IDENTIFYING AN UNKNOWN SPACE-DEPENDENT SOURCE TERM IN A MULTI-TERM TIME-SPACE FRACTIONAL DIFFUSION EQUATION

Y. X. YANG AND Y. S. LI*

ABSTRACT. This paper studies the source identification problem only dependent on space for one-dimensional multi-term time-space fractional diffusion equation(TSFDE) using final value data. We first establish the existence, uniqueness, and regularity of the solution of the direct problem, then transform the inverse problem into an operator equation and prove the important property of the operator . Utilizing the analytical Fredholm theorem, we prove that the source term can be uniquely and dependends continuously determined by additional final value data. Finally, we transform the inverse problem into a variational problem using the Tikhonov regularization method, and provide an approximate solution to the inverse problem using the optimal perturbation algorithm. Numerical examples demonstrate the effectiveness and stability of the algorithm.

1. INTRODUCTION

The classical integer-order partial differential equations and its inverse problems have yielded abundant results and extensive applications. However, with the progress of human civilization, anomalous diffusion phenomena in nature continue to emerge, such as the diffusion of smog, porous mediums, and viscoelastic fluids. It has been found that using classical differential equation models cannot accurately describe these phenomena. Since the beginning of the 21st century, with the establishment of models based on fractional diffusion equations in various fields such as anomalous diffusion, porous media mechanics, non-Newtonian fluid mechanics, and viscoelastic material mechanics, there has been a

Key words and phrases. Inverse space source problem; Multi-term time-space fractional diffusion equation; Optimal perturbation algorithm.

* Corresponding author. This paper was supported by the Natural Science Foundation of Guangxi (No. 2023GXNSFAA026284) and Special Fund for Science and Technological Bases and Talents of Guangxi (No. Guike AD20159058) .

12 Nov 2023 08:11:47 PST
230629-LiYS Version 2 - Submitted to Rocky Mountain J. Math.

strong interest in these models due to their significant application value. Moreover, fractional derivatives can describe non-uniform materials with memory and hereditary properties. Therefore, compared with integer-order diffusion equations, fractional-order diffusion equations are more effective in describing anomalous diffusion phenomena.

However, in some practical problems, some boundary data, or initial data, or diffusion coefficients, or source terms are not given, We aim to solve for these unknown quantities by utilizing additional data, which has led to the emergence of inverse problems for fractional differential equations. In recent years, numerous researchers have conducted extensive studies on inverse problems for fractional diffusion equations, such as the study on the uniqueness of the inverse problem [1, 20, 30, 9, 13, 6], on the numerical computations of the inverse problem [15, 32] .

In this paper, consider the multi-term TSFDE given by:

$$\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,t) = -(-\Delta)^{\frac{\gamma}{2}} u(x,t) + h(x)r(t), \qquad x \in \Omega, t \in (0,T], \tag{1.1}$$

where $\Omega = (0,1)$, a_i $(i = 1;\dots;m)$ be positive constants and $\partial_{0+}^{\beta_i}$ denotes the Caputo fractional left-sided derivative of order $\beta_i (i = 1, 2, \cdots, m)(0 < \beta_m < \cdots < \beta_1 < 1)$:

$$\partial_{0+}^{\beta_i} u(x,t) = \frac{1}{\Gamma(1-\beta_i)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\beta_i}}, \qquad 0 < \beta_i < 1, \quad 0 < t \le T,$$

where Γ is the Gamma function and $T > 0$ is a fixed final time.

The fractional Laplacian operator $(-\Delta)^{\frac{\gamma}{2}}$ is defined for γ $(1 < \gamma \le 2)$ with the spectral decomposition for the Laplace operator $-\Delta$ (see [22, 23, 24]). We defined as the Hilbert space:

$$H_0^{\gamma}(\Omega) := \left\{ u = \sum_{n=1}^{\infty} b_n \phi_n : \|u\|^2_{H_0^{\gamma}(\Omega)} = \sum_{n=1}^{\infty} b_n^2 \bar{\lambda}_n^{\ \gamma} < \infty \right\}.$$

and the operator $(-\Delta)^{\frac{\gamma}{2}}$ by

$$(-\Delta)^{\frac{\gamma}{2}} u := \sum_{n=1}^{\infty} b_n \phi_n \mapsto \sum_{n=1}^{\infty} b_n \bar{\lambda}_n^{\ \gamma/2} \phi_n,$$

which maps $H_0^{\gamma}(\Omega)$ onto $L^2(\Omega)$, where $\overline{\lambda}_n$ and ϕ_n are the appropriate eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ with

$$\begin{cases} -\Delta\phi_n = \overline{\lambda}_n \phi_n, & in \quad \Omega, \\ \phi_n = 0, & on \quad \partial\Omega. \end{cases}$$

Hence we set

$$\|u\|_{H_0^{\gamma}(\Omega)} = \|(-\Delta)^{\frac{\gamma}{2}}u\|_{L^2(\Omega)}.$$

Suppose the unknown function u(x,t) satisfies the following initial and boundary conditions:

$$u(x,0) = 0, \qquad x \in \overline{\Omega}, \tag{1.2}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T]. \tag{1.3}$$

If all functions $h(x)$, $r(t)$ are given appropriately, the problem (1.1)-(1.3) is a direct problem. The inverse problem is to identify the space source term $h(x)$ for problem (1.1)

$$u(x,T) = v(x), \qquad x \in \Omega. \tag{1.4}$$

The source identification problem is a hot point in the current scientific research field, which has a wide range of applications in various fields such as medicine and geological exploration. In classical elliptic and parabolic equations, the source identification problem has been extensively studied and has rich research results. The source identification problem of fractional diffusion equations mainly focuses on the source identification problem of time-fractional diffusion equations, and most literature always assumes that the source term has a separable variable form, and the source term that only depends on spatial variables or time variables is inverted through different measurement data.

In recent years, there have been significant achievements in the research on direct problems of multiterm fractional diffusion equations. However, there is still a lack of extensive attention on the inverse problems, which are mainly concentrated on the inverse problems of multi-term time fractional diffusion equations. For example, Li and Yamamoto proved the uniqueness of identifying the order of two types of multi-term time-fractional diffusion equations in [12]. Li, Imanuvilov and Yamamoto considered the inverse boundary value problem of multi-term time-fractional diffusion equations in [10]. Liu proved the extremum principle of multi-term time-fractional diffusion equations, and gave the uniqueness of identifying the source term in paper[16], but did not provide a numerical method. Sun, Li and Jia simultaneously inverted the diffusion coefficient and source term that only depend on space of multi-term time-fractional diffusion equations using internal observation data, and gave the algorithm and numerical examples in [21]. In paper [14], Li, Huang and Yamamoto researched the initial-boundary value problem of multi-term time-fractional diffusion equations with coefficients that only depend on x, and gave the uniqueness result of identifying the order of multi-term time-fractional diffusion equations using observation data at one internal point. Jiang and Wu studied the identification of the zero-order term that only depends on time in multi-term time-fractional diffusion equations, gave the existence uniqueness of the direct problem and the uniqueness of the inverse problem solution, and used the Levenberg-Marquardt algorithm to give the numerical approximation solution of the zero-order term in [5]. There have also been significant achievements in research on the direct problems of fractional parabolic equations. For example, Tuan et al. in [27] study of the continuity problem is examined in both the linear and nonlinear cases by an order of derivative for conformable parabolic equations. Tuan et al. in [26] investigates the solution of the Kirchhoff parabolic equation involving the Caputo-Fabrizio fractional derivative with a non-singular kernel, the mild solution of equation operators which are defined via Fourier series, the existence, uniqueness and continuity of the mild solution with respect to the derivative order are established. Wang et al. in [28] considers a time-fractional wave equation with an exponential growth source function and proves the local existence and uniqueness of weak solutions in the Orlicz space. Further, it demonstrates the existence of small data solutions exist globally over time in the Orlicz space and the second global-in-time existence of solutions in Besov spaces.

For the time-space fractional diffusion equation, Tatar et al. discussed the inverse space source term and identified simultaneously the orders of the time-space fractional derivatives of the diffusion equation in [22, 24, 23]. Tuan and Long used the truncated Fourier method to invert the space-dependent source term and provided convergence estimates and rules for choosing the regularization parameter in [25], but without numerical example is provided. Regarding the inverse problem of multi-term time-space fractional diffusion equations. Malik et al. [17] studied the identification of the source term and diffusion coefficient that only depend on time in multi-term time-space fractional diffusion equations, where the spatial derivative is a Caputo fractional derivative and the time derivative is a Hilfer fractional derivative, and the additional data is non-local. The uniqueness result of the inverse problem was given, but no algorithm or numerical example was constructed.

In this paper, we study uniqueness for the identification of a space-dependent source term for the multi-term TSFDE on the additional final value data. As far as we know, it is the first issue for the investigation of a space-dependent source term of the multi-term TSFDE.

2. Preliminary

Throughout this paper, we use the following definitions and propositions given in [11, 29].

Definition 2.1. (See [11]) The multinomial Mittag-Leffler function is

$$E_{(\beta_1,\ldots,\beta_m),\beta}(w_1,\ldots,w_m)=\sum_{k=0}^{\infty}\sum_{l_1+\cdots+l_m=l}\frac{(l;l_1,\ldots,l_m)\prod_{i=1}^{m}w_i^{l_i}}{\Gamma(\beta+\sum_{i=1}^{m}\beta_i l_i)},$$

where $0<\beta<2$, $0<\beta_i<1$ and $w_i\in\mathbb{C}(i=1,\ldots,m)$. Here $(l;l_1,\ldots,l_m)$ denotes the multi-term coefficient

$$(l;l_1,\ldots,l_m):=\frac{l!}{l_1!\cdots l_m!}\quad\text{with}\quad l=\sum_{i=1}^{m}l_i,$$

where $l_i(1\le i\le m)$ are non-negative intrgers.

