More solutions to vector nonlinear recurrence equations

by

Christopher S. Withers, Callaghan Innovation, Lower Hutt, New Zealand
email: kit.withers@gmail.com
Saralees Nadarajah, University of Manchester, Manchester M13 9PL, UK
email: mbbsssn2@manchester.ac.uk

Abstract: Let $\mathcal{C}$ denote the complex numbers. Given a function $F(z) : \mathcal{C}^q \to \mathcal{C}^q$, suppose that $w \in \mathcal{C}^q$ is a fixed point, that is, $F(w) = w$, and that $F(z)$ is analytic at w. Then for $1 \leq p \leq q$, the $q \times 1$ vector recurrence equation

$$z_{n+1} = F(z_n)$$

for $n = 0, 1, 2, \ldots$ has a solution of the form

$$z_n = z_n(w, \alpha r^n) = \sum_{i \in \mathcal{N}^p}^{\infty} a_i(w) \ (\alpha r^n)^i,$$

where $a_{0_p}(w) = w$, $0_p = (0, \ldots, 0)$, $\alpha \in \mathcal{C}^p$ is arbitrary,

$$(\alpha r^n)^i = \prod_{k=1}^{p} (\alpha_k r_k^n)^{i_k},$$

and $r_1, \ldots, r_p$ are any distinct eigenvalues of $\dot{F}(w)$, where $\dot{F}(z) = dF(z)/z' \in \mathcal{C}^{q \times q}$. The other $a_i(w)$ are given recursively using a new type of multivariate Bell polynomial.

Keywords: Bell polynomial; Logistic map; Mandelbrot equation

1 Introduction

We are not aware of any research on finding solutions for vector nonlinear recurrence equations. Withers and Nadarajah (2023b) was the first paper giving solutions for vector nonlinear recurrence equations. In this paper, we have provide extensions of results in Withers and Nadarajah (2023b).

Set $\mathcal{N} = \{0, 1, 2, \ldots\}$. Let $\mathcal{R}$ and $\mathcal{C}$ denote the real and complex numbers. Let $F(z) : \mathcal{C}^q \to \mathcal{C}^q$ be a given function. Choose any w such that $w = F(w)$. Set $\dot{F}(z) = dF(z)/z' \in \mathcal{C}^{q \times q}$, $\dot{F} = \dot{F}(w)$, and that $F(z)$ is analytic at w. Section 3 gives solutions to the *general* $q \times 1$ *recurrence equation*

$$z_{n+1} = F(z_n) \tag{1}$$

for $n = 0, 1, \ldots$ when det $\dot{F} \neq 0$.

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Set $0_p = (0, \ldots, 0)$. Our solutions have the form

$$z_n = z_n(w, \alpha r^n) = \sum_i^p a_i \ (\alpha r^n)^i = w + d_n, \ 1 \leq p \leq q, \tag{2}$$

$$d_n = \sum_{i=1}^{\prime\, p} a_i \ (\alpha r^n)^i, \ (\alpha r^n)^i = \prod_{k=1}^p (\alpha_k r_k^n)^{i_k}, \tag{3}$$

where $a_{0_p} = w$, $\sum_i^p$ sums over $i = (i_1, \ldots, i_p) \in \mathcal{N}^p$, $\sum_i^{\prime\, p}$ excludes $i = 0_p$, $r = (r_1, \ldots, r_p)$ are any distinct eigenvalues of $\dot{F} = \dot{F}(w)$, $a_i = a_i(w) \in \mathcal{C}^p$ is given by a recurrence equation, and $\alpha \in \mathcal{C}^p$ is arbitrary. As p increases from 1 to q, this gives an increasingly rich class of solutions. The solutions for different p do not overlap.

Our main results are given in Section 3. Sections 4-5 and the appendix deal with the cases $p = 1$, $(p, q) = (2, 2), (2, 3)$, and $(3, 3)$.

