ON THE INDEX DIVISORS AND MONOGENITY OF CERTAIN NONIC NUMBER FIELDS

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Abstract. In this paper, for any nonic number field K generated by a root α of a monic irreducible trinomial $F(x) = x^9 + ax + b \in \mathbb{Z}[x]$ and for every rational prime p, we characterize when p divides the index of K. We also describe the prime power decomposition of the index $i(K)$. In such a way we give a partial answer of Problem 22 of Narkiewicz ([23]) for this family of number fields. In particular if $i(K) \neq 1$, then K is not mongenic. We illustrate our results by some computational examples.

1. Introduction

Let K be a number field of degree n and $\mathbb{Z}_K$ its ring of integers. For any primitive element $\alpha \in \mathbb{Z}_K$ of K, it is well known that $\mathbb{Z}[\alpha]$ is a free $\mathbb{Z}$-module of rank n, from which it follows that the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ is finite. A well known formula linking $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, $\Delta(\alpha)$, and d_K is given by:

$$\Delta(\alpha) = \pm(\mathbb{Z}_K : \mathbb{Z}[\alpha])^2 \times d_K, \tag{1.1}$$

where d_K is the absolute discriminant of K and $\Delta(\alpha)$ is the discriminant of the minimal polynomial of α over $\mathbb{Q}$. The index of K, denoted by $i(K)$, is the greatest common divisor of the indices of all integral primitive elements of K. Say $i(K) = \gcd\{(\mathbb{Z}_K : \mathbb{Z}[\theta]) \mid K = \mathbb{Q}(\theta) \text{ and } \theta \in \mathbb{Z}_K\}$. A rational prime p dividing $i(K)$ is called a prime common index divisor of K. If K is monogenic, then $\mathbb{Z}_K$ has a power integral basis, i.e., a $\mathbb{Z}$-basis of the form $(1, \theta, \dots, \theta^{n-1})$, and the index of K is trivial, say $i(K) = 1$. Therefore a field having a prime common index divisor is not monogenic. In 1930, Engstrom [10] was the first one who studied the prime power decomposition of the index of a number field. He showed that for number fields of degree $n \leq 7$, $\nu_p(i(K))$ is determined by the form of the factorization of $p\mathbb{Z}_K$, where ν_p is the p-adic valuation of $\mathbb{Q}$. It is an interesting problem to classify these number fields with non-trivial index, which are of course not monogenic. In [27], Śliwa showed that, if p is a non-ramified ideal in K, then $\nu_p(i(K))$ is determined by the factorization of $p\mathbb{Z}_K$. These results were generalized by Nart ([24]), who developed a p-adic characterization of the index of a number field. In [22], Nakahara studied the index of non-cyclic but abelian biquadratic number fields. In [12] Gaál et al. characterized the field indices of biquadratic number fields having Galois group V_4. In [2], for any quartic number field K defined by a trinomial $x^4 + ax + b$, Davis and Spearman characterized when

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$p = 2, 3$ divides $i(K)$. In [5], for any quartic number field K defined by a trinomial $x^4 + ax^2 + b$, El Fadil and Gaál gave necessary and sufficient conditions on a and b, which characterize when a rational prime p divides $i(K)$. In [4], for any rational prime p, El Fadil characterized when p divides the index $i(K)$ for any quintic number field K defined by a trinomial $x^5 + ax^2 + b$. In [7], for every rational prime p, El Fadil and Kchit characterized $\nu_p(i(K))$ for any septic number field defined by a trinomial $x^7 + ax^3 + b$. In [8], they studied the index divisors and monogenity of the number fields defined by the trinomials $x^{12} + ax^m + b$. In ([17], [19]), the authors studied the integral closedness of $\mathbb{Z}[\alpha]$ in a number field K defined by general type of trinomials: $x^n + ax^m + b$. Their results give a partial answer to the problem of monogenity of K, but does not characterize when K is not monogenic. In [18], Jakhar gave sufficient conditions on a, b, m, q, s so that a rational prime p divides the index of the number field defined by the trinomial $x^{q^s} - ax^m - b$. He determined some cases when the field K is not monogenic, but its results does not characterize all the prime divisors of the index of these number fields. In this paper, for any nonic number field K defined by a monic irreducible trinomial $x^9 + ax + b \in \mathbb{Z}[x]$ and for every rational prime p, we characterize when p divides the index $i(K)$. Based on Engstrom's results given in [10], we evaluate $\nu_p(i(K))$ in some cases.

2. Main Results

Throughout this section, K is a number field generated by a complex root α of a monic irreducible trinomial $F(x) = x^9 + ax + b \in \mathbb{Z}[x]$. Without loss of generality, we assume that for every rational prime p, $\nu_p(a) \leq 7$ or $\nu_p(b) \leq 8$.
We start with the following theorem, which characterizes when the ring $\mathbb{Z}[\alpha]$ is integrally closed.

Theorem 2.1. *The ring $\mathbb{Z}[\alpha]$ is integrally closed if and only if the following conditions hold:*

1. *If p divides both a and b, then $\nu_p(b) = 1$.*
2. *If 2 does not divide a and divides b, then $(a, b) \in \{(1, 0), (3, 2)\} \pmod 4$.*
3. *If 3 divides a and does not divide b, then*

$$(a, b) \in \{(0, 2), (0, 5), (3, -1), (3, 2)(6, -1), (6, 5), (0, 4), (0, 7), (3, 1), (3, 7), (6, 1), (6, 4)\} \pmod 9.$$

4. *For every rational prime $p \notin \{2, 3\}$ dividing $2^{24}a^9 + 3^{18}b^8$, if $\nu_p(ab) = 0$, then $\nu_p(2^{24}a^9 + 3^{18}b^8) = 1$.*

If all these conditions hold, then K is monogenic and $i(K) = 1$.

In the next theorems for every rational prime p, we characterize when p divides the index $i(K)$ and evaluate $\nu_p(i(K))$ in some cases.
Let us denote by Δ the discriminant of $F(x)$ and for every rational prime p, let $\Delta_p = \dfrac{\Delta}{p^{\nu_p(\Delta)}}$.

The following theorem characterizes when 2 divides $i(K)$.

Theorem 2.2. *The rational prime* 2 *divides the index* $i(K)$ *if and only if one of the following conditions is satisfied:*

1. $(a, b) \equiv (1, 2) \pmod 4$.
2. $(a, b) \equiv (3, 4) \pmod 8$.
3. $(a, b) \in \{(15, 0), (7, 8)\} \pmod{16}$.
4. $(a, b) \equiv (28, 0) \pmod{32}$.
5. $(a, b) \in \{(4, 0), (52, 32)\} \pmod{64}$.
6. $a \equiv 112 \pmod{128}$ *and* $b \equiv 128 \pmod{256}$.
7. $(a, b) \in \{(368, 256), (112, 256), (240, 0), (496, 0), (448, 0)\} \pmod{512}$.
8. $a \equiv 240 \pmod{256}$ *and* $b \equiv 256 \pmod{512}$.
9. $(a, b) \equiv (64, 0) \pmod{1024}$.

In particular, if one of the above conditions holds, then K is not monogenic.

Remark 1. Based on Engstrom's results given in [10], the following table provides $\nu_2(i(K))$ for some cases of Theorem 2.2:

TABLE 1. $\nu_2(i(K))$

Conditions	$\nu_2(i(K))$
$(a, b) \equiv (1, 2) \pmod 4$	1
$(a, b) \equiv (7, 8) \pmod{16}$ and $\nu_2(\Delta)$ is odd	3
$(a, b) \equiv (7, 8) \pmod{16}$, $\nu_2(\Delta) = 28$, and $\Delta_2 \equiv 3 \pmod 4$	
$(a, b) \equiv (7, 8) \pmod{16}$, $\nu_2(\Delta) = 2k \geq 30$, and $\Delta_2 \equiv 1 \pmod 4$	
$(a, b) \equiv (368, 256) \pmod{512}$	1
$a \equiv 240 \pmod{256}$ and $b \equiv 256 \pmod{512}$	3

The following theorem characterizes when 3 divides $i(K)$.

Theorem 2.3. *The following table provides* $\nu_3(i(K))$*:*

TABLE 2. $\nu_3(i(K))$

Conditions		$\nu_3(i(K))$
$(a, b) \in \{(18, 62), (72, 8)\} \pmod{81}$ *and* $a + b \equiv -1 \pmod{243}$	$\nu_3(\Delta) = 2k$ *and* $\Delta_3 \equiv -1 \pmod 3$	1
$(a, b) \equiv (45, 35) \pmod{81}$ *and* $a + b \equiv 161 \pmod{243}$		
$(a, b) \equiv \{(18, 19), (72, 73)\} \pmod{81}$ *and* $b - a \equiv 1 \pmod{243}$		
$(a, b) \equiv (45, 46) \pmod{81}$ *and* $b - a \equiv 82 \pmod{243}$		
Otherwise		0

In particular, if $i(K) \neq 1$, *then K is not monogenic.*

Theorem 2.4. *For every rational prime* $p \geq 5$ *and for every* $(a, b) \in \mathbb{Z}^2$ *such that* $x^9 + ax + b$ *is irreducible, p does not divide the index* $i(K)$, *where K is the number field defined by* $x^9 + ax + b$.