Proposition 2.2. *(See [11]) Let $0<\beta<2$ and $0<\beta_m<\cdots<\beta_1<1$ be arbitrary. Assume that $\beta_1\pi/2<\eta<\beta_1\pi$, $\eta\le|\arg(w_1)|\le\pi$ and there exists $S>0$ such that $-S\le w_i<0(i=2,\ldots,m)$. Then there exists a constant $C=C(\beta_i(i=2,\ldots,m),\beta,\eta,S)>0$ such that*

$$|\,E_{(\beta_1,\beta_1-\beta_2,\cdots,\beta_1-\beta_m),\beta}(w_1,\ldots,w_m)\,|\le\frac{C}{1+|\,w_1\,|},\qquad\eta\le|\arg(w_1)|\le\pi.$$

For later use, we adopt the abbreviation

$$E^{(n)}_{\beta',\beta}(t):=E_{(\beta_1,\beta_1-\beta_2,\cdots,\beta_1-\beta_m),\beta}(-\lambda_n t^{\beta_1},-a_2t^{\beta_1-\beta_2},\ldots,-a_mt^{\beta_1-\beta_m}),\qquad t>0$$

where $\lambda_n=\overline{\lambda}_n^{\frac{\gamma}{2}}$ is the nth eigenvalue of $(-\Delta)^{\frac{\gamma}{2}}$ $(1<\gamma\le 2)$; $0<\beta<2$, and β_i, a_i are those positive constants in (1.1).

Lemma 2.3. *Let $0<\beta<2$ and $0<\beta_m<\cdots<\beta_1<1$ be arbitrary. Assume that $\beta_1\pi/2<\eta<\beta_1\pi$, $\eta\le|\arg(w_1)|\le\pi$ and there exists $S>0$ such that $-S\le w_i<0(i=2,\ldots,m)$. Then exists a constant $C=C(\beta_i(i=1,\ldots,m),\eta,S)>0$ such that*

$$\frac{d^q}{da_i^q}E^{(n)}_{\beta',\beta_1-\beta_i}(t)\le\frac{Cq!a_i^{-q}}{1+|w_1|^{q+1}}$$

for any fixed a_i $(i=2,\ldots,m)$.

Proof. At first, we rewrite the multinomial Mittag-Leffler function by using the results in [11]:

$$E_{\beta',\gamma}(w_1,\ldots,w_m) = \frac{1}{2\beta_1\pi i}\int_{\kappa(R,\theta)} exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\gamma}{\beta_1}-1}\sum_{h=0}^{\infty}\left(\frac{w_1}{\varphi}+\sum_{l=2}^{m}\frac{w_l}{\varphi^{1-\beta_l/\beta_1}}\right)^h d\varphi. \tag{2.1}$$

We substitute $\gamma = \beta_1 - \beta_i$, $w_1 = -\lambda_n t^{\beta_1}$, $w_l = -a_l t^{\beta_1-\beta_l}$ $(l = 2,\ldots,m)$ in (2.1) to deduce

$$\begin{aligned} E^{(n)}_{\beta',\beta_1-\beta_i}(t) &= E_{\beta',\beta_1-\beta_i}\left(-\lambda_n t^{\beta_1}, -a_2 t^{\beta_1-\beta_2},\ldots,-a_m t^{\beta_1-\beta_m}\right) \\ &= \frac{1}{2\beta_1\pi i}\int_{\kappa(R,\theta)} exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i}{\beta_1}-1}\sum_{h=0}^{\infty}\left(\frac{-\lambda_n t^{\beta_1}}{\varphi}+\sum_{l=2}^{m}\frac{-a_l t^{\beta_1-\beta_l}}{\varphi^{1-\beta_l/\beta_1}}\right)^h d\varphi. \end{aligned} \tag{2.2}$$

Fix any $a_i \in \{a_2,\ldots,a_m\}$. For the q-times derivative of a_i in equation 2.2, we deduce

$$\begin{aligned} &\frac{d^q}{da_i^q}E^{(n)}_{\beta',\beta_1-\beta_i}(t) \\ = &\frac{q!}{2\beta_1\pi i}\int_{\kappa(R,\theta)} exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i}{\beta_1}-1}\sum_{h=0}^{\infty}\frac{(h+q)!}{h!q!}\left(\frac{-\lambda_n t^{\beta_1}}{\varphi}+\sum_{l=2}^{m}\frac{-a_l t^{\beta_1-\beta_l}}{\varphi^{1-\beta_l/\beta_1}}\right)^h\left(\frac{-t^{\beta_1-\beta_i}}{\varphi^{1-\beta_i/\beta_1}}\right)^q d\varphi \\ = &\frac{q!a_i^{-q}}{2\beta_1\pi i}\int_{\kappa(R,\theta)} exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i}{\beta_1}-1}\left(1-\frac{-\lambda_n t^{\beta_1}}{\varphi}-\sum_{l=2}^{m}\frac{-a_l t^{\beta_1-\beta_l}}{\varphi^{1-\beta_l/\beta_1}}\right)^{-q-1}\left(\frac{-a_i t^{\beta_1-\beta_i}}{\varphi^{1-\beta_i/\beta_1}}\right)^q d\varphi \\ = &\frac{q!a_i^{-q}}{2\beta_1\pi i}\int_{\kappa(R,\theta)} exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i}{\beta_1}}\left(\varphi+\lambda_n t^{\beta_1}+\sum_{l=2}^{q}a_l t^{\beta_1-\beta_l}\varphi^{\beta_l/\beta_1}\right)^{-q-1}\left(-a_i t^{\beta_1-\beta_i}\varphi^{\beta_i/\beta_1}\right)^q d\varphi \\ = &\frac{q!a_i^{-q}w_i^q}{2\beta_1\pi i}\int_{\kappa(R,\theta)}\frac{exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i+q\beta_i}{\beta_1}}}{\left(\varphi-w_1-\sum_{l=2}^{m}w_l\varphi^{\beta_l/\beta_1}\right)^{q+1}}d\varphi. \end{aligned}$$

We refer to the proof results of Lemma 2.3 in [11]. We have the estimate

$$\begin{aligned} \left|\frac{d^q}{da_i^q}E^{(n)}_{\beta',\beta_1-\beta_i}(t)\right| &= \frac{q!a_i^{-q}|w_i|^q}{2\beta_1\pi i}\left|\int_{\kappa(R,\theta)}\frac{exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i+q\beta_i}{\beta_1}}}{\left(\varphi-w_1-\sum_{l=2}^{m}w_l\varphi^{\beta_l/\beta_1}\right)^{q+1}}d\varphi\right| \\ &\le \frac{q!a_i^{-q}}{|w_1|^{q+1}}\left(\frac{|w_i|^q}{2\beta_1\pi i}\int_{\kappa(R,\theta)} exp\left(\varphi^{\frac{1}{\beta_1}}\right)\varphi^{\frac{1-\beta_1+\beta_i+q\beta_i}{\beta_1}}d\varphi\right) \\ &\le \frac{Cq!a_i^{-q}}{|w_1|^{q+1}}, \qquad |w_1| > R, \end{aligned}$$

and on the other hand, by $|w_1| \leq R$ estimate can be directly verified that

$$\begin{aligned}\left|\frac{d^q}{da_i^q}E^{(n)}_{\beta',\beta_1-\beta_i}(t)\right| &\leq Cq!a_i^{-q}|w_i|^q\sum_{h=0}^{\infty}\frac{\frac{(h+q)!}{h!q!}\left(|w_1|+\sum\limits_{l=1}^{m}|w_l|\right)^h}{\Gamma\left(\beta_1-\beta_i+(\beta_1-\beta_2)(h-q)\right)}\\ &\leq Cq!a_i^{-q},\end{aligned}$$

the constants C above are positive constants depending on β_j $(j = 1, 2, \ldots, m)$, η, S. The proof is completed. □

Proposition 2.4. *(See* [11]*) Let* $1 > \beta_1 > \cdots > \beta_m > 0$*, then we have*

$$\frac{d}{dt}\{t^{\beta_1}E^{(n)}_{\beta',1+\beta_1}(t)\} = t^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t), \quad t > 0.$$

Definition 2.5. (See [29]) If $z(t) \in L(0, T)$, then for $\alpha > 0$ the RiemannCLiouville fractional left integral $I^{\alpha}_{0+}f$ and right integral $I^{\alpha}_{T-}f$ are defined by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)ds}{(t-s)^{1-\alpha}}, \quad 0 < t \leq T,$$

and

$$I^{\alpha}_{T-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^T\frac{f(s)ds}{(s-t)^{1-\alpha}}, \quad 0 \leq t < T.$$

Definition 2.6. (See [29]) Let $z(t) \in AC[0, T]$, then for $0 < \alpha < 1$ the Caputo fractional left derivative $\partial^{\alpha}_{0+}y(t)$ and right derivative $\partial^{\alpha}_{T-}y(t)$ of order α are defined by

$$\partial^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{y'(s)ds}{(t-s)^{\alpha}} =: (I^{1-\alpha}_{0+}y')(t), \quad 0 < t \leq T,$$

and

$$\partial^{\alpha}_{T-}y(t) = -\frac{1}{\Gamma(1-\alpha)}\int_t^T\frac{y'(s)ds}{(s-t)^{\alpha}} =: (I^{1-\alpha}_{T-}y')(t), \quad 0 \leq t < T.$$

And for $0 < \alpha < 1$ the RiemannCLiouville fractional left derivative of order α is defined by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{y'(s)}{(t-s)^{\alpha}}ds =: \frac{d}{dt}(I^{1-\alpha}_{0+}y)(t), \quad 0 < t \leq T.$$

Proposition 2.7. *(See* [29]*) Let* $z(t) \in AC[0, T]$*. Then the Caputo fractional left derivative* $\partial^{\beta}_{0+}z(t)$ *and the Riemann-Liouville fractional left derivative* $D^{\beta}_{0+}z(t)$ *exist almost everywhere on* $(0, T]$*, such that*

$$\partial^{\beta}_{0+}z(t) = D^{\beta}_{0+}z(t) - \frac{z(0)}{\Gamma(1-\beta)}t^{-\beta}, \quad a.e. \quad t \in (0, T].$$

Lemma 2.8. *Assume $\lambda_n > 0$, and $r(t) \in AC[0,T]$. Then we have*

$$\partial_{0+}^{\beta_i} \int_0^t r(\tau)(t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau = \int_0^t r(\tau)(t-\tau)^{\beta_1-\beta_i-1} E^{(n)}_{\beta',\beta_1-\beta_i}(t-\tau)d\tau,$$

for any $0 < \beta_i < 1$ ($i = 1, \ldots, m$).