If the initial value z_0 is given, the solution (2) works if

$$z_0 = \sum_i^p a_i \ \alpha^i$$

for some α in $\mathcal{C}^p$. If $p = q$, this can be inverted to find α by multivariate Lagrange inversion if z_0 is not too far from w. See, for example, Gessel (1987). If $q = 1$, (2) can be simplified further: see Withers and Nadarajah (2022). The case $p = 1$ was treated in Withers and Nadarajah (2023b): see Section 4. Note that $I \in \mathcal{N}$, $r_k^I \equiv 1$ implies $x_{n+I} = x_n$. For $j = (j_1, \ldots, j_q)$ any row vector in $\mathcal{N}^q$, set

$$j! = \prod_{i=1}^q j_i!, \ \partial_i = \partial/\partial z_{j_i}, \ F_{\boldsymbol{\cdot} j}(z) = \partial_1^{j_1} \cdots \partial_q^{j_q} F(z), \ f(j) = F_{\boldsymbol{\cdot} j}(w)/j!. \tag{4}$$

$F(z)$ and $f(j)$ are column vectors with kth components $F_k(z)$ and $f_k(j)$ for $1 \leq k \leq q$. The kth components of a_i, d_n and z_n of (2) are $a_{i,k}$, $d_{n,k}$, $z_{n,k}$. To minimise double subscripts we sometimes use

$$a(i) = a_i : \ \mathcal{N}^p \Rightarrow \mathcal{C}^q, \ a_k(i) = a_{i,k} : \ \mathcal{N}^p \Rightarrow \mathcal{C}. \tag{5}$$

2 A new type of Bell polynomial

Expressions for powers of a power series with coefficients $a_i : \mathcal{N} \to \mathcal{C}$ are given in terms of the *partial ordinary Bell polynomial* $\widehat{B}_{i,k}(a)$. These are defined in terms of

$$S(x, a) = \sum_{i=1}^\infty a_i x^i$$

for x in $\mathcal{C}$ and $a = (a_1, a_2, \ldots)$ any sequence in $\mathcal{C}$ by

$$S(x, a)^j = \sum_{i=j}^\infty x^i \ \widehat{B}_{i,j}(a) \tag{6}$$

for $j = 0, 1, \ldots$. They are tabled on page 309 of Comtet (1974) for $i \leq 10$. We drop the hat and write $B_{i,j}(a)$ or $B^{1,1}_{i,j}(a)$ for his $\widehat{B}_{i,j}(a)$.

We now extend (6) to $x \in \mathcal{C}^p$ and $i \in \mathcal{N}^p$. Set $I(A) = 1$ or 0 for $A = 0$ true or false. For $x \in \mathcal{C}^p$ and $a = \{a_i \in \mathcal{C} : \ i \in \mathcal{N}^p\}$ excluding $i = 0_p$, or with $a_{0_p} = 0$, set

$$x^i = \prod_{k=1}^p x_k^{i_k}, \ S^{p,1}(x, a) = \sum_i^p a_i \ x^i \in \mathcal{C}.$$

Define the $(p, 1)$-**Bell polynomial** $B^{p,1}_{i,j}(a)$ **on** $a_i : \mathcal{N}^p \to \mathcal{C}$ by

$$S^{p,1}(x, a)^j = \sum_i^p B^{p,1}_{i,j}(a) \ x^i \tag{7}$$

for $j = 0, 1, \ldots$. We now show that

$$B^{p,1}_{i,j}(a) = 0 \text{ if } j > |i| = i_1 + \cdots + i_p. \tag{8}$$

To see this, take $p = 2$ and set $B_i = B^{p,1}_{i,3}(a)$. Then

$$\begin{aligned} S(x, a) &= x_1 a_{1,0} + x_2 a_{0,1} + x_1^2 a_{2,0} + x_1 x_2 a_{1,1} + x_2^2 a_{0,2} + \cdots, \\ S(x, a)^3 &= x_1 B_{1,0} + x_2 B_{0,1} + x_1^2 B_{2,0} + x_1 x_2 B_{1,1} + x_2^2 B_{0,2} + \cdots. \end{aligned}$$

So, $B_i = 0$ for $|i| < 3$. By (8), we can replace $\sum_i^p$ in (7) by $\sum_{|i| \geq j}^p$. This form of multivariate Bell polynomial was introduced by Withers and Nadarajah (2010). See also Withers and Nadarajah (2013a, 2013b).