3. Preliminaries

Our proofs are based on Newton polygon techniques applied on prime ideal factorization, which is a standard method which is rather technical but very efficient to apply. We have introduced the corresponding concepts in several former papers. Here we only give the theorem of index of Ore which plays a key role for proving our main results.

Let $K = \mathbb{Q}(\alpha)$ be a number field generated by a complex root α of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$. We shall use Dedekind's theorem [25, Chapter I, Proposition 8.3] and Dedekind's criterion [1, Theorem 6.1.4]. Let $\phi \in \mathbb{Z}_p[x]$ be a monic lift to an irreducible factor of $F(x)$ modulo p, $F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_l(x)\phi(x)^l$ the ϕ-expansion of $F(x)$, and $N_\phi^+(F)$ the principal ϕ-Newton polygon of $F(x)$. Let $\mathbb{F}_\phi$ be the field $\mathbb{F}_p[x]/(\overline{\phi})$, then to every side S of $N_\phi^+(F)$ with initial point (i, u_i), and every $i = 0, \ldots, l$, let the residue coefficient $c_i \in \mathbb{F}_\phi$ defined as follows:

$$c_i = \begin{cases} 0, & \text{if } (s+i, u_{s+i}) \text{ lies strictly above } S, \\ \left(\dfrac{a_{s+i}(x)}{p^{u_{s+i}}}\right) \mod (p, \phi(x)), & \text{if } (s+i, u_{s+i}) \text{ lies on } S. \end{cases}$$

Let $-\lambda = -h/e$ be the slope of S, where h and e are two positive coprime integers and $l = l(S)$ its length. Then $d = l/e$ is the degree of S. Hence, if i is not a multiple of e, then $(s+i, u_{s+i})$ does not lie on S, and so $c_i = 0$. Let $R_1(F)(y) = t_d y^d + t_{d-1} y^{d-1} + \cdots + t_1 y + t_0 \in \mathbb{F}_\phi[y]$, called the residual polynomial of $F(x)$ associated to the side S, where for every $i = 0, \ldots, d$, $t_i = c_{ie}$. If $R_1(F)(y)$ is square free for each side of the polygon $N_\phi^+(F)$, then we say that $F(x)$ is ϕ-regular.

Let $\overline{F(x)} = \prod_{i=1}^{r} \overline{\phi_i}^{l_i}$ be the factorization of $F(x)$ into powers of monic irreducible coprime polynomials over $\mathbb{F}_p$, we say that the polynomial $F(x)$ is p-regular if $F(x)$ is a ϕ_i-regular polynomial with respect to p for every $i = 1, \ldots, r$. Let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{ir_i}$ be the ϕ_i-principal Newton polygon of $F(x)$ with respect to p. For every $j = 1, \ldots, r_i$, let $R_{1_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{a_{ijs}}(y)$ be the factorization of $R_{1_{ij}}(F)(y)$ in $\mathbb{F}_{\phi_i}[y]$. Then we have the following theorem of index of Ore:

Theorem 3.1. ([9, Theorem 1.7 and Theorem 1.9])
Under the above hypothesis, we have the following:

(1)

$$\nu_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \geq \sum_{i=1}^{r} ind_{\phi_i}(F).$$

The equality holds if $F(x)$ is p-regular.

(2) *If $F(x)$ is p-regular, then*

$$p\mathbb{Z}_K = \prod_{i=1}^{r}\prod_{j=1}^{t_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}$$

is the factorization of $p\mathbb{Z}_K$ into powers of prime ideals of $\mathbb{Z}_K$, where e_{ij} is the smallest positive integer satisfying $e_{ij}\lambda_{ij} \in \mathbb{Z}$ and the residue degree of $\mathfrak{p}_{ijs}$ over p is given by $f_{ijs} = \deg(\phi_i) \times \deg(\psi_{ijs})$ for every (i, j, s).

If the theorem of Ore fails, that is, $F(x)$ is not p-regular, then in order to complete the factorization of $F(x)$, Guàrdia, Montes, and Nart introduced the notion of high order Newton polygon ([13]). Similar to first order, for each order r, they introduced the valuation ω_r, the key polynomial $\phi_r(x)$ for this valuation, the Newton polygon $N_r(F)$ of any polynomial $F(x)$ with respect to ω_r and $\phi_r(x)$, and for each side T_i of $N_r(F)$, the residual polynomial $R_r(F)(y)$, and the index of $F(x)$ in order r. For more details, we refer to [13].

In [10], Engstrom determined $\nu_p(i(K))$, from the factorization of $p\mathbb{Z}_K$, for every number filed of degree $n \leq 7$. Moreover, his results characterize $\nu_p(i(K))$ for an arbitrary number field in certain particular cases, according to the factorization of p in K. Here are the results used to calculate $\nu_p(i(K))$ in the case of number fields defined by $x^9 + ax + b$.

Theorem 3.2. *(*[10, Corollary, p. 230]*)*
Let p be a rational prime. For every positive integer f, let $\mathcal{P}_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p with residue degree f and $\mathcal{N}_f$ the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree f. If K is a number field of degree n in which

$$p\mathbb{Z}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_s^{e_s},$$

where $\mathcal{P}_{f_i} < \mathcal{N}_{f_i}$ for $f_i \neq 1$, and $e_i = 1$ for $f_i = 1$ except for one prime ideal, then

$$\nu_p(i(K)) = s_1 + \sum_{i=1}^{s} s_i\left(r - p^i\left(\frac{s_i+1}{2}\right)\right),$$

where r is the number of first residue degree and first ramification index prime ideals dividing p and $s_i = \lfloor r/p^i \rfloor$.

Theorem 3.3. *(*[10, Theorem 6]*)*
If K is a number field of degree n in which

$$2\mathbb{Z}_K = \mathfrak{p}_1^{a}\mathfrak{p}_2^{b}\mathfrak{p}_3^{2}\mathfrak{p}_4^{c}\mathfrak{p}_5^{d}, \textit{ with residue degree } 1 \textit{ each ideal factor,}$$

where $a \geq b \geq 2 > c \geq d$, then

$$\nu_2(i(K)) = \begin{cases} 2 + c + 2d & \textit{if } a > 2, \\ 3 + c + 2d & \textit{if } a = 2. \end{cases}$$

4. Proofs of Main Results

Throughout this section, if $p\mathbb{Z}_K = \displaystyle\prod_{i=1}^{r}\prod_{j=1}^{t_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}$, then e_{ij} denotes the ramification index of $\mathfrak{p}_{ijs}$ and f_{ijs} denotes its residue degree for every (i,j,s).
For every rational prime p and every integer m let $m_p = m/p^{\nu_p(m)}$.

Proof of Theorem 2.1.
Let $K = \mathbb{Q}(\alpha)$ be a number field defined by a monic irreducible trinomial $F(x) = x^9 + ax + b \in \mathbb{Z}[x]$. Since $\Delta = 2^{24}a^9 + 3^{18}b^8$ is the discriminant of $F(x)$, thanks to the index formula (1.1), we have the following:

(1) If p divides both a and b, then p does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ if and only if $\nu_p(b) = 1$.
(2) If $p = 2$ and 2 does not divide b, then 2 does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$.
(3) If $p = 2$, 2 divides b and does not divide a, then $F(x) \equiv x(x-1)^8 \pmod 2$. Let $\phi_1 = x$ and $\phi_2 = x - 1$, then $F(x) = \cdots + 36\phi_2^2 + (a+9)\phi_2 + a + b + 1$. Hence 2 does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ if and only if $\nu_2(a+b+1) = 1$; that is $(a,b) \in \{(1,0),(3,2)\} \pmod 4$.
(4) If $p = 3$ and 3 does not divide a, then 3 does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$.
(5) If $p = 3$, 3 divides a, and $b \equiv -1 \pmod 3$, then $F(x) \equiv (x-1)^9 \pmod 3$. Let $\phi = x - 1$. Then $F(x) = \cdots + 36\phi^2 + (a+9)\phi + a + b + 1$. Hence 3 does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ if and only if $\nu_3(a+b+1) = 1$; that is $(a,b) \in \{(0,2),(0,5),(3,-1),(3,2)(6,-1),(6,5)\} \pmod 9$.
(6) If $p = 3$, 3 divides a, and $b \equiv 1 \pmod 3$, then $F(x) \equiv (x+1)^9 \pmod 3$. Let $\phi = x + 1$. Then $F(x) = \cdots - 36\phi^2 + (a+9)\phi - a + b - 1$. Hence 3 does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ if and only if $\nu_3(-a+b-1) = 1$; that is $(a,b) \in \{(0,4),(0,7),(3,1),(3,7),(6,1),(6,4)\} \pmod 9$.
(7) If $p \notin \{2,3\}$, p^2 divides $\Delta = 2^{24}a^9 + 3^{18}b^8$, and $\nu_p(ab) = 0$, then $F(x)$ admits a multiple root $\overline{u}$ modulo p if and only if $F(u) = u^9 + au + b \equiv 0 \pmod p$ and $F'(u) = 9u^8 + a \equiv 0 \pmod p$. That is $8au + 9b \equiv 0 \pmod p$. Let $u = \dfrac{-9b}{8a} \in \mathbb{Q}$. Since $\nu_p(8a) = 0$, then $u \in \mathbb{Z}_p$. Let $\phi = x - u$. Then $F(x) = \cdots + 36u^7\phi^2 + A\phi + B$, with

$$\begin{array}{rclcl} A &=& a + 9u^8 &=& \dfrac{2^{24}a^7 + 3^{18}b^8}{2^{24}a^8} = \dfrac{\Delta}{2^{24}a^8}, \text{and} \\ B &=& au + b + u^9 &=& \dfrac{-3^{18}b^9 - 2^{24}a^9 b}{2^{27}a^9} = \dfrac{-b\Delta}{2^{27}a^9}. \end{array}$$

Since $\nu_p(A) = \nu_p(B) = \nu_p(\Delta)$ and $\nu_p(36u^7) = 0$, then $N_\phi^+(F) = S_1$ has a single side joining $(0,\nu_p(\Delta))$ and $(2,0)$. Since $\nu_p(\Delta) \geq 2$, by Theorem 3.1, $\nu_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \geq \lfloor \nu_p(\Delta)/2 \rfloor \geq 1$. Hence p divides the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$.

□

For the proof of Theorems 2.2, 2.3, and 2.4, we need the following lemma, which characterizes the prime common index divisors of K.

Lemma 4.1. ([10])
Let p be a rational prime and K a number field. For every positive integer f, let $\mathcal{P}_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p with residue degree f and $\mathcal{N}_f$ the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree f. Then p divides the index $i(K)$ if and only if $\mathcal{P}_f > \mathcal{N}_f$ for some positive integer f.

Proof of Theorem 2.2.
Since $\Delta = 2^{24}a^9 + 3^{18}b^8$ is the discriminant of $F(x)$, thanks to the index formula (1.1), if 2 does not divide b, then $\nu_2((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = 0$ and so $\nu_2(i(K)) = 0$. Assume that 2 divides b. Then we have the following cases:

(1) If $\nu_2(a) = 0$, then $F(x) \equiv x(x-1)^8 \pmod{2}$. Let $\phi = x - 1$. Then x provides a unique prime ideal of $\mathbb{Z}_K$ lying above 2 with residue degree 1. For ϕ, let $F(x) = \phi^9 + 9\phi^8 + 36\phi^7 + 84\phi^6 + 126\phi^5 + 126\phi^4 + 84\phi^3 + 36\phi^2 + (a+9)\phi + a + b + 1$. Then we have the following cases:

 (i) If $(a, b) \in \{(1, 0), (3, 2)\} \pmod{4}$, then by Theorem 2.1, $\nu_2(i(K)) = 0$.

 (ii) If $(a, b) \equiv (1, 2) \pmod{4}$, then $N_\phi^+(F) = S_1 + S_2$ has two sides joining $(0, w)$, $(1, 1)$, and $(8, 0)$ with $w \geq 2$. Thus the degree of each side is 1 and so $2\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{21}\mathfrak{p}_{22}^7$ with residue degree 1 each ideal factor. Since there are just two monic irreducible polynomials of degree 1 in $\mathbb{F}_2[x]$, by Lemma 4.1, 2 divides $i(K)$. Applying Theorem 3.2, we get $\nu_2(i(K)) = 1$.

 (iii) For $(a, b) \equiv (3, 0) \pmod{4}$, we have the following sub-cases:

  (a) If $(a, b) \not\equiv (7, 8) \pmod{16}$, then the treatment of this case is similar the case (ii) above, and Table 3 summarizes the obtained results.

Table 3

Cases	$2\mathbb{Z}_K$	f_i	$\nu_2(i(K))$
$(a, b) \in \{(3, 0), (7, 4)\} \pmod{8}$	$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1$, $f_2 = 2$	0
$(a, b) \equiv (3, 4) \pmod{8}$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^3\mathfrak{p}_4^4$	$f_i = 1$	≥ 1
$(a, b) \in \{(7, 0), (15, 8)\} \pmod{16}$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^4$	$f_1 = f_3 = 1$, $f_2 = 2$	0
$(a, b) \equiv (15, 0) \pmod{16}$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^3\mathfrak{p}_4^4$	$f_i = 1$	≥ 1

  (b) If $(a, b) \equiv (7, 8) \pmod{16}$, then $\nu_2(\Delta) \geq 28$. As in the proof of Theorem 2.1, let $b_2 = \dfrac{b}{8}$ and $u = \dfrac{-9b_2}{a}$. Since 2 does not divide a, then $u \in \mathbb{Z}_2$. Let $\phi = x - u$, then $F(x) = \phi^9 + 9u\phi^8 + 36u^2\phi^7 + 84u^3\phi^6 + 126u^4\phi^5 + 126u^5\phi^4 + 84u^6\phi^3 + 36u^7\phi^2 + A\phi + B$ with

$$\begin{aligned} A &= a + 9u^8 &&= \frac{\Delta}{2^{24}a^8}, \text{ and} \\ B &= au + b + u^9 &&= \frac{-b\Delta}{2^{27}a^9} &&= \frac{-b_2\Delta}{2^{24}a^9}. \end{aligned}$$

Thus $\nu_2(A) = \nu_2(B) = \nu_2(\Delta) - 24$.

b$_1$- If $\nu_2(\Delta)$ is odd, then $N_{\phi}^{+}(F) = S_1 + S_2 + S_3$ has three sides joining $(0, \nu_2(\Delta) - 24)$, $(2, 2)$, $(4, 1)$, and $(8, 0)$. Thus the degree of each side is 1 and so $2\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{21}^2\mathfrak{p}_{22}^2\mathfrak{p}_{23}^4$ with residue degree 1 each ideal factor. Hence 2 divides $i(K)$. Applying Theorem 3.3, we get $\nu_2(i(K)) = 3$.

b$_2$- If $\nu_2(\Delta) = 2k + 26$ is even ($k \geq 1$), then $N_{\phi_2}^{+}(F) = S_1 + S_2 + S_3$ has three sides joining $(0, 2k+2)$ $(2, 2)$, $(4, 1)$, and $(8, 0)$ with $d(S_2) = d(S_3) = 1$ and $R_{1_1}(F)(y) = (y+1)^2 \in \mathbb{F}_{\phi}[y]$. Let us replace ϕ by $x - (u + 2^k)$. Then $F(x) = \cdots + A_4(x - (u + 2^k))^4 + A_3(x - (u + 2^k))^3 + A_2(x - (u + 2^k))^2 + A_1(x - (u + 2^k)) + A_0$ with $\nu_2(A_4) = 1$, $\nu_2(A_3) = \nu_2(A_2) = 2$,

$$\begin{aligned} A_1 &= A + 72u^7 2^k + 252u^6(2^k)^2 + 504u^5(2^k)^3 + 630u^4(2^k)^4 + 504u^3(2^k)^5 + 252u^2(2^k)^6 \\ &\quad +72u(2^k)^7 + 9(2^k)^8, \text{ and} \\ A_0 &= B + 2^k A + 36u^7(2^k)^2 + 84u^6(2^k)^3 + 126u^5(2^k)^4 + 126u^4(2^k)^5 \\ &\quad +84u^3(2^k)^6 + 36u^2(2^k)^7 + 9u(2^k)^8 + (2^k)^9. \end{aligned}$$

Clearly, $\nu_2(A_1) = k + 3$.