Proof. Denote $g(t) = \int_0^t r(\tau)(t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau$. Then for $0 < t \leq T$, according to Proposition 2.4, we get

$$\begin{aligned} |g(t)| &\leq \|r\|_{C[0,T]} \left| \int_0^t (t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau \right| \\ &\leq \|r\|_{C[0,T]} \frac{Ct^{\beta_1}}{1+\lambda_n t^{\beta_1}} \\ &\leq CT^{\beta_1} \|r\|_{C[0,T]} . \end{aligned}$$

Define $g(0) = 0$, then easily prove $g(t) \in C[0,T]$. According to $r(t) \in AC[0,T]$ and Proposition 2.4, it follows that

$$g(t) = r(0)t^{\beta_1} E^{(n)}_{\beta',\beta_1+1}(t) + \int_0^t r'(\tau)(t-\tau)^{\beta_1} E^{(n)}_{\beta',\beta_1+1}(t-\tau)d\tau.$$

Hence

$$g'(t) = r(0)t^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t) + \int_0^t r'(\tau)(t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau.$$

Consequently $g \in AC[0,T]$. Further, we have

$$\begin{aligned}
&\int_0^t \frac{g(\tau)}{(t-\tau)^{\beta_i}}d\tau \\
= &\int_0^t (t-\tau)^{-\beta_i}\int_0^\tau r(s)(\tau-s)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(\tau-s)dsd\tau \\
= &\int_0^t r(s)\int_s^t (t-\tau)^{-\beta_i}(\tau-s)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(\tau-s)d\tau ds \\
= &\int_0^t r(s)\int_s^t (t-\tau)^{-\beta_i}(\tau-s)^{\beta_1-1}\sum_{l=0}^{\infty}\sum_{l_1+\cdots+l_m=l}\frac{(l;l_1,\ldots,l_m)(-1)^l\lambda_n^{l_1}(\tau-s)^{\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h}}{\Gamma(\beta_1+\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h)}d\tau ds \\
= &\int_0^t r(s)\sum_{l=0}^{\infty}\sum_{l_1+\cdots+l_m=l}\frac{(l;l_1,\ldots,l_m)(-1)^l\lambda_n^{l_1}}{\Gamma(\beta_1+\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h)}\int_s^t (t-\tau)^{-\beta_i}(\tau-s)^{\beta_1-1+\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h}d\tau ds \\
= &\int_0^t r(s)\sum_{l=0}^{\infty}\sum_{l_1+\cdots+l_m=l}\frac{(l;l_1,\ldots,l_m)(-1)^l\lambda_n^{l_1}\Gamma(1-\beta_i)}{\Gamma(\beta_1+\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h-(\beta_i-1))}(t-s)^{\beta_1-1+\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h-(\beta_i-1)}ds \\
= &\,\Gamma(1-\beta_i)\int_0^t r(s)(t-s)^{\beta_1-\beta_i}\sum_{l=0}^{\infty}\sum_{l_1+\cdots+l_m=l}\frac{(l;l_1,\ldots,l_m)(-1)^l\lambda_n^{l_1}(t-s)^{\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)k_h}}{\Gamma(\beta_1-\beta_i+1+\beta_1 l_1+\sum\limits_{h=2}^{m}(\beta_1-\beta_h)l_h)}ds \\
= &\,\Gamma(1-\beta_i)\int_0^t r(\tau)(t-\tau)^{\beta_1-\beta_i}E^{(n)}_{\beta',\beta_1-\beta_i+1}(t-\tau)d\tau.
\end{aligned}$$

Therefore, by Proposition 2.7, we can conclude

$$\begin{aligned}
\partial_{0+}^{\beta_i}g(t) &= D_t^{\beta_i}g(t) = \frac{1}{\Gamma(1-\beta_i)}\frac{d}{dt}\int_0^t \frac{g(\tau)}{(t-\tau)^{\beta_i}}d\tau \\
&= \frac{1}{\Gamma(1-\beta_i)}\frac{d}{dt}\left[\Gamma(1-\beta_i)\int_0^t r(\tau)(t-\tau)^{\beta_1-\beta_i}E^{(n)}_{\beta',\beta_1-\beta_i+1}(t-\tau)d\tau\right] \\
&= \int_0^t r(\tau)(t-\tau)^{\beta_1-\beta_i-1}E^{(n)}_{\beta',\beta_1-\beta_i}(t-\tau)d\tau.
\end{aligned}$$

The lemma is proved. □

Lemma 2.9. *Assuming that $r(t) \in L^\infty(0,T)$, $1 > \beta_1 > \cdots > \beta_m > 0$, $\lambda_n \geq 0$, denote*

$$W_n(t) = \int_0^t r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau, \quad t \in (0,T] \tag{2.3}$$

and define $W_n(0) = 0$, then $W_n(t) \in C[0,T]$. Moreover, we can obtain the following estimates:

$$|W_n(t)| \leq C_1\frac{\| r \|_{L^\infty(0,T)}}{\lambda_n}, \tag{2.4}$$

$$|W_n(t)| \leq C_2 \| r \|_{L^\infty(0,T)}\,. \tag{2.5}$$

where $C_1, C_2 > 0$ *is a constant.*

Proof. Let's first prove the continuity of $W_n(t)$. Suppose $h > 0$, for any t, $t + h \in (0, T]$, we have

$$\begin{aligned}
|W_n(t+h) - W_n(t)| &= \left|\int_0^{t+h} r(\tau)(t+h-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t+h-\tau)d\tau - \int_0^t r(\tau)(t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\right| \\
&= \Big|\int_0^t r(\tau)[(t+h-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t+h-\tau) - (t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)]d\tau \\
&\quad + \int_t^{t+h} r(\tau)(t+h-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t+h-\tau)d\tau\Big| \\
&\leq \|r\|_{L^\infty(0,T)} \left|(t+h)^{\beta_1} E^{(n)}_{\beta',\beta_1+1}(t+h) - t^{\beta_1} E^{(n)}_{\beta',\beta_1+1}(t)\right|.
\end{aligned}$$

This means $\lim_{h\to 0^+} W_n(t+h) = W_n(t)$. By a similar deduction, we have $\lim_{h\to 0^-} W_n(t+h) = W_n(t)$. Therefore, we can obtain $W_n(t) \in C[0, T]$. Next, prove two estimates.

$$\begin{aligned}
|W_n(t)| &= \Big|\int_0^t r(\tau)(t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\Big| \\
&\leq \| r \|_{L^\infty(0,T)} \left|\int_0^t (t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\right| \\
&\leq \| r \|_{L^\infty(0,T)}\, t^{\beta_1} E^{(n)}_{\beta',\beta_1+1}(t) \\
&\leq \| r \|_{L^\infty(0,T)} \frac{C_1 t^{\beta_1}}{1+\lambda_n t^{\beta_1}} \\
&\leq C_1 \frac{\| r \|_{L^\infty(0,T)}}{\lambda_n},
\end{aligned}$$

and

$$\begin{aligned}
|W_n(t)| &= \left|\int_0^t r(\tau)(t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\right| \\
&\leq \| r \|_{L^\infty(0,T)} \left|\int_0^t (t-\tau)^{\beta_1-1} E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\right| \\
&\leq \| r \|_{L^\infty(0,T)} \left|\int_0^t (t-\tau)^{\beta_1-1} \frac{C}{1+\lambda_n(t-\tau)^{\beta_1}} d\tau\right| \\
&\leq C \| r \|_{L^\infty(0,T)} \left|\int_0^t (t-\tau)^{\beta_1-1} d\tau\right| \\
&\leq C_2 \| r \|_{L^\infty(0,T)}\, .
\end{aligned}$$

The lemma is proved. □

3. Existence and uniqueness of a strong solution for the direct problem

Denote the eigenvalues of $-\triangle$ with homogeneous Dirichlet boundary condition as $\bar{\lambda}_n$ and the corresponding eigenfunctions as $\phi_n \in H^2(\Omega)\bigcap H_0^1(\Omega)$, that means we have $-\triangle\phi_n = \bar{\lambda}_n\phi_n$ and $\phi_n|_{\partial\Omega} = 0$. Counting according to the multiplicities, we can set: $0 < \bar{\lambda}_1 \le \bar{\lambda}_2 \le \cdots \le \bar{\lambda}_n \le \cdots$ and $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$.

Let us define a strong solution to the direct problem (1.1)-(1.3), then we prove its existence and uniqueness based on the methods in [20].

Definition 3.1. We call $u \in C([0,T];L^2(\Omega))\bigcap L^2(0,T;H_0^{\gamma}(\Omega))$ such that $(-\Delta)^{\frac{\gamma}{2}}u \in C(([0,T];L^2(\Omega))\bigcap L^2(0,T;L^2(\Omega)))$ is a strong solution to (1.1)-(1.3) if (1.1)hold in $L^2(\Omega)$ for $0 \le t \le T$, (1.2)holds in $L^2(\Omega)$ as $t \to 0^+$ and (1.3) holds in trace sense.