Now suppose that $a_i \in \mathcal{C}^q$, not $\mathcal{C}$, where again $i \in \mathcal{N}^p$ excluding $i = 0_p$, or equivalently with $a_i = 0_q$ for $i = 0_p$. Then we can extend (7) to the (p, q)-**Bell polynomial** $B^{p,q}_{i,j}(a)$ **on** $a_i : \mathcal{N}^p \to \mathcal{C}^q$ by noting that for $j \in \mathcal{N}^q$

$$S^{p,q}(x, a) = \sum_i^p a_i x^i$$

implies

$$[S^{p,q}(x, a)]^j = \prod_{k=1}^q \left[S^{p,1}(x, a_{\cdot k})\right]^{j_k} = \sum_i^q B^{p,q}_{i,j}(a) \ x^i, \tag{9}$$

where

$$B^{p,q}_{i,j}(a) = \sum_{i_1 + \cdots + i_q = i}^p \prod_{k=1}^q B^{p,1}_{i_k, j_k}(a_{\cdot k}), \ a_{\cdot k} = (a_{i,k}, \ i \in \mathcal{N}^p). \tag{10}$$

By (8), $j_k \leq |i_k|$ for $k = 1, \ldots, q$ implies $|j| = j_1 + \cdots + j_q \leq |i|$. So,

$$B^{p,q}_{i,j}(a) = 0 \text{ if } |j| > |i|.$$

So, $\sum_i^p$ in (9) can be replaced by $\sum_{|i| \geq |j|}^p$. A recurrence equation for $B^{p,q}_{i,j}(a)$ is given in terms of $e_{k,q}$, the kth unit vector in $\mathcal{N}^q$, by

$$B^{p,q}_{i,j+e_{k,q}}(a) = \left[\text{ coefficient of } x^i \text{ in } S^{p,q}(x, a)^j S^{p,1}(x, a_{\cdot k})\right] = \sum_{i_1 + i_2 = i} B^{p,q}_{i_1,j}(a) \ a_{i_2,k}$$

for $1 \leq k \leq p$ and $j \in \mathcal{N}^q$. For example,

$$B^{p,q}_{0_p,j}(a) = I\left(j = 0_q\right), \; B^{p,q}_{i,0_p}(a) = I\left(i = 0_p\right), \; B^{p,q}_{i,e_{b,q}}(a) = a_{i,b}, \tag{11}$$

and $j = e_{b_1,q} + \cdots + e_{b_k,q}$ implies

$$B^{p,q}_{i,j}(a) = \sum^{p}_{i_1+\cdots+i_k=i} a_{i_1,b_1} \cdots a_{i_k,b_k}.$$

$B^{p,q}_{i,j}(a)$ is a new type of Bell polynomial. A different type of multivariate Bell polynomial was used in Section 5 of Withers and Nadarajah (2012).

3 Main results

Let $e_{k,q}$ be the kth unit row vector in $\mathcal{N}^{1\times q}$. Set $I_q = diag(1,\ldots,1) \in \mathcal{C}^{q\times q}$. Given $w = F(w)$, we seek a solution of (1) of the form (2). Define the **order** of a_i as $|i| = i_1 + \cdots + i_p$. We now give a_i of order 1, then a_i of order $|i| \geq 2$ in terms of a_i of lower order. We switch to the notation of (5). Set $B_{i,j} = B^{p,q}_{i,j}(a)$, $a(i) = a_i$, $a_k(i) = a_{i,k}$.

Theorem 3.1 *Let w be any solution of $F(w) = w$. Let F be analytic at w. Fix $p \in \{1,2,\ldots,q\}$. Let $r_1,\ldots,r_p$ be any distinct eigenvalues of*

$$\dot{F} = \dot{F}(w) = \left(f\left(e_{1,q}\right),\ldots,f\left(e_{q,q}\right)\right)$$

of (4). For $1 \leq k \leq p$, let $a\left(e_{k,p}\right)$ be a right eigenvector of $\dot{F}$ with eigenvalue r_k. Assume that for $i \notin \{e_{1,p},\ldots,e_{p,p}\}$,

$$r^i = \prod_{k=1}^{p} r_k^{i_k} \text{ is not an eigenvalue of } \dot{F}. \tag{12}$$