- For $\nu_2(\Delta) = 28$; $k = 1$, we have $A_0 \equiv B + 2A + 36u^7 2^2 + 84u^6 2^3 + 126u^5 2^4 + 126u^4 2^5 \pmod{128}$. Hence $A_0 \equiv \frac{2^4}{a^9}(-b_2\Delta_2 + 2a\Delta_2 + 63a^2b_2^7 + 10a^3b_2^6 + 82a^4b_2^5 + 124a^5b_2^4) \pmod{128}$. Since $\frac{A_0}{2^4} \equiv \frac{1}{a^9}(-b_2\Delta_2 + 2a\Delta_2 + 7a^2b_2^7 + 2a^3b_2^6 + 2a^4b_2^5 + 4a^5b_2^4) \pmod{8} \equiv -(-b_2\Delta_2 + 6\Delta_2 + b_2 + 2) \pmod{8}$, three cases arise are summarized in Table 4.

TABLE 4. $\nu_2(\Delta) = 28$

Cases	$\nu_2(A_0)$	$2\mathbb{Z}_K$	f_i	$\nu_2(i(K))$
$\Delta_2 \equiv 1 \pmod{8}$	≥ 7	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^2\mathfrak{p}_5^4$	$f_i = 1$	≥ 1
$\Delta_2 \equiv 5 \pmod{8}$	6	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_4^4$	$f_1 = f_3 = f_4 = 1$, $f_2 = 2$	
$\Delta_2 \equiv 3, 7 \pmod{8}$	5	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4^4$	$f_i = 1$	3

- For $\nu_2(\Delta) \geq 30$; $k \geq 2$, we have $A_0 \equiv B + 36u^7(2^k)^2 \pmod{2^{3k+3}}$. Hence $A_0 \equiv \frac{2^{2k+2}}{a^9}(-b_2\Delta_2 + 9^8a^3b_2^7)) \pmod{2^{3k+3}}$. Since $\frac{A_0}{2^{2k+2}} \equiv -(-b_2\Delta_2 + 7b_2) \pmod{8}$, three cases arise are summarized in Table 5.

TABLE 5. $\nu_2(\Delta) \geq 30$ is even

Cases	$\nu_2(A_0)$	$2\mathbb{Z}_K$	f_i	$\nu_2(i(K))$
$\Delta_2 \equiv 7 \pmod{8}$,	$\geq 2k + 5$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^2\mathfrak{p}_5^4$	$f_i = 1$	≥ 1
$\Delta_2 \equiv 3 \pmod{8}$	$2k + 4$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_4^4$	$f_1 = f_3 = f_4 = 1$, $f_2 = 2$	
$\Delta_2 \equiv 1, 5 \pmod{8}$	$2k + 3$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4^4$	$f_i = 1$	3

(2) If $\nu_2(a) \geq 1$, then $F(x) \equiv x^9 \pmod{2}$. Let $\phi = x$. Then $F(x) = \phi^9 + a\phi + b$. By assumption, $\nu_2(b) \leq 8$ or $\nu_2(a) \leq 7$.
If $8\nu_2(b) < 9\nu_2(a)$, then $N_\phi(F) = S_1$ has a single side of degree $d \in \{1, 3\}$.
  (i) If $d = 1$ then $R_{1_1}(F)(y)$ is irreducible as it is of degree 1. Thus $2\mathbb{Z}_K = \mathfrak{p}_1^9$ with residue degree 1. Hence $\nu_2(i(K)) = 0$.
  (ii) If $d = 3$, then $R_{1_1}(F)(y) = y^3 + 1 = (y + 1)(y^2 + y + 1) \in \mathbb{F}_\phi[y]$. Thus $2\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2^3$ with $f_1 = 1$ and $f_2 = 2$. Hence $\nu_2(i(K)) = 0$.

If $8\nu_2(b) > 9\nu_2(a)$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, \nu_2(b))$, $(1, \nu_2(a))$, and $(0, 9)$ with $d(S_1) = 1$ and $d(S_2) \in \{1, 2, 4\}$ since $\nu_2(a) \leq 7$. Let $d = d(S_2)$. Then we have the following cases:
  (i) If $d = 1$; $\nu_2(a) \in \{1, 3, 5, 7\}$, then $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^8$ with residue degree 1 each ideal factor. Hence $\nu_2(i(K)) = 0$.
  (ii) If $d = 2$ with $\nu_2(b) \geq 3$ and $\nu_2(a) = 2$; $a \equiv 4 \pmod{8}$ and $b \equiv 0 \pmod{8}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, \nu_2(b))$, $(1, 2)$ and $(9, 0)$ with $d(S_1) = 1$ and $R_{1_2}(F)(y) = (y + 1)^2 \in \mathbb{F}_\phi[y]$. In this case, we have to use second order Newton polygon. Let $\omega_2 = 4[\nu_2, \phi, 1/4]$ be the valuation of second order Newton polygon and $g_2 = x^4 - 2$ the key polynomial of ω_2, where $[\nu_2, \phi, 1/4]$ is the augmented valuation of ν_2 with respect to ϕ and $\lambda = 1/4$. Let $F(x) = xg_2^2 + 4xg_2 + (a + 4)x + b$, then we have $\omega_2(x) = 1$, $\omega_2(g_2) = 4$, and $\omega_2(m) = 4 \times \nu_2(m)$ for every $m \in \mathbb{Q}_2$. Table 6 summarizes the obtained results.

TABLE 6

Cases	g_2	$2\mathbb{Z}_K$	f_i	$\nu_2(i(K))$
$(a, b) \in \{(4, 8), (12, 8)\} \pmod{16}$	$x^4 - 2$	$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a, b) \in \{(12, 16), (28, 16)\} \pmod{32}$				
$(a, b) \equiv (12, 0) \pmod{32}$		$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1, f_2 = 2$	
$(a, b) \equiv (28, 0) \pmod{32}$		$\mathfrak{p}_{11}\mathfrak{p}_{21}^4\mathfrak{p}_{22}^4$	$f_i = 1$	≥ 1
$(a, b) \in \{(4, 16), (20, 16)\} \pmod{32}$	$x^4 - 2x^2 - 2$	$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a, b) \in \{(4, 32), (36, 32)\} \pmod{64}$	$x^4 - 2x^2 - 6$			
$(a, b) \equiv (36, 0) \pmod{64}$		$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1, f_2 = 2$	
$(a, b) \equiv (4, 0) \pmod{64}$		$\mathfrak{p}_{11}\mathfrak{p}_{21}^4\mathfrak{p}_{22}^4$	$f_i = 1$	≥ 1
$(a, b) \in \{(20, 0), (52, 0)\} \pmod{64}$	$x^4 - 2x^2 - 4x - 2$	$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a, b) \equiv (20, 32) \pmod{64}$		$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1, f_2 = 2$	
$(a, b) \equiv (52, 32) \pmod{64}$		$\mathfrak{p}_{11}\mathfrak{p}_{21}^4\mathfrak{p}_{22}^4$	$f_i = 1$	≥ 1

  (iii) If $d = 4$; $a \equiv 16 \pmod{32}$ and $b \equiv 0 \pmod{32}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, \nu_2(b))$, $(1, 4)$ and $(9, 0)$ with $d(S_1) = 1$ and $R_{1_2}(F)(y) = (y + 1)^4 \in \mathbb{F}_\phi[y]$. In this case, we have to use second order Newton polygon. Let $\omega_2 = 2[\nu_2, \phi, 1/2]$ be the valuation of second order Newton polygon and $g_2 = x^2 - 2$ the key polynomial of ω_2, where $[\nu_2, \phi, 1/2]$ is the augmented valuation of ν_2 with respect to ϕ and $\lambda = 1/2$. Let

$F(x) = xg_2^4 + 8xg_2^3 + 24xg_2^2 + 32xg_2 + (a+16)x + b$, then we have $\omega_2(x) = 1$, $\omega_2(g_2) = 2$, and $\omega_2(m) = 2 \times \nu_2(m)$ for every $m \in \mathbb{Q}_2$.

(a) If $(a, b) \in \{(16, 32), (48, 32)\} \pmod{64}$, then $N_2^+(F) = T_1$ has a single side joining $(0, 10)$ and $(4, 9)$. Thus $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^8$ with residue degree 1 each ideal factor. Hence $\nu_2(i(K)) = 0$.

(b) If $(a, b) \equiv (16, 0) \pmod{64}$, then $N_2^+(F) = T_1$ has a single side joining $(0, 11)$ and $(2, 9)$ with $R_{2_1}(F)(y) = (y+1)^2 \in \mathbb{F}_2[y]$. In this case, we have to use third order Newton polygon. Let $\omega_3 = 2[\omega_2, g_2, 1/2]$ be the valuation of third order Newton polygon and $g_3 = x^4-4x^2-4x+4$ the key polynomial of ω_3, where $[\omega_2, g_2, 1/2]$ is the augmented valuation of ω_2 with respect to g_2, and $\lambda' = 1/2$. Let

$$F(x) = xg_3^2 + ((8x+8)g_2 + 24x + 48)g_3 + (48x+160)g_2 + (a+176)x + b + 192,$$

then we have $\omega_3(x) = 2$, $\omega_3(g_2) = 5$, $\omega_3(g_3) = 10$, and $\omega_3(m) = 4 \times \nu_2(m)$ for every $m \in \mathbb{Q}_2$. Thus $N_3^+(F) = T_1'$ has a single side joining $(0, 23)$ and $(2, 22)$. It follows that $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^8$ with residue degree 1 each ideal factor. Hence $\nu_2(i(K)) = 0$.