Theorem 3.2. *Let $g \in H_0^{\gamma}(\Omega)$, $q \in L^2(\Omega)$, $f \in L^{\infty}(0,T)$, then exists a unique solution $u(x,t)$ given by*

$$u(x,t) = \sum_{n=1}^{\infty}\int_0^t r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\,(q,\phi_n)\,\phi_n(x), \tag{3.1}$$

where $\lambda_n = \overline{\lambda}_n^{\frac{\gamma}{2}}$; moreover, the following estimates hold:

$$\|u\|_{C([0,T];L^2(\Omega))} \le C\|r\|_{L^{\infty}(0,T)}\|h\|_{L^2(\Omega)}, \tag{3.2}$$

$$\|u\|_{L^2(0,T;H_0^{\gamma}(\Omega))} \le C\|r\|_{L^{\infty}(0,T)}\|h\|_{L^2(\Omega)}, \tag{3.3}$$

where C are positive constants depending on β_j ($j = 1,2,\cdots,m$), T, Ω.

Proof. We first verify $u \in C\left([0,T];L^2(\Omega)\right)$ and $\lim\limits_{t\to 0}\|u(\cdot,0)\|_{L^2(\Omega)} = 0$. By using the result of Proposition 2.2 and together with (2.5), we can obtain

$$\|u(x,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}\left(\int_0^t r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\right)^2(h,\phi_n)^2 \le \left(C\,\|r\|_{L^{\infty}(0,T)}\,\|h\|_{L^2(\Omega)}\right)^2.$$

where C is positive constants depending on β_j ($j = 1,2,\cdots,m$), T, Ω.

For $t, t+k \in [0,T]$, we have

$$\begin{aligned}
&u(x,t+k)-u(x,t)\\
=&\sum_{n=1}^{\infty}\int_0^{t+k} r(\tau)(t+k-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t+k-\tau)d\tau(h,\phi_n)\phi_n(x)-\\
&\sum_{n=1}^{\infty}\int_0^{t} r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau(h,\phi_n)\phi_n(x)\\
=&\sum_{n=1}^{\infty}\left(\int_0^{t+k} r(\tau)(t+k-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t+k-\tau)d\tau-\int_0^{t} r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau\right)(h,\phi_n)\phi_n(x).
\end{aligned}$$

Combining 2.2 and Lemma 2.9, we have the following estimate:

$$\begin{aligned}
\|u(x,t+k)-u(x,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}(W_n(t+k)-W_n(t))^2(h,\phi_n)^2\\
&\le 2\sum_{n=1}^{\infty}\left(W_n^2(t+k)+W_n^2(t)\right)(h,\phi_n)^2\\
&\le C\|r\|^2_{L^\infty(0,T)}\|h\|^2_{L^2(\Omega)},
\end{aligned}$$

where$W_n(t)=\int_0^t r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau$. Since $\lim_{k\to 0}\left|(t+k)^{\beta_1}E^{(n)}_{\beta',\beta_1+1}(t+k)-t^{\beta_1}E^{(n)}_{\beta',\beta_1+1}(t)\right|=0$ for each $n\in\mathbb{N}$, by using the Lebesgue theorem, we can arrive at

$$\lim_{k\to 0}\|u(x,t+k)-u(x,t)\|_{L^2(\Omega)}=0, \tag{3.4}$$

it follows that $u\in C([0,T];L^2(\Omega))$.

Utilizing $\lim_{t\to 0}\lambda_n t^{\beta_1}E^{(n)}_{\beta',\beta_1+1}(t)=0$ and (3.4), we deduce that

$$\lim_{t\to 0^+}\|u(\cdot,t)\|_{L^2(\Omega)}=0.$$

Next, we prove that $(-\Delta)^{\frac{\gamma}{2}}u\in C\left(((0,T];L^2(\Omega))\cap L^2(0,T;L^2(\Omega))\right)$ and $u\in L^2(0,T;H_0^{\gamma}(\Omega))$. Using (3.1), we have

$$(-\Delta)^{\frac{\gamma}{2}}u(x,t) = \sum_{n=1}^{\infty}\lambda_n\int_0^t r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau(h,\phi_n)\phi_n(x).$$

For $0<t\le T$, by using (2.4), derive the estimate

$$\begin{aligned}
\left\|(-\Delta)^{\frac{\gamma}{2}}u(x,t)\right\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^2(h,\phi_n)^2(\int_0^t r(\tau)(t-\tau)^{\beta_1-1}E^{(n)}_{\beta',\beta_1}(t-\tau)d\tau)^2\\
&\le C\sum_{n=1}^{\infty}\lambda_n^2\frac{\|r\|^2_{L^\infty(0,T)}}{\lambda_n^2}(h,\phi_n)^2\\
&\le C\|r\|^2_{L^\infty(0,T)}\|h\|^2_{L^2(\Omega)}.
\end{aligned}$$

Since $(-\Delta)^{\frac{\gamma}{2}}u(x,t)$ are convergent in $L^2(\Omega)$ uniformly on $t \in [t_0, T]$ for any given $t_0 > 0$, $(-\triangle)^{\frac{\gamma}{2}}u \in C\left([0,T]; L^2(\Omega)\right)$ can be arrived at similarly to the first part of the proof, and we have an estimate of

$$\|u\|_{L^2(0,T;H_0^\gamma(\Omega))} = \left\|(-\triangle)^{\frac{\gamma}{2}}u\right\|_{L^2(0,T;L^2(\Omega))} \le C\|r\|_{L^\infty(0,T)}\|h\|_{L^2(\Omega)},$$

where C are positive constants depending on β_j $(j = 1, 2, \cdots, m)$, T, Ω. □

In the next section, we reformulate the inverse source problem (1.1)-(1.4). For the inverse source problem, we demonstrate the existence and uniqueness of the solution and provide a stability estimate.

4. Uniqueness and a stability estimate for the inverse space source problem

We next reformulate the inverse problem for (1.1)-(1.4). Hence, we define the following operator equation

$$Qh(x) + \Phi(x) = h(x), \tag{4.1}$$

where $Q(h) : L^2(\Omega) \to L^2(\Omega)$, $Q(h)$ and $\Phi(x)$ are respectively defined by

$$Qh(x) = \frac{\sum\limits_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T)}{r(T)}, \tag{4.2}$$

and

$$\Phi(x) = \frac{(-\Delta)^{\frac{\gamma}{2}} v(x)}{r(T)}. \tag{4.3}$$

Throughout this paper, we will assume that $|r(T)| > k > 0$. The operator equation (4.1) has a solution (a unique solution) $h \in L^2(\Omega)$ if and only if the inverse problem (1.1)-(1.4) has a solution (a unique solution) for any fixed $a_i \in J \subset (0, \infty)$ $(i = 2, \ldots, m)$, according to Lemma 2.4 of [24].

Theorem 4.1. *For any fixed $a_i \in J$ $(i = 2, \ldots, m)$ and let $r(t) \in AC[0,T]$, the operator $Q(h) : L^2(\Omega) \to L^2(\Omega)$ is a compact operator.*

Proof. By use (2.9), we have

$$\begin{aligned}
\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,t) &= \sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} \sum_{n=1}^{\infty} (h, \phi_n) \int_0^t r(\tau)(t-\tau)^{\beta_1 - 1} E^{(n)}_{\beta', \beta_1}(t-\tau) d\tau \phi_n(x) \\
&= \sum_{i=1}^{m} \sum_{n=1}^{\infty} a_i (h, \phi_n) \partial_{0+}^{\beta_i} \int_0^t r(\tau)(t-\tau)^{\beta_1 - 1} E^{(n)}_{\beta', \beta_1}(t-\tau) d\tau \phi_n(x).
\end{aligned}$$

According to the lemma 2.3, it follows that

$$\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,t) = \sum_{n=1}^{\infty}\sum_{i=1}^{m} a_i (h,\phi_n) \int_0^t r(\tau)(t-\tau)^{\beta_1-\beta_i-1} E^{(n)}_{\beta',\beta_1-\beta_i}(t-\tau) d\tau \phi_n(x). \tag{4.4}$$

This together with Proposition2.2 and Proposition2.4 mean that

$$\begin{aligned}
\left\| \sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T) \right\|^2_{H_0^\gamma} &= \left\| \sum_{n=1}^{\infty}\sum_{i=1}^{m} a_i (h,\phi_n) \int_0^T r(\tau)(T-\tau)^{\beta_1-\beta_i-1} E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau) d\tau \phi_n(x) \right\|^2_{H_0^\gamma} \\
&= \left\| \sum_{n=1}^{\infty}\sum_{i=1}^{m} a_i (h,\phi_n) \int_0^T r(\tau)(T-\tau)^{\beta_1-\beta_i-1} E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau) d\tau \lambda_n \phi_n(x) \right\|^2_{L^2(\Omega)} \\
&= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{m} a_i \int_0^T r(\tau)(T-\tau)^{\beta_1-\beta_i-1} E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau) d\tau \right)^2 (h,\phi_n)^2 \lambda_n^2 \\
&\le \|r\|^2_{C[0,T]} \sum_{n=1}^{\infty} m\lambda_n^2 \sum_{i=1}^{m} a_i^2 \left(\int_0^T (T-\tau)^{\beta_1-\beta_i-1} E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau) d\tau \right)^2 (h,\phi_n)^2 \\
&\le m\|r\|^2_{C[0,T]} \sum_{i=1}^{m} a_i^2 \left(\lambda_n T^{\beta_1-\beta_i} E^{(n)}_{\beta',\beta_1-\beta_i+1}(T) \right)^2 \sum_{n=1}^{\infty} (h,\phi_n)^2 \\
&\le C_1^2 \|h\|^2_{L^2(\Omega)}.
\end{aligned}$$

Since $H_0^\gamma(\Omega)$ is compactly embedded in $L^2(\Omega)$, the operator $r(T)Q(h)$ is compact. Finally, because $|r(T)| > k$, we can conclude that the operator $Q(h) : L^2(\Omega) \to L^2(\Omega)$ is compact.