Then for all $\alpha \in \mathcal{C}^p$, a solution of (1) is given by (2), where for $|i| \geq 2$,

$$a(i) = \left(r^i I_q - \dot{F}\right)^{-1} E_i, \; E_i = \sum^{q}_{2\leq|j|\leq|i|} B_{i,j} \; f(j) = \sum_{k=2}^{|i|} C_{i,k}, \tag{13}$$

$$C_{i,k} = \sum_{b/k} \left[B_{i,j} f(j)\right]$$

at $j = e_{b_1,q} + \cdots + e_{b_k,q}$, and $\sum_{b/k}$ sums over $1 \leq b_1 \leq \cdots \leq b_k \leq q$.

(12) implies that roots of 1 are not eigenvalues of $\dot{F}$. For, if $1^{1/I}, r_2,\ldots,r_p$ are eigenvalues, then $1^{1/I} = \left(1^{1/I}\right)^{I+1} r_2^0 \cdots r_p^0$ is not an eigenvalue.

Proof Note that

$$d_{n,k} = S\left(\alpha r^n, a_{\cdot k}\right) = \sum_{i}^{p} a_{i,k} \; \left(\alpha r^n\right)^i$$

for $(\alpha r^n)^i$ of (3) and $a_{\bullet k}$ of (10). The Taylor series expansion gives

$$z_{n+1} - w = d_{n+1} = F(z_n) - F(w) = \sum_j^q d_n^j \ f(j), \tag{14}$$

where

$$d_n^j = \prod_{k=1}^q d_{n,k}^{j_k} = \sum_{|i| \geq |j|} B_{i,j} \ (\alpha r^n)^i$$

by (9). For $i \in \mathcal{N}^p$ the coefficient of $(\alpha r^n)^i$ in (14) is $r^i \ a(i) = C_i$, where

$$C_i = \sum_{|j| \leq |i|}^q B_{i,j} \ f(j) = \sum_{k=1}^{|i|} C_{i,k}. \tag{15}$$

Consider the case $i = e_m$, where $1 \leq m \leq p$. By (11) and (15),

$$r_m a(e_{m,p}) = C_{e_{m,p}} = \sum_{k=1}^q B_{e_{m,p}, e_{k,q}} \ f(e_{k,q}) = \sum_{k=1}^q a_k(e_{m,p}) \ f(e_{k,q}) = \dot{F} a(e_{m,p}).$$

So, for $m = 1, \ldots, q$, $a(e_{m,p})$ is a right eigenvector of $\dot{F}$ with eigenvalue r_m. Now take $|i| \geq 2$. Then $C_{1,k} = \dot{F} a(i)$. So, (15) implies (13). The proof is complete. $\square$

We now illustrate how $B_{i,j}$ can be calculated as needed using the fact that $j = e_{b_1,q} + \cdots + e_{b_k,q}$ implies

$$B_{i,j} = \sum_{i_1 + \cdots + i_k = i}^p a_{b_1}(i_1) \cdots a_{b_k}(i_k).$$

For $|i| = r$, we write $i = I_1 + \cdots + I_r$, where $I_k = e_{D_k,p}$, $1 \leq D_1 \cdots \leq D_r \leq p$.

Consider the case $|i| = 2$. The partitions of 2 are 2 and 11. So, $a(i) = \left(r^i I_q - \dot{F}\right)^{-1} E_2$, where, for $j = e_{b,q} + e_{c,q}$,

$$E_2 = C_{i,2} = \sum_{b \leq c} B_{i,j} \ f(j), \ B_{2I_1,j} = a_b(I_1) a_c(I_1), \ B_{I_1+I_2,j} = \sum_{b,c}^2 a_b(I_1) a_c(I_2),$$

where

$$\sum_{b,c}^2 A_{b,c} = A_{b,c} + A_{c,b}.$$

For example, if $i = 2e_{1,p}$, $a(i) = \left(r_1^2 I_q - \dot{F}\right)^{-1} C_{i,2}$, where

$$C_{i,2} = \sum_{1 \leq b \leq c \leq q} a_b(e_{1,p}) a_c(e_{1,p}) \ f(e_{b,q} + e_{c,q}). \tag{16}$$