(c) For the other cases, we need also to use third order Newton polygon, and its treatment is similar to the case (b) above. The obtained results are summarized in Table 7.

TABLE 7

Cases	g_2	g_3	$2\mathbb{Z}_K$	f_i	$\nu_2(i(K))$
$(a, b) \in \{(48, 64), (112, 64)\} \pmod{128}$	x^2-2	$-$	$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a, b) \equiv (48, 0) \pmod{128}$		$x^4-2x^3-4x^2+4x-4$	$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1$, $f_2 = 2$	
$(a, b) \in \{(112, 128), (240, 128)\} \pmod{256}$		x^2-2x-2	$\mathfrak{p}_1\mathfrak{p}_2^4\mathfrak{p}_3^4$	$f_i = 1$	≥ 1
$(a, b) \in \{(112, 0), (368, 0)\} \pmod{512}$		x^2-2x-2	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^4$	$f_1 = f_3 = 1$ $f_2 = 2$	0
$(a, b) \equiv (368, 256) \pmod{512}$		x^2-2x-2	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2$	$f_1 = 1$ $f_2 = f_3 = 2$	1
$(a, b) \equiv (112, 256) \pmod{512}$		x^2-2x-2	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4^2$	$f_1 = f_3 = f_4 = 1$ $f_2 = 2$	≥ 1
$(a, b) \in \{(240, 256), (496, 256)\} \pmod{512}$		x^2-2x-6	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4^4$	$f_i = 1$	3
$(a, b) \equiv (240, 0) \pmod{512}$			$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4^2$	$f_1 = f_2 = f_3 = 1$ $f_4 = 2$	≥ 1
$(a, b) \equiv (496, 0) \pmod{512}$			$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4^2\mathfrak{p}_5^2$	$f_i = 1$	≥ 1

(iv) If $d = 2$ with $\nu_2(b) \geq 7$ and $\nu_2(a) = 6$; $a \equiv 64 \pmod{128}$ and $b \equiv 0 \pmod{128}$, then $N_\phi^+(F) = S_1 + S_2$ has two sides joining $(0, \nu_2(b))$, $(1, 6)$, and $(9, 0)$ with $d(S_1) = 1$ and $R_{1_2}(F)(y) = (y+1)^2 \in \mathbb{F}_\phi[y]$. In this case, we have to use second order Newton polygon. Let $\omega_2 = 4[\nu_2, \phi, 3/4]$ be the valuation of second order Newton polygon and $g_2 = x^4 - 8$ the key polynomial of ω_2, where $[\nu_2, \phi, 3/4]$ is the augmented valuation of ν_2 with respect to ϕ and $\lambda = 3/4$. Let $F(x) = xg_2^2 + 16xg_2 + (a + 64)x + b$, then we have $\omega_2(x) = 3$, $\omega_2(g_2) = 12$, and $\omega_2(m) = 4 \times \nu_2(m)$ for every $m \in \mathbb{Q}_2$. Table 8 summarizes the obtained results.

TABLE 8

Cases	g_2	$2\mathbb{Z}_K$	f_i	$\nu_2(i(K))$
$(a, b) \in \{(64, 128), (192, 128)\} \pmod{256}$	$x^4 - 8$	$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a, b) \in \{(64, 256), (320, 256)\} \pmod{512}$	$x^4 - 4x^2 - 8$			
$(a, b) \in \{(64, 512), (566, 512)\} \pmod{1024}$	$x^4 - 4x - 24$			
$(a, b) \equiv (576, 0) \pmod{1024}$	$x^4 - 4x - 24$	$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1, f_2 = 2$	
$(a, b) \equiv (64, 0) \pmod{1024}$		$\mathfrak{p}_1\mathfrak{p}_2^4\mathfrak{p}_3^4$	$f_i = 1$	≥ 1
$(a, b) \in \{(320, 512), (832, 512)\} \pmod{1024}$	$x^4 - 4x^2 - 8$	$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a, b) \in \{(320, 0), (832, 0)\} \pmod{1024}$	$x^4 - 2x^3 - 4x^2 - 8$			
$(a, b) \in \{(192, 256), (448, 256)\} \pmod{512}$	$x^4 - 8$			
$(a, b) \equiv (192, 0) \pmod{512}$		$\mathfrak{p}_1\mathfrak{p}_2^4$	$f_1 = 1, f_2 = 2$	
$(a, b) \equiv (448, 0) \pmod{512}$		$\mathfrak{p}_1\mathfrak{p}_2^4\mathfrak{p}_3^4$	$f_i = 1$	≥ 1

□

Proof of Theorem 2.3.
If $\nu_3(a) = 0$, then since $\Delta = 2^{24}a^9 + 3^{18}b^8$ is the discriminant of $F(x)$, thanks to the index formula (1.1), $\nu_3((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = 0$ and so $\nu_3(i(K)) = 0$. Now assume that 3 divides a. Then we have the following cases:

1. If $b \equiv -1 \pmod 3$, then $F(x) \equiv (x-1)^9 \pmod 3$. Let $\phi = x - 1$. Then $F(x) = \phi^9 + 9\phi^8 + 36\phi^7 + 84\phi^6 + 126\phi^5 + 126\phi^4 + 84\phi^3 + 36\phi^2 + (a+9)\phi + a + b + 1$.
   - (i) If $(a, b) \in \{(0, 2), (0, 5), (3, -1), (3, 2)(6, -1), (6, 5)\} \pmod 9$, then by Theorem 2.1, $\nu_3(i(K)) = 0$.
   - (ii) If $(a, b) \in \{(3, 5), (6, 2)\} \pmod 9$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, w)$, $(1, 1)$, and $(9, 0)$ with $w \geq 2$. Thus the degree of each side is 1 and so $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^8$ with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.
   - (iii) If $(a, b) \in \{(0, 8), (0, 17), (9, -1), (9, 8), (18, -1), (18, 17)\} \pmod{27}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, 2)$, $(3, 1)$, and $(9, 0)$. Thus $3\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2^6$ with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.
   - (iv) If $(a, b) \in \{(0, -1), (9, 17)\} \pmod{27}$, then $N_\phi(F) = S_1 + S_2 + S_3$ has three sides joining $(0, w)$, $(1, 2)$, $(3, 1)$, and $(9, 0)$ with $w \geq 3$. It follows that $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^6$ with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.
   - (v) If $(a, b) \in \{(18, 8), (18, 35), (45, 8), (45, 62), (72, 35), (72, 62)\} \pmod{81}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, 3)$, $(3, 1)$, and $(9, 0)$. It follows that $3\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2^6$ with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.
   - (vi) If $(a, b) \equiv (18, 62) \pmod{81}$, then we have the following sub-cases:
     - (a) If $a + b \equiv 80 \pmod{243}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, 4)$, $(3, 1)$, and $(9, 0)$ with $d(S_2) = 1$ and $R_{1_1}(F)(y) = y^3 + y^2 + y + 1 =$

$(y^2+1)(y+1) \in \mathbb{F}_\phi[y]$. Thus $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{12}\mathfrak{p}_{21}^6$ with $f_{11} = f_{21} = 1$ and $f_{12} = 2$. Hence $\nu_3(i(K)) = 0$.

(b) If $a+b \equiv 161 \pmod{243}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0,4)$, $(3,1)$, and $(9,0)$ with $d(S_2) = 1$ and $R_{1_1}(F)(y) = y^3 + y^2 + y - 1$ which is irreducible over $\mathbb{F}_\phi$. Thus $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^6$ with $f_1 = 3$ and $f_2 = 1$. Hence $\nu_3(i(K)) = 0$.

(c) If $a+b \equiv -1 \pmod{243}$, then $\nu_3(\Delta) \geq 23$. As in the proof of Theorem 2.1, let $a_3 = \dfrac{a}{9}$ and $u = \dfrac{-b}{8a_3}$. Since 3 does not divide $8a_3$, then $u \in \mathbb{Z}_3$. Let $\phi = x - u$, then $F(x) = \phi^9 + 9u\phi^8 + 36u^2\phi^7 + 84u^3\phi^6 + 126u^4\phi^5 + 126u^5\phi^4 + 84u^6\phi^3 + 36u^7\phi^2 + A\phi + B$ with $A = a + 9u^8 = \dfrac{\Delta}{2^{24}a^8}$ and $B = au + b + u^9 = \dfrac{-b\Delta}{2^{27}a^9}$. Thus $\nu_3(A) = \nu_3(\Delta) - 16$ and $\nu_3(B) = \nu_3(\Delta) - 18$. It follows that $N_\phi(F) = S_1 + S_2 + S_3$ has three sides joining $(0, \nu_3(\Delta) - 18)$, $(2,2)$, $(3,1)$, and $(9,0)$ with $d(S_2) = d(S_3) = 1$.