□

Theorem 4.2. *According to the definition of $Q(h)$, the operator $Qh : J \to L^2(\Omega)$ is real analytic for any fixed $h \in L^2(\Omega)$ and $a_l \in J$ $(l = 2, \ldots, m)$.*

Proof. Fix any $h \in L^2(\Omega)$. To check the operator Qh is real analytic, let we demonstrate first

$$\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T) \in C^\infty(0,\infty; L^2(\Omega)). \tag{4.5}$$

Indeed according to the proof of Theorem 4.1 and the compact embedding, we have

$$\left\| \sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T) \right\|_{L^2(\Omega)} \le K.$$

This confirms (4.5).

Next, we need to claim

$$\left\| \frac{d^q}{da_l^q} \left(\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T) \right) \right\|_{L^2(\Omega)} \le K a_l^{-q} q!, \quad q \in \mathbb{N}, a_l \in I; K > 0, a_i > 0. \tag{4.6}$$

For any fixed $a_l \in J$ $(l = 2, \ldots, m)$, one has

$$\begin{aligned}
&\frac{d^q}{da_l^q}\left(\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T)\right) \\
= \ & \frac{d^q}{da_l^q}\left[\sum_{n=1}^{\infty}\sum_{i=1}^{m} a_i\,(h,\phi_n)\int_0^T r(\tau)(T-\tau)^{\beta_1-\beta_i-1}E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau)d\tau\phi_n(x)\right] \\
\leq \ & \|r\|_{C[0,T]}\sum_{n=1}^{\infty}\frac{d^q}{da_l^q}\left[\sum_{i=1}^{m} a_i\int_0^T (T-\tau)^{\beta_1-\beta_i-1}E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau)d\tau\right](h,\phi_n)\,\phi_n(x) \\
= \ & \|r\|_{C[0,T]}\sum_{n=1}^{\infty}\frac{d^q}{da_l^q}\left[\sum_{i=1}^{m} a_i T^{\beta_1-\beta_i}E^{(n)}_{\beta',\beta_1-\beta_i+1}(T)\right](h,\phi_n)\,\phi_n(x) \\
= \ & \|r\|_{C[0,T]}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m} T^{\beta_1-\beta_i}a_i\frac{d^q}{da_l^q}E^{(n)}_{\beta',\beta_1-\beta_i+1}(T) + qT^{\beta_1-\beta_l}\frac{d^{q-1}}{da_l^{q-1}}E^{(n)}_{\beta',\beta_1-\beta_l+1}(T)\right)(h,\phi_n)\,\phi_n(x).
\end{aligned}$$

According to Lemma 2.3, it follows that

$$\begin{aligned}
&\left\|\frac{d^q}{da_l^q}\left(\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T)\right)\right\|^2_{L^2(\Omega)} \\
= \ & \|r\|^2_{C[0,T]}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m} T^{\beta_1-\beta_i}a_i\frac{d^q}{da_l^q}E^{(n)}_{\beta',\beta_1-\beta_i+1}(T) + qT^{\beta_1-\beta_l}\frac{d^{q-1}}{da_l^{q-1}}E^{(n)}_{\beta',\beta_1-\beta_l+1}(T)\right)^2 (h,\phi_n)^2 \\
\leq \ & \|r\|^2_{C[0,T]}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m} T^{\beta_1-\beta_i}a_i\frac{Cq!a_l^{-q}}{1+|w_1|^{q+1}} + a_l T^{\beta_1-\beta_l}\frac{Cq!a_l^{-q}}{1+|w_1|^{q}}\right)^2 (h,\phi_n)^2 \\
\leq \ & \left[C(m+1)A\|r\|_{C[0,T]}\|h\|_{L^2(\Omega)}\right]^2\left(q!a_l^{-q}\right)^2,
\end{aligned}$$

where $A = \max\{a_1, a_2T^{\beta_1-\beta_2}, \ldots, a_mT^{\beta_1-\beta_m}\}$. Donated $K = C(m+1)A\|r\|_{C[0,T]}\|h\|_{L^2(\Omega)}$, the (4.6) can be confirmed.

Thus, we can easily prove $Qh : J \to L^2(\Omega)$ is real analytic for arbitrarily fixed $h \in L^2(\Omega)$ and $a_l \in J$ (see the last theorem on page 65 of [7]).

□

Theorem 4.3. *Let $r(t) \in C[0,T]$. For sufficiently small a_l $(l = 1, \ldots, m)$. There exists a constant $0 < M < 1$ such that*

$$\|Qh\|_{L^2(\Omega)} \leq M\|h\|_{L^2(\Omega)}.$$

Moreover, 1 is not an eigenvalue of the operator Q.

Proof. By (4.1) we have

$$\|Qh\|_{L^2(\Omega)} = \frac{1}{r(T)}\left\|\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T)\right\|_{L^2(\Omega)}.$$

According to Proposition 2.2 and (4.4), one has

$$\begin{aligned}
\left\|\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u(x,T)\right\|_{L^2(\Omega)}^2 &= \left\|\sum_{n=1}^{\infty}\sum_{i=1}^{m} a_i\,(h,\phi_n)\int_0^T r(\tau)(T-\tau)^{\beta_1-\beta_i-1}E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau)d\tau\phi_n(x)\right\|_{L^2(\Omega)}^2 \\
&= \sum_{n=1}^{\infty}(h,\phi_n)^2\left(\sum_{i=1}^{m} a_i\int_0^T r(\tau)(T-\tau)^{\beta_1-\beta_i-1}E^{(n)}_{\beta',\beta_1-\beta_i}(T-\tau)d\tau\right)^2 \\
&\le \|r\|_{C[0,T]}^2\sum_{n=1}^{\infty}(h,\phi_n)^2\left(\sum_{i=1}^{m} a_i T^{\beta_1-\beta_i}E^{(n)}_{\beta',\beta_1-\beta_i+1}(T)\right)^2 \\
&\le \|r\|_{C[0,T]}^2\sum_{n=1}^{\infty}(h,\phi_n)^2\left(\sum_{i=1}^{m}\frac{Ca_i}{\lambda_n T^{\beta_i}}\right)^2 \\
&\le \|r\|_{C[0,T]}^2\left(m\frac{Ca_l}{\lambda_n T^{\beta_l}}\right)^2\|h\|_{L^2(\Omega)}^2,
\end{aligned}$$

where $\frac{a_l}{T^{\beta_l}} = \max\{\frac{a_1}{T^{\beta_1}},\ldots,\frac{a_m}{T^{\beta_m}}\}$. Since $|r(T)| > k > 0$, we have

$$\|Qh\|_{L^2(\Omega)} \le \frac{Ca_l\|r\|_{C[0,T]}}{\lambda_n T^{\beta_l}k}\|h\|_{L^2(\Omega)}.$$

Let $M = \frac{Ca_l\|r\|_{C[0,T]}}{\lambda_n T^{\beta_l}k}$, and sufficiently small $a_l > 0$, we can get $M < 1$. Consequently, 1 is not an eigenvalue of the operator Q.

This completes the proof. □

Theorem 4.4. *Assume D is a finite set in J and that it satisfies for any $a_i \in J\backslash D$. Suppose $v \in H_0^{\gamma}(\Omega)$ is the additional data. Hence, the inverse problem (1.1)-(1.4) has a unique solution. Furthermore, there exists a constant $C_2 > 0$ such that*

$$\|h\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H_0^{\gamma}(\Omega))} + \left\|\sum_{i=1}^{m} a_i\partial_{0+}^{\beta_i}u\right\|_{L^2(0,T;L^2(\Omega))} \le C_2\|v\|_{H_0^{\gamma}(\Omega)}.$$

Proof. According to Theorem 4.1 and Theorem 4.3, it follows that the operator Q can not occur in the first alternative of the Analytic Fredholm Theorem (see page 266, Theorem 8.92 in [19]). It means that $(I-Q)^{-1}$ exists for every $a_i \in J\backslash D$ $(i = 1,\ldots,m)$, where $D \subset J$ is a discrete set. Apply the Analytic Fredholm Theorem, and the inverse problem (1.1)-(1.4) is uniquely solvable.

12 Nov 2023 08:11:47 PST
230629-LiYS Version 2 - Submitted to Rocky Mountain J. Math.

By (3.1) we can conclude

$$\begin{aligned}
\|u\|_{H_0^\gamma(\Omega)} &= \|(-\Delta)^{\frac{\gamma}{2}} u\|_{L^2(\Omega)} \\
&\leq \sum_{n=1}^{\infty} \lambda_n^2 \left(\int_0^t r(\tau)(t-\tau)^{\beta_1 - 1} E^{(n)}_{\beta',\beta_1}(t-\tau) d\tau \right)^2 (h, \phi_n)^2 \\
&\leq C^2 \|r\|^2_{C[0,T]} \sum_{n=1}^{\infty} \left(\lambda_n t^{\beta_1} E^{(n)}_{\beta',\beta_1+1}(t) \right)^2 (h, \phi_n)^2 \\
&\leq C^2 \|h\|^2_{L^2(\Omega)}.
\end{aligned}$$

According to Theorem 4.3, it follows that

$$\|h\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H_0^\gamma(\Omega))} + \left\| \sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} u \right\|_{L^2(0,T;L^2(\Omega))} \leq C_3 \|h\|_{L^2(\Omega)}.$$

By (4.1) and Theorem 4.3, we can obtain

$$\begin{aligned}
\|h\|_{L^2(\Omega)} &\leq \|Qh(x) + \Phi(x)\|_{L^2(\Omega)} \\
&\leq \|Qh(x)\|_{L^2(\Omega)} + \|\Phi(x)\|_{L^2(\Omega)} \\
&\leq M\|h(x)\|_{L^2(\Omega)} + \frac{1}{|r(T)|} \|(-\Delta)^{\frac{\gamma}{2}} v(x)\|_{L^2(\Omega).}
\end{aligned}$$

Consequently

$$\begin{aligned}
\|h\|_{L^2(\Omega)} &\leq \frac{1}{k(1-M)} \|(-\Delta)^{\frac{\gamma}{2}} v(x)\|_{L^2(\Omega)} \\
&\leq C_2 \|v\|_{H_0^\gamma(\Omega)}.
\end{aligned}$$

The proof is completed. □

5. The inversion algorithm

In the following, we compute the space-dependent source term $h(x)$ by the optimal perturbation algorithm. Suppose that $\{\phi_k(x), k = 1, 2, \cdots, \infty\} \in C^2[0, 1]$ is a set of basis functions, let

$$h(x) \approx h^K(x) = \sum_{k=1}^{K} w_k \phi_k(x), \tag{5.1}$$

where $h^K(x)$ is the K dimensional approximate solution to $h(x)$, $K \in \mathbf{N}$ is a truncated level of $h(x)$, and $w_k, k = 1, 2, \cdots K$ are the coefficients of expansion. Using a space with finite dimensions as

$$\Phi^K = span\{\phi_1, \phi_2, \cdots, \phi_K\},$$

and a K-dimensional vector as $\boldsymbol{w} = (w_1, w_2, \cdots, w_K) \in R^K$ is feasible. We identify an approximation $h^K(x) \in \Phi^K$ with a vector $\boldsymbol{w} \in R^K$.

