula

Theorem 2.3. *Let $\alpha, \beta \in \mathbf{C}$ with $\Re(\alpha) > 0$ and $\Re(\beta) > -1$. Then the following Rodrigues formula holds true:*

$$L_n^{\alpha,\beta}(x) = \frac{x^{-\beta}\Gamma(\alpha n+\beta+1)}{n!(\beta+1)_n} \frac{d^n}{dz^n}\left[z^{\beta+n} {}_1R_0\left[\begin{array}{c} \beta+1 \\ - \end{array}\middle| \alpha, \beta+1; x-z\right]\right]_{z=x}. \tag{2.5}$$

Proof. From (1.4) and employing the Taylor's theorem [7], we get

$$\frac{(\beta+1)_n}{\Gamma(\alpha n+\beta+1)} L_n^{\alpha,\beta}(x) = \frac{1}{2\pi i}\int_C (1-t)^{-(\beta+1)} {}_1R_0\left[\begin{array}{c} \beta+1 \\ - \end{array}\middle| \alpha, \beta+1; \frac{xt}{t-1}\right] t^{-n-1} dt,$$

where C being a closed contour surrounding at $t = 0$ and lying within the disk $|t| < 1$. On setting $z = \frac{x}{1-t}$, we get

$$L_n^{\alpha,\beta}(x) = \frac{x^{-\beta}\Gamma(\alpha n+\beta+1)}{(\beta+1)_n\, 2\pi i}\int_{C'} \frac{z^{\beta+n}}{(z-x)^{n+1}} {}_1R_0\left[\begin{array}{c} \beta+1 \\ - \end{array}\middle| \alpha, \beta+1; x-z\right] dz,$$

where C' is a circle $|z - x| = r$ of small radius r. Now by Cauchy's integral theorem [7], this yields the desired result (2.5). □

Remark 2.1. *For $\alpha = 1$ in (2.5), this reduces to the Rodrigues formula for the generalized Laguerre polynomial [2],*

$$L_n^{\beta}(x) = \frac{x^{-\beta}e^x}{n!}\frac{d^n}{dx^n}(x^{\beta+n}e^{-x}). \tag{2.6}$$

2.3 Recurrence Relation

Theorem 2.4. *Let $\alpha, \beta \in \mathbf{C}$ with $\Re(\alpha) > 0$ and $\Re(\beta) > -1$. Then the following differential recurrence relation holds true:*

$$xDL_n^{\alpha,\beta}(x^{\alpha}) = \alpha n L_n^{\alpha,\beta}(x^{\alpha}) - \frac{\alpha\Gamma(\alpha n+\beta+1)}{\Gamma(\alpha n-\alpha+\beta+1)} L_{n-1}^{\alpha,\beta}(x^{\alpha}). \tag{2.7}$$

Proof. From the right hand side of (2.7), we get

$$\begin{aligned}
&\alpha n L_n^{\alpha,\beta}(x^{\alpha}) - \frac{\alpha\Gamma(\alpha n+\beta+1)}{\Gamma(\alpha n-\alpha+\beta+1)} L_{n-1}^{\alpha,\beta}(x^{\alpha}) \\
&= \frac{\alpha\Gamma(\alpha n+\beta+1)}{(n-1)!}\sum_{k=0}^{n}(-1)^k\left[\binom{n}{k} - \binom{n-1}{k}\right]\frac{x^{\alpha k}}{\Gamma(\alpha k+\beta+1)} \\
&= \frac{x\Gamma(\alpha n+\beta+1)}{(n)!}\sum_{k=0}^{n}\frac{(-n)_k(\alpha k)x^{\alpha k-1}}{k!\Gamma(\alpha k+\beta+1)} \\
&= xDL_n^{\alpha,\beta}(x^{\alpha}).
\end{aligned}$$

This completes the proof of the result (2.7). □

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Remark 2.2. *For $\alpha = 1$ in (2.7), this reduces to the differential recurrence relation for generalized Laguerre polynomials [2],*

$$xDL_n^{\beta}(x) = nL_n^{\beta}(x) - (\beta + n)L_{n-1}^{\beta}(x). \tag{2.8}$$

2.4 Fractional Differential Equation

Theorem 2.5. *Let $\alpha, \beta \in \mathbf{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > -1$. Then the following fractional differential equation holds true:*

$$D^{\alpha}\left[x^{\beta+1}DL_n^{\alpha,\beta}(x^{\alpha})\right] - x^{\beta+1}DL_n^{\alpha,\beta}(x^{\alpha}) + n\alpha x^{\beta}L_n^{\alpha,\beta}(x^{\alpha}) = 0. \tag{2.9}$$

Proof. From (2.7), one can write

$$xDL_n^{\alpha,\beta}(x^{\alpha}) = \frac{\alpha\Gamma(\alpha n + \beta + 1)}{(n-1)!}\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)\frac{x^{\alpha k}}{\Gamma(\alpha k + \beta + 1)}. \tag{2.10}$$