If $i = e_{1,p} + e_{2,p}$, $a(i) = \left(r_1 r_2 I_q - \dot{F}\right)^{-1} C_{i,2}$, where

$$C_{i,2} = \sum_{1 \leq b \leq c \leq q} \left[\sum_{b,c}^{2} a_b\left(e_{1,p}\right) a_c\left(e_{2,p}\right) \right] f\left(e_{b,q} + e_{c,q}\right). \qquad (17)$$

Consider the case $|i| = 3$. The partitions of 3 are 3,21 and 111. So, $a(i) = \left(r^i I_q - \dot{F}\right)^{-1} (C_{i,2} + C_{i,3})$, where, for $j = e_{b,q} + e_{c,q}$,

$$C_{i,2} = \sum_{b \leq c} B_{i,j} f(j),$$

$$B_{3I_1,j} = \sum_{b,c}^{2} a_b\left(I_1\right) a_c\left(2I_1\right), \ B_{2I_1+I_2,j} = \sum_{b,c}^{2} \left[a_b\left(2I_1\right) a_c\left(I_2\right) + a_b\left(I_1\right) a_c\left(I_1 + I_2\right)\right],$$

$$B_{I_1+I_2+I_3,j} = \sum_{b,c}^{2} \sum_{1,2,3}^{3} a_b\left(I_1\right) a_c\left(I_2 + I_3\right),$$

and, for $j = e_{b,q} + e_{c,q} + e_{d,q}$,

$$C_{i,3} = \sum_{b \leq c \leq d} B_{i,j} f(j), \ B_{3I_1,j} = a_b\left(I_1\right) a_c\left(I_1\right) a_d\left(I_1\right),$$

$$B_{2I_1+I_2,j} = \sum_{b,c,d}^{3} a_b\left(I_1\right) a_c\left(I_1\right) a_d\left(I_2\right), \ B_{I_1+I_2+I_3,j} = \sum_{b,c,d}^{6} a_b\left(I_1\right) a_c\left(I_2\right) a_d\left(I_3\right),$$

and $\sum_{b,c,d}^{N}$ sums over all permutations of b, c, d giving N distinct terms. N is the multinomial coefficient.

Consider the case $|i| = 4$. The partitions of 4 are 4,31,22,211 and 1111. So, $a(i) = \left(r^i I_q - \dot{F}\right)^{-1} \sum_{k=2}^{4} C_{i,k}$, where, for $j = e_{b,q} + e_{c,q}$,

$$C_{i,2} = \sum_{b \leq c} B_{i,j} f(j),$$

$$B_{4I_1,j} = \sum_{b,c}^{2} a_b\left(I_1\right) a_c\left(3I_1\right) + a_b\left(2I_1\right) a_c\left(2I_1\right),$$

$$B_{3I_1+I_2,j} = \sum_{b,c}^{2} \left[a_b\left(3I_1\right) a_c\left(I_2\right) + a_b\left(2I_1\right) a_c\left(I_1 + I_2\right) + a_b\left(I_1\right) a_c\left(I_1 + I_2\right)\right],$$

$$B_{2I_1+2I_2,j} = \sum_{b,c}^{2} \left[a_b\left(2I_1 + I_2\right) a_c\left(I_2\right) + a_b\left(2I_1\right) a_c\left(2I_2\right) + a_b\left(I_1\right) a_c\left(I_1 + 2I_2\right)\right],$$

$$B_{2I_1+I_2+I_3,j} = \sum_{b,c}^{2} \left[a_b\left(2I_1\right) a_c\left(I_2 + I_3\right) + \sum_{1,2,3}^{6} a_b\left(2I_1 + I_2\right) a_c\left(I_3\right) \right],$$

$$B_{I_1+I_2+I_3+I_4,j} = \sum_{b,c}^{2} \left[\sum_{1,2,3}^{4} a_b\left(I_1\right) a_c\left(I_2 + I_3 + I_4\right) + \sum_{1,2,3}^{6} a_b\left(I_1 + I_2\right) a_c\left(I_3 + I_4\right) \right],$$