To overcome ill-conditioning and ensure the numerical stability of the solution, we use Tikhonov regularization to solve the following minimization problem.

$$\min J(\boldsymbol{w}) = \frac{1}{2}||u(x, T; \boldsymbol{w}) - v(x)||^2_{L^2(0,T)} + \frac{\mu}{2}||\boldsymbol{w}||^2, \tag{5.2}$$

where $\mu > 0$ is a regularization parameter, $u(x, T; \boldsymbol{w})$ is the solution of the direct problem (1.1)-(1.3) for any prescribed $h^K(x)$ given by (5.1).

Next, the problem (5.2) is solved using the optimal perturbation algorithm to determine $h(x)$. For any given $\boldsymbol{w}_k \in R^K$, set

$$\boldsymbol{w}_{j+1} = \boldsymbol{w}_j + \delta\boldsymbol{w}_j, \quad j = 0, 1, \cdots, \tag{5.3}$$

where $\delta\boldsymbol{w}_j$ is referred to as the given perturbation. Therefore, to derive $\boldsymbol{w}_{j+1}$ from the given $\boldsymbol{w}_j$, we only need to obtain an ideal perturbation $\delta\boldsymbol{w}_j$. For ease of writing, they are denoted by $\boldsymbol{w}$ and $\delta\boldsymbol{w}$, respectively. By expanding $u(x, T; \boldsymbol{w} + \delta\boldsymbol{w})$ at $\boldsymbol{w}$ in a Taylor expansion , neglecting high-order terms, we obtain

$$u(x, T; \boldsymbol{w} + \delta\boldsymbol{w}) \approx u(x, T; \boldsymbol{w}) + \nabla^T_{\boldsymbol{w}} u(x, T; \boldsymbol{w}) \cdot \delta\boldsymbol{w}.$$

The error functional with perturbation is defined as:

$$F(\delta\boldsymbol{w}) = \frac{1}{2}||\nabla^T_{\boldsymbol{w}} u(x, T; \boldsymbol{w}) \cdot \delta\boldsymbol{w} - [v(x) - u(x, T; \boldsymbol{w})]||^2_{L^2(0,T)} + \frac{\mu}{2}||\delta\boldsymbol{w}||^2. \tag{5.4}$$

The regularization parameter is provided by

$$\mu = \mu(n) = \frac{1}{1 + exp(\theta(n - n_0))},$$

where n is the number of iteration, n_0 is an a priori chosen number and $\theta \geq 0$ is the adjust parameter, which is based on the characteristics of the sigmoid-type function (see[2]). We will choose $n_0 = 5$, $\theta = 0.8$ in all of the following numerical examples.

The spatial domain $[0, 1]$ is discretized into $0 = x_1 < x_2 < \cdots < x_S = 1$ where S is the number of grids, and the L^2 norm in 5.4 is converted to the discrete Euclidean norm, given by

$$F(\delta\boldsymbol{w}) = \frac{1}{2}||B\delta\boldsymbol{w} - (\eta - \beta)||^2_2 + \frac{\mu}{2}||\delta\boldsymbol{w}||^2_2, \tag{5.5}$$

where

$$
\begin{aligned}
B &= (b_{sk})_{S\times K},\\
b_{sk} &= \frac{u(x_s, T; w_1, \cdots, w_k + \tau, \cdots, w_K) - u(x_s, T; \boldsymbol{w})}{\tau},\\
&\quad s = 1, 2, \cdots, S,
\end{aligned}
$$

and τ is the numerical differentiation step, where

$$
\begin{aligned}
\beta &= (u(x_1, T; \boldsymbol{w}), u(x_2, T; \boldsymbol{w}), \cdots, u(x_S, T; \boldsymbol{w})),\\
\eta &= (\nu(x_1), \nu(x_2), \cdots, \nu(x_S)).
\end{aligned}
$$

Using the method in [8]), (5.5) is transformed into the following normal equations:

$$(\mu I + B^T B)\delta \boldsymbol{w} = B^T(\eta - \beta). \tag{5.6}$$

Consequently, an optimal perturbation can be solved by the use of the following formula:

$$\delta \boldsymbol{w} = (\mu I + B^T B)^{-1} B^T(\eta - \beta). \tag{5.7}$$

The following iteration stopping rule is selected:

$$\parallel \delta \boldsymbol{w} \parallel \leq eps, \tag{5.8}$$

where eps is a given convergent precision.

The direct problem (1.1)-(1.3) should be resolved so as to use the inversion technique to solve the inverse diffusion coefficient problem at each stage. Thus in the following, we give the implicit finite difference scheme with matrix transfer technique [3, 4, 31] for solving the direct problem (1.1)-(1.3).

The grid sizes for time and space in the finite difference algorithm are $\Delta t = \frac{T}{N}$ and $\Delta x = \frac{1}{M}$, respectively. Time is discretized by $t_n = n\Delta t$ ($n = 0, 1, \cdots, N$), while space is discretized by $x_i = i\Delta x$ ($i = 0, 1, \cdots, M$). The values denoted $u_i^n \approx u(x_i, t_n)$ are the approximate values of function u at the grid points.

First, Consider the following standard diffusion equation with initial boundary value conditions

$$
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x)r(t), \quad 0 < x < 1, \quad t > 0,\\
u(x, 0) = 0,\\
u(0, t) = u(1, t) = 0,
\end{cases} \tag{5.9}
$$

Introducing a finite difference approximation, we obtain

$$\begin{cases} \frac{du_j}{\partial t} = \frac{1}{h}(u_{j-1} - 2u_i + u_{j+1}) + r(t)h_j, \quad 0 < t < T, \\ u_j = 0 \\ u_0 = u_N = 0, \end{cases}$$

where $u_j = u(x_j, t)$, $q_j = q(x_j)$, $j = 1, 2, \cdots M$, τ is the space step defined as $\tau = \frac{1}{M}$. The above equations can be expressed as the following system of ordinary differential equations:

$$\frac{\partial \boldsymbol{U}}{\partial t} = -\eta \boldsymbol{B}\boldsymbol{U} + r(t)H, \tag{5.10}$$

where $\eta = \frac{1}{\tau^2}$ and $\boldsymbol{U}, b \in \mathbb{R}^{N-1}$, $\boldsymbol{B} \in \mathbb{R}^{N-1\times N-1}$,

$$\boldsymbol{U} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad \boldsymbol{U}^0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N-1} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

For a real nonsingular, symmetric matrix $\boldsymbol{B}_{N-1\times N-1}$, there exits a nonsingular matrix $P_{N-1\times N-1}$ such that

$$\boldsymbol{B} = P\Lambda P^{-1},$$

where $\Lambda = diag(\lambda_1, \lambda_2, \cdots \lambda_{N-1})$, $\lambda_j (j = 1, 2, \cdots N-1)$ are the eigenvalues of $\boldsymbol{B}$.

Then, we consider the direct problem (1.1)-(1.3) rewritten in the following matrix form:

$$\sum_{i=1}^{m} a_i \partial_{0+}^{\beta_i} = -\bar{\eta}\boldsymbol{B}^{\frac{\alpha}{2}}\boldsymbol{U} + r(t)H, \tag{5.11}$$

where $\bar{\eta} = \frac{1}{h^\alpha}$, $A^{\frac{\alpha}{2}} = P\Lambda^{\frac{\alpha}{2}}P^T$. The time-fractional derivative is approximated by

$$\partial_{0^+}^{\beta_i} u(x, t_n) \approx \frac{\tau^{-\beta_i}}{\Gamma(2-\beta_i)} \left(b_0^{(\beta_i)} u(x, t_n) - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\beta_i)} - b_{n-k}^{(\beta_i)}) u(x, t_k) - b_{n-1}^{(\beta_i)} u(x, t_0) \right),$$

where $b_l^{(\beta_i)} = (l+1)^{1-\beta_i} - l^{1-\beta_i}, l \geq 0$, this scheme was used in [18]. We have $\omega_k^{\beta_i} = b_{n-k-1}^{(\beta_i)} - b_{n-k}^{(\beta_i)}$, and

$$(5.12)\qquad \begin{cases} \partial_{0^+}^{\beta_i} u_1^n = \frac{\tau^{-\beta_i}}{\Gamma(2-\beta_i)}\left(b_0^{(\beta_i)} u_1^n - \sum_{k=1}^{n-1} \omega_k^{\beta_i} u_1^k - b_{n-1}^{(\beta_i)} u_1^0\right), \\ \partial_{0^+}^{\beta_i} u_2^n = \frac{\tau^{-\beta_i}}{\Gamma(2-\beta_i)}\left(b_0^{(\beta_i)} u_2^n - \sum_{k=1}^{n-1} \omega_k^{\beta_i} u_2^k - b_{n-1}^{(\beta_i)} u_2^0\right), \\ \cdots \qquad \cdots \qquad \cdots \\ \partial_{0^+}^{\beta_i} u_{N-1}^n = \frac{\tau^{-\beta_i}}{\Gamma(2-\beta_i)}\left(b_0^{(\beta_i)} u_{N-1}^n - \sum_{k=1}^{n-1} \omega_k^{\beta_i} u_{N-1}^k - b_{n-1}^{(\beta_i)} u_{N-1}^0\right). \end{cases}$$