Now on multiplying both side by x^{α} and then apply the operator D^{α}, where $D = \frac{d}{dx}$, this gets

$$\begin{aligned} D^{\alpha}\left[x^{\beta+1}DL_n^{\alpha,\beta}(x^{\alpha})\right] &= \frac{\alpha\Gamma(\alpha n + \beta + 1)}{(n-1)!}\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)\frac{x^{\alpha(k-1)+\beta}}{\Gamma(\alpha k - \alpha + \beta + 1)} \\ &= -\frac{\alpha\Gamma(\alpha n + \beta + 1)x^{\beta}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k \left(\begin{array}{c} n-1 \\ k \end{array}\right)\frac{x^{\alpha k}}{\Gamma(\alpha k + \beta + 1)} \\ &= -\frac{\alpha\Gamma(\alpha n + \beta + 1)x^{\beta}}{\Gamma(\alpha n - \alpha + \beta + 1)}L_{n-1}^{\alpha,\beta}(x^{\alpha}). \end{aligned} \tag{2.11}$$

From (2.7) and (2.11), further simplification by removing $L_{n-1}^{\alpha,\beta}(x^{\alpha})$, this yields the fractional differential equation of order $\alpha + 1$ (2.9) for polynomials (1.1). □

Remark 2.3. *For $\alpha = 1$ in (2.9), this reduces to the differential equation for the generalized Laguerre polynomial [2].*

$$xD^2L_n^{\beta}(x) + (1 + \beta - x)DL_n^{\beta}(x) + nL_n^{\beta}(x) = 0. \tag{2.12}$$

2.5 Orthogonality

Theorem 2.6. *If the polynomials $L_n^{\alpha,\beta}(x)$ form a simple set of real polynomials and weight function $x^{\beta}e^{-x} > 0$ over interval $0 < x < \infty$, then necessary and sufficient condition that the set $L_n^{\alpha,\beta}(x)$ to be orthogonal with respect to weight function $x^{\beta}e^{-x}$ over the interval $0 < x < \infty$ is that*

$$\int_0^{\infty} x^{\beta}e^{-x}L_n^{\alpha,\beta}(x^{\alpha})x^m dx = \frac{\Gamma(\alpha n + \beta + 1)}{n!}\delta'_{mn}, \tag{2.13}$$

where $\alpha, \beta \in \mathbf{C}$, $\Re(\alpha) > 0, \Re(\beta) > -1$ and $\delta'_{mn} = \left\{\begin{array}{ll} 0, & \textit{if } m = 0,1,2,...,n-1 \\ \neq 0, & \textit{if } m = n. \end{array}\right.$

Proof. From the left-hand side of (2.13), we obtain

$$\int_0^{\infty} x^{\beta}e^{-x}L_n^{\alpha,\beta}(x^{\alpha})x^m dx = \frac{\Gamma(\alpha n + \beta + 1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{\Gamma(\alpha k + \beta + 1)k!}\left(\int_0^{\infty} e^{-x}x^{\alpha k + \beta + m}dx\right)$$

$$= \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\Gamma(\alpha k+\beta+m+1)}{\Gamma(\alpha k+\beta+1)k!}$$
$$= \frac{\Gamma(\alpha n+\beta+1)}{n!}\left[\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}D^m(y^{\alpha k+\beta+m})\right]_{y=1}, \quad \textit{where } D=\frac{d}{dy}.$$

In which $y^{\alpha k+\beta+m}$ has been inserted for convenience and will be removed later by replacing $y=1$ [2].

$$\int_0^\infty x^\beta e^{-x}L_n^{\alpha,\beta}(x^\alpha)x^m dx = \frac{\Gamma(\alpha n+\beta+1)}{n!}\left[D^m(y^{\beta+m})\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}y^{\alpha k}\right]_{y=1}$$
$$= \frac{\Gamma(\alpha n+\beta+1)}{n!}\left[D^m(y^{\beta+m})(1-y^\alpha)^n\right]_{y=1}$$
$$= \frac{\Gamma(\alpha n+\beta+1)}{n!}\delta'_{mn},$$
$$\textit{where} \quad \delta'_{mn} = \begin{cases} 0, & \textit{if } m=0,1,2,...,n-1 \\ \neq 0, & \textit{if } m=n. \end{cases}$$

Which is zero for $m=0,1,2,\cdots n-1$ and non zero for $m=n$. We consider $L_n^{\alpha,\beta}(x^\alpha)$ to observe the othoganality for different values of $\alpha,(0<\alpha<1)$. For $\alpha=1$, (2.13) reduces to the orthogonal condition for the generalized Laguerre polynomial [2]. □

Concluding Remark

In this paper, we established integral representations, Rodrigues formula, recurrence relation and fractional differential equation. We used the technique discussed by Rainville [2] to obtained orthogonal condition for polynomials $L_n^{\alpha,\beta}(x)$. These established results may play significant role in mathematical physics, classical analysis, quantum mechanics, fractional calculus and engineering mathematics.

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