If $\nu_3(\Delta)$ is odd, then $d(S_1) = 1$ and so $3\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$ with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.

If $\nu_3(\Delta)$ is even, then $d(S_1) = 2$ with $R_{1_1}(F)(y) = y^2 + B_3 \in \mathbb{F}_\phi[y]$. Since $(36u^7)_3 \equiv 1 \pmod 3$, then two cases arise:

- If $\Delta_3 \not\equiv a_3 \pmod 3$; $\Delta_3 \equiv 1 \pmod 3$, then $R_{1_1}(F)(y)$ is irreducible over $\mathbb{F}_\phi$ and so $3\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$ with $f_1 = 2$ and $f_2 = f_3 = 1$. Hence $\nu_3(i(K)) = 0$.
- If $\Delta_3 \equiv a_3 \pmod 3$; $\Delta_3 \equiv -1 \pmod 3$, then $R_{1_1}(F)(y) = (y-1)(y+1) \in \mathbb{F}_\phi[y]$ and so $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{12}\mathfrak{p}_{21}\mathfrak{p}_{31}^6$ with residue degree 1 each ideal factor. Hence 3 divides $i(K)$. Applying Theorem 3.2, we get $\nu_3(i(K)) = 1$.

(vii) If $(a,b) \equiv (45,35) \pmod{81}$, then we have the following sub-cases:

(a) If $a+b \equiv 80 \pmod{243}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0,4)$, $(3,1)$, and $(9,0)$ with $d(S_2) = 1$ and $R_{1_1}(F)(y) = y^3 + y^2 - y + 1$ which is irreducible over $\mathbb{F}_\phi$. Thus $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^6$ with $f_1 = 3$ and $f_2 = 1$. Hence $\nu_3(i(K)) = 0$.

(b) If $a+b \equiv 161 \pmod{243}$, then $\nu_3(\Delta) \geq 23$. As in the case (iv$_c$) above, let $a_3 = \dfrac{a}{9}$, $u = \dfrac{-b}{8a_3} \in \mathbb{Z}_3$, and $\phi = x - u$, then 3 divides $i(K)$ if and only if $\nu_3(\Delta)$ is even and $\Delta_3 \equiv -1 \pmod 3$. In this case also, we have $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{12}\mathfrak{p}_{21}\mathfrak{p}_{31}^6$ with residue degree 1 each ideal factor. Applying Theorem 3.2, we get $\nu_3(i(K)) = 1$.

(c) If $a+b \equiv -1 \pmod{243}$, then $N_\phi(F) = S_1 + S_2 + S_3$ has three sides joining $(0,w)$, $(1,3)$, $(3,1)$, and $(9,0)$ with $w \geq 5$, $d(S_1) = d(S_3) = 1$, and $R_{1_2}(F)(y) = y^2 + y - 1$ which is irreducible over $\mathbb{F}_\phi$. Thus $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^6$ with $f_1 = f_3 = 1$ and $f_2 = 2$. Hence $\nu_3(i(K)) = 0$.

(viii) If $(a,b) \equiv (72,8) \pmod{81}$, then we have the following sub-cases:

    (a) If $a+b \equiv 80 \pmod{243}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0,4)$, $(3,1)$, and $(9,0)$ with $d(S_2) = 1$ and $R_{1_1}(F)(y) = y^3 + y^2 + 1 = (y-1)(y^2 - y - 1) \in \mathbb{F}_\phi[y]$. Thus $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{12}\mathfrak{p}_{21}^6$ with $f_{11} = f_{21} = 1$ and $f_{12} = 2$. Hence $\nu_3(i(K)) = 0$.
    (b) If $a+b \equiv 161 \pmod{243}$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0,4)$, $(3,1)$, and $(9,0)$ with $d(S_2) = 1$ and $R_{1_1}(F)(y) = y^3 + y^2 - 1$ which is irreducible over $\mathbb{F}_\phi$. Thus $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^6$ with $f_1 = 3$ and $f_2 = 1$. Hence $\nu_3(i(K)) = 0$.
    (c) If $a+b \equiv -1 \pmod{243}$, then $\nu_3(\Delta) \geq 23$. As in the case (iv$_c$) above, let $a_3 = \dfrac{a}{9}$, $u = \dfrac{-b}{8a_3} \in \mathbb{Z}_3$, and $\phi = x - u$, then 3 divides $i(K)$ if and only if $\nu_3(\Delta)$ is even and $\Delta_3 \equiv -1 \pmod 3$. In this case also, we have $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{12}\mathfrak{p}_{21}\mathfrak{p}_{31}^6$ with residue degree 1 each ideal factor. Applying Theorem 3.2, we get $\nu_3(i(K)) = 1$.

(2) If $b \equiv 1 \pmod 3$, then $F(x) \equiv (x+1)^9 \pmod 3$. Let $\phi = x + 1$. Then $F(x) = \phi^9 - 9\phi^8 + 36\phi^7 - 84\phi^6 + 126\phi^5 - 126\phi^4 + 84\phi^3 - 36\phi^2 + (a+9)\phi - a + b - 1$.
  (i) If $(a,b) \in \{(0,4),(0,7),(3,1),(3,7),(6,1),(6,4)\} \pmod 9$, then by Theorem 2.1, $\nu_3(i(K)) = 0$.
  (ii) The treatment of the other cases is similar to the case $b \equiv -1 \pmod 3$ above. Table 9 summarizes the obtained results.

(3) If $b \equiv 0 \pmod 3$, then $F(x) \equiv x^9 \pmod 3$. Let $\phi = x$. Then $F(x) = \phi^9 + a\phi + b$. By assumption, $\nu_3(b) \leq 8$ or $\nu_3(a) \leq 7$.
If $8\nu_3(b) < 9\nu_3(a)$, then $N_\phi(F) = S_1$ has a single side joining $(0, \nu_3(b))$ and $(9,0)$ with degree $d \in \{1,3\}$.
  (i) If $d = 1$, then $R_{1_1}(F)(y)$ is irreducible as it is of degree 1. Thus $3\mathbb{Z}_K = \mathfrak{p}_1^9$ with residue degree 1. Hence $\nu_3(i(K)) = 0$.
  (ii) If $d = 3$, then $-\lambda_1 = -1/3, -2/3$ is the slope of S_1. Since $e_1 = 3$ divides the the ramification index of any prime ideal of $\mathbb{Z}_K$ lying above 3, then there is at most three prime ideals of $\mathbb{Z}_K$ lying above 3 with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.

If $8\nu_3(b) > 9\nu_3(a)$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, \nu_3(b))$, $(1, \nu_3(a))$, and $(9,0)$ with $d(S_1) = 1$ and $d(S_2) \in \{1,2,4\}$ since $\nu_3(a) \leq 7$. Let $d = d(S_2)$. Then we have the following cases:
  (i) If $d = 1$; $\nu_3(a) \in \{1,3,5,7\}$, then $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^8$ with residue degree 1 each ideal factor. Hence $\nu_3(i(K)) = 0$.
  (ii) If $d = 2$, then $R_{1_2}(F)(y) = a_3y^2 + b_3 \in \mathbb{F}_\phi[y]$; that is $R_{1_2}(F)(y) = \pm(y^2+1)$ or $R_{1_2}(F)(y) = \pm(y-1)(y+1)$. Thus $3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^4$ with $f_1 = 1$ and $f_2 = 2$ or $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{21}^4\mathfrak{p}_{22}^4$ with $f_{11} = f_{21} = f_{22} = 1$ respectively. Hence $\nu_3(i(K)) = 0$.
  (iii) If $d = 4$, then $R_{1_2}(F)(y) = a_3y^4 + b_3 \in \mathbb{F}_\phi[y]$; that is $R_{1_2}(F)(y) = \pm(y^4+1) = \pm(y^2 - y - 1)(y^2 + y - 1)$ or $R_{1_2}(F)(y) = \pm(y^4 - 1) = \pm(y-1)(y+1)(y^2+1)$. Thus $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{21}^2\mathfrak{p}_{22}^2$ with $f_{11} = 1$ and $f_{21} = f_{22} = 2$ or $3\mathbb{Z}_K = \mathfrak{p}_{11}\mathfrak{p}_{21}^2\mathfrak{p}_{22}^2\mathfrak{p}_{23}^2$ with $f_{11} = f_{21} = f_{22} = 1$ and $f_{23} = 2$ respectively. Hence $\nu_3(i(K)) = 0$.