Then, the implicit difference scheme (5.11)-(5.12) can be rewritten in matrix form given as

$$(5.13)\qquad \begin{cases} \boldsymbol{A}\boldsymbol{U}^n = b, \\ U^0 = \mathbf{0}, \end{cases}$$

where

$$U^n = \left(u_1^n, u_2^n, \cdots, u_{N-1}^n\right),$$

$$A = \sum_{i=1}^{m}\left(b_0^{(\beta_i)} \frac{\tau^{-\beta_i}}{\Gamma(2-\beta_j)} I_{(N-1)\times(N-1)}\right) + \bar{\eta} B^{\frac{\alpha}{2}},$$

and

$$b = \sum_{i=1}^{m}\left(\frac{\tau^{-\beta_i}}{\Gamma(2-\beta_i)}\begin{pmatrix} u_1^1 & u_1^2 & \cdots & u_1^{n-1} \\ u_2^1 & u_2^2 & \cdots & u_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ u_{N-1}^1 & u_{N-1}^2 & \cdots & u_{N-1}^{n-1} \end{pmatrix}\begin{pmatrix} \omega_1^{(\beta_i)} \\ \omega_2^{(\beta_i)} \\ \vdots \\ \omega_{n-1}^{(\beta_i)} \end{pmatrix}\right) + r(t_n)H.$$

6. Numerical experiments

In this part, we prove the effectiveness of the optimal perturbation algorithm with numerical results by using five examples for the one-dimensional scenarios and two-dimensional scenarios. The algorithms convergence and stability are examined.

The effectiveness of the optimal perturbation algorithm is demonstrated through numerical results from three examples in this section, and the convergence and stability of the algorithm are analyzed. In all experiments, we set $T = 1$, and the number of grid points on both space and time axes is 51. The accurate data is perturbed to create the noisy data randomly, i.e.,

$$v^\delta = v + \delta v \cdot (2 \cdot rand(size(v)) - 1),$$

the corresponding noise level is calculated by $\delta = \|v^\delta - v\|_{L^2(0,1)}$.

Intending to show the accuracy of the numerical solution, the relative root mean square(RRMS) error is estimated to be

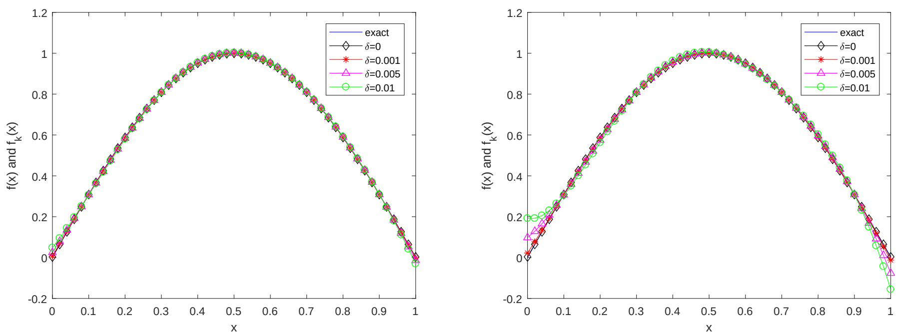
$$\varepsilon(h) = \left(\frac{\sum_{j=1}^{n} \left(h^K(x_j) - h(x_j) \right)^2}{\sum_{j=1}^{n} h(x_j)^2} \right)^{1/2}, \tag{6.1}$$

where n is the total number of the uniformly distributed point on time internal $[0, 1]$, $h^K(x)$ is the space source term reconstructed at the final iteration and $h(t)$ is the precise solution.

Unless otherwise specified, we let $m = 3$ and $a_i = 1(i = 1, 2, 3)$, select $eps = 10^{-6}$ as the convergent precision, the numerical differential step $\tau = 0.01$, the first iteration is zeros, i.e., $\boldsymbol{w} = \mathbf{0}$.

Example 1: Consider the source function $h(x) = sin(\pi x)$, $r(t) = e^{-t}$ and $\beta_1 = 0.8, \beta_2 = 0.5, \beta_3 = 0.2$, and use the finite difference method to resolve the direct problem (1.1)-(1.3) to obtain the final value data and setting $K = 6$ and $\Phi^k = \{1, t, \cdots, t^2\}$.

Figure 1 presents the numerical comparison results of inverting the source term $h(x)$ for different values of γ and different levels of relative noise $\delta = 0, 0.1\%, 0.5\%, 1\%$. It can be observed that the numerical approximation closely matches the exact source term $h(x)$ very well, except for a slight deviation near $x = 0$ when the relative noise level is $\delta = 10\%$. This indicates that our proposed regularization method is highly effective and the identification of the source term $h(x)$ is stable.



(a) $\alpha = 1.2$ **(b)** $\alpha = 1.8$

FIGURE 1. Numerical solutions for Example 1 .

In Table 1, with a fixed relative noise level $\delta = 1\%$ and $\gamma = 1.2$, the relationship between different values of β and the RRMS error $\varepsilon(h)$ is provided for Example 1. It can be observed

Table 1. The RRMS errors of Example 1 for various values of $\boldsymbol{\beta}$ with fixed $\gamma = 1.2$, $\delta = 1\%$

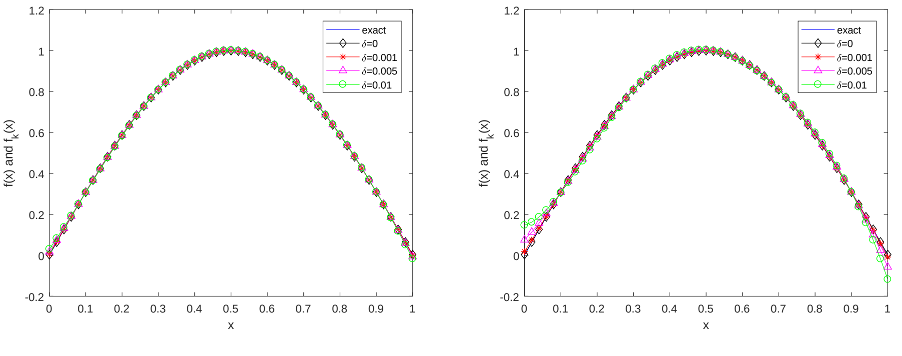
$\boldsymbol{\beta}$	(0.3, 0.2, 0.1)	(0.5, 0.4, 0.3)	(0.7, 0.6, 0.5)	(0.9, 0.8, 0.7)
$\varepsilon(h)$	0.0127	0.0143	0.0160	0.0177

Table 2. The RRMS errors of Example 1 for various values of γ with fixed $\beta_1 = 0.8, \beta_2 = 0.5, \beta_3 = 0.2$, $\delta = 1\%$

γ	1.1	1.3	1.5	1.7	1.9
$\varepsilon(h)$	0.0119	0.0200	0.0329	0.0528	0.0831

that different values of β have little impact on the numerical accuracy. In Table 2, with a fixed relative noise level $\delta = 1\%$ and $\beta_1 = 0.8, \beta_2 = 0.5, \beta_3 = 0.2$, the relationship between different values of γ and the RRMS error $\varepsilon(h)$ is presented for Example 1. It can be roughly observed that the precision of the numerical results decreases as γ increases.

In Figure 2, we show numerical results for Example 1 with various γ and various noise levels $\delta = 0, 0.1\%, 0.5\%, 1\%$ in which we fixed $\beta_1 = 0.9, \beta_2 = 0.8, \cdots, \beta_9 = 0.1$. It can be observed that the numerical results match very well. Table 3 displays the RRMS error for different values of γ in Example 1, with a fixed error level $\beta_1 = 0.9, \beta_2 = 0.8, \cdots, \beta_9 = 0.1$ and $\delta = 1\%$. It can be observed that the numerical accuracy improves as γ decreases.



(a) $\alpha = 1.2$ **(b)** $\alpha = 1.8$

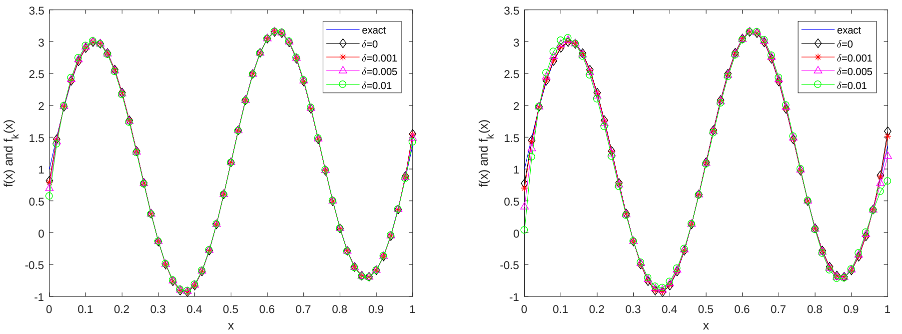
Figure 2. Numerical solutions for Example 1 with $\beta_1 = 0.9$, $\beta_2 = 0.8$, $\cdots, \beta_9 = 0.1$.

Table 3. The RRMS errors of Example 1 for various values of γ with fixed $\beta_1 = 0.9, \beta_2 = 0.8, \cdots, \beta_9 = 0.1$, $\delta = 1\%$

γ	1.1	1.3	1.5	1.7	1.9
$\varepsilon(h)$	0.0072	0.0127	0.0224	0.0388	0.0655

Example 2: Consider the source function $h(x) = 2sin(4\pi x) + e^{-x} + x$, $r(t) = e^{-t}$ and $\beta_1 = 0.8, \beta_2 = 0.5, \beta_3 = 0.2$, and use the finite difference method to resolve the direct problem (1.1)-(1.3) to obtain the final value data and setting $K = 10$ and $\Phi^k = \{1, t, \cdots, t^1 0\}$.

In Figure 3, a comparison of the numerical results for the inverse source term $h(x)$ is shown for different values of γ and different levels of relative noise $\delta = 0, 0.1\%, 0.5\%, 1\%$. From the figure, it can be observed that when $\gamma = 1.2$, the numerical accuracy remains high even at a relative noise level of $\delta = 10\%$. However, for $\gamma = 1.8$, there is a decline in the numerical results at the endpoints as delta increases.

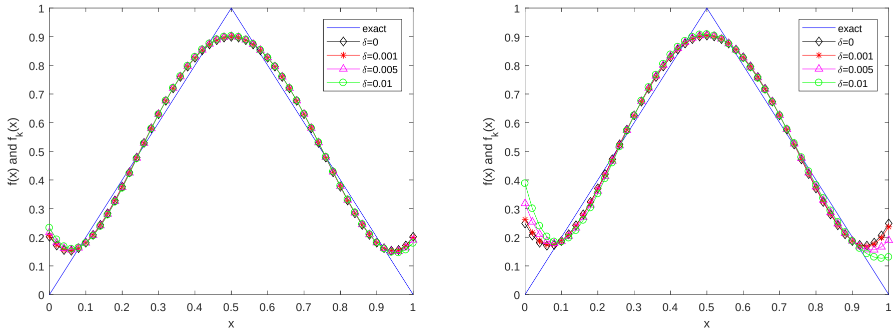


(a) $\alpha = 1.2$ **(b)** $\alpha = 1.8$

Figure 3. Numerical solutions for Example 2 .

Example 3: Consider the source function $h(x) = 1 - |2t - 1|$, $r(t) = e^{-t}$ and $\beta_1 = 0.8, \beta_2 = 0.5, \beta_3 = 0.2$, and use the finite difference method to resolve the direct problem (1.1)-(1.3) to obtain the final value data and setting $K = 10$ and $\Phi^k = \{1, t, \cdots, t^1 0\}$.

Figure 4 presents a comparison of numerical results for the identification of the space source term $h(x)$, considering different relative noise levels δ. The numerical result is good except at the endpoints and sharp point.



(a) $\alpha = 1.2$ **(b)** $\alpha = 1.8$

FIGURE 4. Numerical solutions for Example 2 .

7. CONCLUSIONS

In this paper, we investigates the inversion of a space-dependent source term in a multi-term TSFDE using final value data. Based on the equation, the operator equation $Qh(x) + \Phi(x) = h(x)$ is constructed, and some important properties of the operator Q are proved. Using these properties and the analytical Fredholm theorem, it is shown that the space source term can be uniquely and continuously dependent on the additional final value data. Additionally, this chapter also uses the Tikhonov regularization method to convert the inverse problem into a variational problem, and provides an approximate solution to the inverse problem using the best perturbation algorithm. Numerical examples show the feasibility and stability of the algorithm in identifying space source terms.

ACKNOWLEDGMENTS

This paper was supported by the Natural Science Foundation of Guangxi (2023GXNSFAA026284)) and Special Fund for Science and Technological Bases and Talents of Guangxi (No. Guike AD20159058).

REFERENCES

[1] J. Cheng, J. Nakagawa, M. Yamamoto, and T. Yamazaki. Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. *Inverse probl.*, 25(11):115002, 2009.

[2] S. Haykin. *Neural Networks: A Comprehensive Foundation*. Prentice Hall PTR, 1994.

[3] M. Ilic, F. Liu, I. Turner, and V. Anh. Numerical approximation of a fractional-in-space diffusion equation. I. *Fract. Calc. Appl. Anal.*, 8(3):323–341, 2005.

[4] M. Ilic, F. Liu, I. Turner, and V. Anh. Numerical approximation of a fractional-in-space diffusion equation. II-with nonhomogeneous boundary conditions. *Fract. Calc. Appl. Anal.*, 9(4):333–349, 2006.

[5] S. Z. Jiang and Y. J. Wu. Recovering a time-dependent potential function in a multi-term time fractional diffusion equation by using a nonlinear condition. *J. Inverse Ill-Posed Probl.*, 29(2):233–248, 2021.

[6] B.T. Jin and W. Rundell. A tutorial on inverse problems for anomalous diffusion processes. *Inverse Probl.*, 31(3):035003, 40, 2015.

[7] F. John. *Partial differential equations*. Berlin: Springer-Verlag; 1982, 1982.

[8] A. Kirsch. *An introduction to the mathematical theory of inverse problems*. New York: Springer, 1996.

[9] G. S. Li, D. L. Zhang, X. Z. Jia, and M. Yamamoto. Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation. *Inverse Probl.*, 29(6):065014, 2013.

[10] Z. Li, O. Y. Imanuvilov, and M. Yamamoto. Uniqueness in inverse boundary value problems for fractional diffusion equations. *Inverse probl.*, 32(1):015004, 2016.

[11] Z. Li, Y. Liu, and M. Yamamoto. Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. *Appl. Math. Comput.*, 257:381–397, 2015.

[12] Z. Li and M. Yamamoto. Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Appl. Anal.*, 94(3):570–579, 2015.

[13] Z. Y. Li and Z. D. Zhang. Unique determination of fractional order and source term in a fractional diffusion equation from sparse boundary data. *Inverse Probl.*, 36(11):115013, 2020.

[14] Z.Y. Li, X. C. Huang, and M. Yamamoto. Initial-boundary value problems for multi-term time-fractional diffusion equations with x-dependent coefficients. *Evol. Equ. Control. The.*, 9(1):153–179, 2020.

[15] J.J. Liu and M. Yamamoto. A backward problem for the time-fractional diffusion equation. *Applicable Analysis*, 89(11):1769–1788, 2010.

[16] Y. Liu. Strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse source problem. *Comput. Math. Appl.*, 73(1):96–108, 2017.

[17] S. A. Malik, A. Ilyas, and A. Samreen. Simultaneous determination of a source term and diffusion concentration for a multi-term space-time fractional diffusion equation. *Math. Model. Anal.*, 26(3):411–431, 2021.

[18] D.A. Murio. Time fractional IHCP with caputo fractional derivatives. *Comput. Math. Appl.*, 56(9):2371–2381, 2008.

[19] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*. New York:Springer, 2004.

[20] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.*, 382(1):426–447, 2011.

[21] C. L. Sun, G. S.Li, and X.Z. Jia. Simultaneous inversion for the diffusion and source coefficients in the multi-term tfde. *Inverse Probl. Sci. En.*, 25(11):1618–1638, 2017.

[22] S. Tatar, R. Tnaztepe, and S. Ulusoy. Determination of an unknown source term in a space-time fractional diffusion equation. *J. Fract. Calc. Appl.*, 6(1):83–90, 2015.

[23] S. Tatar, R. Tnaztepe, and S. Ulusoy. Simultaneous inversion for the exponents of the fractional time and space derivatives in the space-time fractional diffusion equation. *Appl. Anal.*, 95(1):1–23, 2016.

[24] S. Tatar and S. Ulusoy. An inverse source problem for a one-dimensional space-time fractional diffusion equation. *Appl. Anal.*, 94(11):2233–2244, 2015.

[25] N. H. Tuan and L. D. Long. Fourier truncation method for an inverse source problem for space-time fractional diffusion equation. *Electron. J. Differential Equations*, page 122, 2017.

[26] N. H. Tuan, A. T. Nguyen, and N. H. Can. Existence and continuity results for Kirchhoff parabolic equation with Caputo-Fabrizio operator. *Chaos Solitons Fractals*, 167:113028, 2023.

[27] N. H. Tuan, V. T. Nguyen, D. O. N. A. L. OREGAN, and N. Can. New results on continuity by order of derivative for conformable parabolic equations. *Fractals*, page 2340014, 2023.

[28] R. Wang, N. H. Can, A. T. Nguyen, and N. H. Tuan. Local and global existence of solutions to a time-fractional wave equation with an exponential growth. *Commun. Nonlinear Sci. Numer. Simul.*, 118:107050, 2023.

[29] T. Wei, X. L. Li, and Y. S. Li. An inverse time-dependent source problem for a time-fractional diffusion equation. *Inverse Probl.*, 32(8):085003, 2016.

[30] M. Yamamoto and Y. Zhang. Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a carleman estimate. *Inverse Probl.*, 28(10):105010, 2012.

[31] Q. Q. Yang, I. Turner, F. Liu, and Milos Ilić. Novel numerical methods for solving the time-space fractional diffusion equation in two dimensions. *SIAM J. Sci. Comput.*, 33(3):1159–1180, 2011.

[32] Zhang Z. D and Z. Zhou. Recovering the potential term in a fractional diffusion equation. *IMA J. Appl. Math.*, 82(3):579–600, 2017.

School of Mathematics and Computing Science, Guangxi Colleges and Universities Key Laboratory of Data Analysisand Computation, Guilin University of Electronic Technology, Guilin 541002,P.R.China;

E-mail address: `1921084939@qq.com (Y. X. Yang)`

School of Mathematics and Computing Sciences, Guangxi Colleges and Universities Key Laboratory of DataAnalysis and Computation, Guilin University of Electronic Technology, Guilin 541002,P.R.China; Center for Applied Mathematics of Guangxi (GUET), Guilin 541002,P.R.China;

E-mail address: `lys0311@163.com ( Y. S. Li)`