□

TABLE 9

Cases		$3\mathbb{Z}_K$	f_i	$\nu_3(i(K))$
$(a,b) \in \{(3,4),(6,7)\} \pmod 9$		$\mathfrak{p}_1\mathfrak{p}_2^8$	$f_i = 1$	0
$(a,b) \in \{(0,10),(0,19),(9,1),(9,19),(18,1),(18,10)\} \pmod{27}$		$\mathfrak{p}_1^3\mathfrak{p}_2^6$	$f_i = 1$	
$(a,b) \in \{(0,1),(9,10)\} \pmod{27}$		$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^6$	$f_i = 1$	
$(a,b) \in \{(18,46),(18,73),(45,19),(45,73),(72,19),(72,46)\} \pmod{81}$		$\mathfrak{p}_1^3\mathfrak{p}_2^6$	$f_i = 1$	
$(a,b) \equiv (18,19) \pmod{81}$ and $b-a \equiv 82 \pmod{243}$		$\mathfrak{p}_1\mathfrak{p}_2^6$	$f_1 = 3$, $f_2 = 1$	
$(a,b) \equiv (18,19) \pmod{81}$ and $b-a \equiv 163 \pmod{243}$		$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^6$	$f_1 = f_3 = 1$, $f_2 = 2$	
$(a,b) \equiv (18,19) \pmod{81}$ and $b-a \equiv 1 \pmod{243}$	$\nu_3(\Delta)$ is odd	$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$	$f_i = 1$	
	$\nu_3(\Delta)$ is even, $\Delta_3 \equiv 1 \pmod 3$	$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$	$f_1 = 2$, $f_2 = f_3 = 1$	
	$\nu_3(\Delta)$ is even, $\Delta_3 \equiv -1 \pmod 3$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^6$	$f_i = 1$	1
$(a,b) \equiv (45,46) \pmod{81}$ and $b-a \equiv 163 \pmod{243}$		$\mathfrak{p}_1\mathfrak{p}_2^6$	$f_1 = 3$, $f_2 = 1$	0
$(a,b) \equiv (45,46) \pmod{81}$ and $b-a \equiv 1 \pmod{243}$		$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^6$	$f_1 = f_3 = 1$, $f_2 = 2$	
$(a,b) \equiv (45,46) \pmod{81}$ and $b-a \equiv 82 \pmod{243}$	$\nu_3(\Delta)$ is odd	$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$	$f_i = 1$	
	$\nu_3(\Delta)$ is even, $\Delta_3 \equiv 1 \pmod 3$	$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$	$f_1 = 2$, $f_2 = f_3 = 1$	
	$\nu_3(\Delta)$ is even, $\Delta_3 \equiv -1 \pmod 3$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^6$	$f_i = 1$	1
$(a,b) \equiv (72,73) \pmod{81}$ and $b-a \equiv 82 \pmod{243}$		$\mathfrak{p}_1\mathfrak{p}_2^6$	$f_1 = 3$, $f_2 = 1$	0
$(a,b) \equiv (72,73) \pmod{81}$ and $b-a \equiv 163 \pmod{243}$		$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^6$	$f_1 = f_3 = 1$, $f_2 = 2$	
$(a,b) \equiv (72,73) \pmod{81}$ and $b-a \equiv 1 \pmod{243}$	$\nu_3(\Delta)$ is odd	$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$	$f_i = 1$	
	$\nu_3(\Delta)$ is even, $\Delta_3 \equiv 1 \pmod 3$	$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3^6$	$f_1 = 2$, $f_2 = f_3 = 1$	
	$\nu_3(\Delta)$ is even, $\Delta_3 \equiv -1 \pmod 3$	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^6$	$f_i = 1$	1

Proof of Theorem 2.4.
For $p = 5$. Since $\Delta = 2^{24}a^9 + 3^{18}b^8$, $\nu_5(\Delta) \geq 1$ if and only if $(a,b) \in \{(0,0),(1,1),(1,2),(1,3),(1,4)\} \pmod 5$. Thanks to the index formula (1.1), 5 can divides the index $i(K)$ only if $(a,b) \in \{(0,0),(1,1),(1,2),(1,3),(1,4)\} \pmod 5$.

(1) For $(a,b) \in \{(1,1),(1,2),(1,3),(1,4)\} \pmod 5$, one can easily check that $F(x) \equiv \phi_{i1} \cdot \phi_{i2}^2 \pmod 5$ with $\deg(\phi_{i1}) = 7$, $\deg(\phi_{i2}) = 1$, and ϕ_{ij} is irreducible over $\mathbb{F}_5$ for every $i = 1,\ldots,4$ and $j = 1,2$. Thus there is at most two prime ideals of $\mathbb{Z}_K$ lying above 5 with residue degree 1 each ideal factor. Hence $\nu_5(i(K)) = 0$.

(2) If $(a,b) \equiv (0,0) \pmod 5$, then $F(x) \equiv x^9 \pmod 5$. Let $\phi = x$. Then $F(x) = \phi^9 + a\phi + b$.

   (i) If $8\nu_5(b) < 9\nu_5(a)$, then $N_\phi(F) = S_1$ has a single side joining $(0,\nu_5(b))$ and $(9,0)$ with degree $d \in \{1,3\}$.

      (a) If $d = 1$, then $R_{1_1}(F)(y)$ is irreducible as it is of degree 1. Thus $5\mathbb{Z}_K = \mathfrak{p}_1^9$ with residue degree 1. Hence $\nu_5(i(K)) = 0$.

      (b) If $d = 3$, then $R_{1_1}(F)(y) = y^3 + b_5 \in \mathbb{F}_\phi[y]$. One can check easily that, for every value $b_5 \in \mathbb{F}_\phi^*$, $R_{1_1}(F)(y) = \psi_1 \cdot \psi_2$ with $\deg(\psi_1) = 1$, $\deg(\psi_2) = 2$, and ψ_i is irreducible over $\mathbb{F}_\phi$. Thus $5\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2^3$ with $f_1 = 1$ and $f_2 = 2$. Hence $\nu_5(i(K)) = 0$.

   (ii) If $8\nu_5(b) > 9\nu_5(a)$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, \nu_5(b))$, $(1, \nu_5(a))$, and $(9, 0)$ with $d(S_1) = 1$ and $d(S_2) \in \{1, 2, 4\}$ since $\nu_5(a) \le 7$. Thus ϕ can provides at most five prime ideal of $\mathbb{Z}_K$ lying above 5 with residue degree 1 each ideal factor. Hence $\nu_5(i(K)) = 0$.

For $p = 7$. Since $\Delta = 2^{24}a^9 + 3^{18}b^8$, $\nu_7(\Delta) \ge 1$ if and only if $(a, b) \in \{(0, 0), (3, 1), (3, 6), (5, 1), (5, 6), (6, 1), (6, 6)\} \pmod 7$. Thanks to the index formula (1.1), 7 can divides the index $i(K)$ only if $(a, b) \in \{(0, 0), (3, 1), (3, 6), (5, 1), (5, 6), (6, 1), (6, 6)\} \pmod 7$.

1. For $(a, b) \in \{(3, 1), (3, 6), (5, 1), (5, 6), (6, 1), (6, 6)\} \pmod 7$, one can easily check that $F(x) \equiv \phi_{i1} \cdot \phi_{i2} \cdot \phi_{i3}^2 \pmod 7$ with $\deg(\phi_{i1}) = 4$, $\deg(\phi_{i2}) = 3$, $\deg(\phi_{i2}) = 1$, and ϕ_{ij} is irreducible over $\mathbb{F}_7$ for every $i = 1, \ldots, 6$ and $j = 1, 2, 3$. Thus there is at most two prime ideals of $\mathbb{Z}_K$ lying above 7 with residue degree 1 each ideal factor. Hence $\nu_7(i(K)) = 0$.
2. If $(a, b) \equiv (0, 0) \pmod 7$, then $F(x) \equiv x^9 \pmod 7$. Let $\phi = x$. Then $F(x) = \phi^9 + a\phi + b$.
   - (i) If $8\nu_7(b) < 9\nu_7(a)$, then $N_\phi(F) = S_1$ has a single side joining $(0, \nu_7(b))$ and $(9, 0)$ with degree $d \in \{1, 3\}$.
     - (a) If $d = 1$, then $R_{1_1}(F)(y)$ is irreducible as it is of degree 1. Thus $7\mathbb{Z}_K = \mathfrak{p}_1^9$ with residue degree 1. Hence $\nu_7(i(K)) = 0$.
     - (b) If $d = 3$, then $R_{1_1}(F)(y) = y^3 + b_5 \in \mathbb{F}_\phi[y]$. Thus there is at most three prime ideals of $\mathbb{Z}_K$ lying above 7 with residue degree 1 each ideal factor. Hence $\nu_7(i(K)) = 0$.
   - (ii) If $8\nu_7(b) > 9\nu_7(a)$, then $N_\phi(F) = S_1 + S_2$ has two sides joining $(0, \nu_7(b))$, $(1, \nu_7(a))$, and $(9, 0)$ with $d(S_1) = 1$ and $d(S_2) \in \{1, 2, 4\}$ since $\nu_7(a) \le 7$. Thus ϕ can provides at most five prime ideal of $\mathbb{Z}_K$ lying above 7 with residue degree 1 each ideal factor. Hence $\nu_7(i(K)) = 0$.

For $p \ge 11$, since there is at most 9 prime ideals of $\mathbb{Z}_K$ lying above p with residue degree 1 each, and there is at least $p \ge 11$ monic irreducible polynomial of degree f in $\mathbb{F}_p[x]$ for every positive integer f, we conclude that p does not divide $i(K)$.

□

5. EXAMPLES

Let $F(x) = x^9 + ax + b \in \mathbb{Z}[x]$ be a monic irreducible polynomial and K the nonic number field generated by a complex root of $F(x)$.

1. For $a = 51$ and $b = 122$, we have $(a, b) \equiv (3, 2) \pmod 4$, $(a, b) \equiv (6, 5) \pmod 9$, and for every rational prime $p \notin \{2, 3\}$, $\nu_p(\Delta) \le 1$. By Theorem 2.1, $\mathbb{Z}[\alpha]$ is integrally closed and so K is monogenic. Hence $i(K) = 1$.
2. For $a = 35$ and $b = 20$, we have $(a, b) \equiv (3, 4) \pmod 8$, then by Theorem 2.2, $i(K)$ is even. Hence K is not monogenic.
3. For $a = 1392$ and $b = 768$, we have $(a, b) \equiv (368, 256) \pmod{512}$, then by Theorem 2.2, $\nu_2(i(K)) = 1$. On the other hand, $F(x)$ is 3-Eisenstein, then $\nu_3(i(K)) = 0$. We conclude that $i(K) = 2$. Hence K is not monogenic.
4. For $a = 126$ and $b = 40130$, we have $(a, b) \equiv (45, 35) \pmod{81}$, $a + b \equiv 161 \pmod{243}$, $\nu_3(\Delta) = 26$, and $\Delta_3 \equiv -1 \pmod 3$, then by Theorem 2.3,

$\nu_3(i(K)) = 1$. On the other hand, $F(x)$ is 2-Eisenstein, then $\nu_2(i(K)) = 0$. We conclude that $i(K) = 3$. Hence K is not monogenic.

(5) For $a = 15381$ and $b = 6634$, we have $(a, b) \equiv (1, 2) \pmod 4$, then by Theorem 2.2, $\nu_2(i(K)) = 1$. On the other hand, $(a, b) \equiv (72, 73) \pmod{81}$, $b - a \equiv 1 \pmod{243}$, $\nu_3(\Delta) = 24$, and $\Delta_3 \equiv -1 \pmod 3$, then by Theorem 2.3, $\nu_3(i(K)) = 1$. We conclude that $i(K) = 6$. Hence K is not monogenic.

(6) For $a = 183$ and $b = 296$, we have $(a, b) \equiv (7, 8) \pmod{16}$ and $\nu_2(\Delta) = 29$, then by Theorem 2.2, $\nu_2(i(K)) = 3$. On the other hand, $(a, b) \equiv (21, 53) \pmod{81}$, then by Theorem 2.3, $\nu_3(i(K)) = 0$. We conclude that $i(K) = 8$. Hence K is not monogenic.

(7) For $a = 7335$ and $b = 24184$, we have $(a, b) \equiv (7, 8) \pmod{16}$, $\nu_2(\Delta) = 28$, and $\Delta_2 \equiv 3 \pmod 8$, then by Theorem 2.2, $\nu_2(i(K)) = 3$. On the other hand, $(a, b) \equiv (45, 46) \pmod{243}$, $b - a \equiv 82 \pmod{243}$, $\nu_3(\Delta) = 24$, and $\Delta_3 \equiv -1 \pmod 3$, then by Theorem 2.3, $\nu_3(i(K)) = 1$. We conclude that $i(K) = 24$. Hence K is not monogenic.

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References

[1] H. Cohen, *A Course in Computational Algebraic Number Theory*, GTM 138, Springer-Verlag Berlin Heidelberg, (1993).

[2] C. T. Davis and B. K. Spearman, *The index of a quartic field defined by a trinomial* $x^4 + ax + b$, J. Algera Appl., 17(10) (2018), 185–197.

[3] R. Dedekind, *Über den Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Kongruenzen*, Göttingen Abhandlungen, **23** (1878), 1–23.

[4] L. El Fadil, *On common index divisors and monogenity of certain number fields defined by* $x^5 + ax^2 + b$, Commun. Algebra, (2022), doi: 10.1080/00927872.2022.2025820

[5] L. El Fadil and I. Gaál, *On non-monogenity of certain number fields defined by trinomials* $x^4 + ax^2 + b$, arXiv:2204.03226

[6] L. El Fadil and O. Kchit, *On index divisors and monogenity of certain sextic number fields defined by* $x^6 + ax^5 + b$, arXiv:2206.05529

[7] L. El Fadil and O. Kchit, *On index divisors and monogenity of certain septic number fields defined by* $x^7 + ax^3 + b$, Commun. Algebra, (2022), 1–15. doi: 10.1080/00927872.2022.2159035

[8] L. El Fadil and O. Kchit, *On index divisors and monogenity of certain number fields defined by* $x^{12} + ax^m + b$, Ramanujan J., (2023).(To appear)

[9] L. El Fadil, J. Montes, and E. Nart, *Newton polygons and p-integral bases of quartic number fields*, J. Algebra Appl., 11(4) (2012), 1250073.

[10] H. T. Engstrom, *On the common index divisor of an algebraic number field*, Trans. Amer. Math. Soc., 32 (1930), 223–237.

[11] I. Gaál, Diophantine equations and power integral bases, Theory and algorithm, Second edition, Boston, Birkhäuser, (2019).

[12] I. Gaál, A. Pethö, and M. Pohst, *On the indices of biquadratic number fields having Galois group V_4*, Arch. Math., 57 (1991), 357–361.

[13] J. Guàrdia, J. Montes, and E. Nart, *Newton polygons of higher order in algebraic number theory*, Trans. Amer. Math. Soc., **364**(1) (2012), 361–416.

[14] J. Guàrdia and E. Nart, *Genetics of polynomials over local fields*, Contemp. Math., 637 (2015), 207–241.

[15] K. Hensel, *Arithmetische Untersuchungen über die gemeinsamen ausserwesentlichen Discriminantentheiler einer Gattung*, J. Reine Angew. Math., 113 (1894), 128–160. ISSN 0075-4102. doi: 10.1515/crll.1894.113.128.

[16] K. Hensel, *Theorie der algebraischen Zahlen*, Teubner Verlag, Leipzig, Berlin, (1908).

[17] A. Jakhar, *On nonmonogenic algebraic number fields*, Rocky Mt. J. Math. 53(1) (2023), 103–110.

[18] A. Jakhar, S. Khanduja, and N. Sangwan, *Characterization of primes dividing the index of a trinomial*, Int. J. Number Theory, 13(10) (2017), 2505–2514.

[19] L. Jones and D. White, *Monogenic trinomials with non-squarefree discriminant*, Int. J. Math., 32(13) (2021). ID: 2150089, 21 p.

[20] S. Maclane, *A construction for absolute values in polynomial rings*, Trans. Amer. Math. Soc., 40 (1936), 363–395.

[21] J. Montes and E. Nart, *On a theorem of Ore*, J. Algebra, **146**(2) (1992), 318–334.

[22] T. Nakahara, *On the indices and integral bases of non-cyclic but abelian biquadratic fields*, Arch. Math., 41(6) (1983), 504-508.

[23] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Springer Verlag, 3. Auflage, (2004).

[24] E. Nart, *On the index of a number field*, Trans. Amer. Math. Soc., 289 (1985), 171-183.

[25] J. Neukirch, *Algebraic Number Theory*, Springer-Verlag, Berlin, (1999).

[26] O. Ore, *Newtonsche Polygone in der Theorie der algebraischen Korper*, Math. Ann., 99 (1928), 84–117.

[27] J. Śliwa, *On the nonessential discriminant divisor of an algebraic number field*. Acta Arith. 42 (1982), 57-72.

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