and, for $j = e_{b,q} + e_{c,q} + e_{d,q}$,

$$C_{i,3} = \sum_{b \le c \le d} B_{i,j} f(j),\ B_{4I_1,j} = \sum_{b,c,d}^{3} a_b\left(2I_1\right) a_c\left(I_1\right) a_d\left(I_1\right),$$

$$B_{3I_1+I_2,j} = \sum_{b,c,d}^{6} a_b\left(2I_1\right) a_c\left(I_1\right) a_d\left(I_2\right),$$

$$B_{2I_1+2I_2,j} = \sum_{b,c,d}^{3} \left[a_b\left(2I_1\right) a_c\left(I_2\right) a_d\left(I_2\right) + a_b\left(I_1\right) a_c\left(I_1\right) a_d\left(2I_2\right)\right],$$

$$B_{2I_1+I_2+I_3,j} = \sum_{b,c,d}^{6} \left[a_b\left(2I_1\right) a_c\left(I_2\right) a_d\left(I_3\right) + a_b\left(I_1\right) a_c\left(I_1+I_2\right) a_d\left(I_3\right)\right],$$

$$B_{I_1+I_2+I_3+I_4,j} = \sum_{b,c,d}^{12} \left[a_b\left(I_1+I_2\right) a_c\left(I_3\right) a_d\left(I_4\right)\right],$$

and, for $j = e_{b_1} + e_{b_2} + e_{b_3} + e_{b_4}$,

$$C_{i,4} = \sum_{b_1 \le \cdots \le b_4} B_{i,j} f(j),\ B_{4I_1,j} = \prod_{k=1}^{4} a_{b_k}\left(I_k\right),$$

$$B_{3I_1+I_2,j} = \sum_{b_1 \cdots b_4}^{6} a_{b_1}\left(I_1\right) a_{b_2}\left(I_1\right) a_{b_3}\left(I_1\right) a_{b_4}\left(I_2\right),$$

$$B_{2I_1+2I_2,j} = \sum_{b_1 \cdots b_4}^{6} a_{b_1}\left(I_1\right) a_{b_2}\left(I_1\right) a_{b_3}\left(I_2\right) a_{b_4}\left(I_2\right),$$

$$B_{2I_1+I_2+I_3,j} = \sum_{b_1 \cdots b_4}^{12} a_{b_1}\left(I_1\right) a_{b_2}\left(I_1\right) a_{b_3}\left(I_2\right) a_{b_4}\left(I_3\right),$$

$$B_{I_1+I_2+I_3+I_4,j} = \sum_{b_1 \cdots b_4}^{24} \prod_{k=1}^{4} a_{b_k}\left(I_k\right).$$

4 The case $p = 1$

We now show that the solution with $p = 1$ reduces to that of Withers and Nadarajah (2023b), since E_i of (13) agrees with that given there by two methods, the second being (2.7) there. Withers and Nadarajah (2023b) gave many examples. We now give some values of $B_{i,j} = B_{i,j}^{1,q}$.

Set $\delta_{r,s} = I(r = s)$. Then $e_{1,1} = 1$, r_1 is any eigenvalue of $\dot{F}$ with right eigenvector $a_1 \in \mathcal{C}^q$.

$j = e_{b_1,q} + \cdots + e_{b_k,q}$ implies that

$$B_{i,j} = \sum_{i_1+\cdots+i_k=i}^{1} a_{b_1}(i_1) \cdots a_{b_k}(i_k),$$

$$B_{1,j} = \delta_{k,1}\ a_{b_1}(1),\ B_{2,j} = \delta_{k,1}\ a_{b_1}(2) + \delta_{k,2}\ a_{b_1}(1) a_{b_2}(1),$$

$$B_{3,j} = \delta_{k,1}\ a_{b_1}(3) + \delta_{k,2} \sum_{b_1,b_2}^{2} a_{b_1}(1) a_{b_2}(2) + \delta_{k,3}\ a_{b_1}(1) a_{b_2}(1) a_{b_3}(1),$$

$$B_{4,j} = \delta_{k,1} a_{b_1}(4) + \delta_{k,2} \left[\sum_{b_1,b_2}^{2} a_{b_1}(1) a_{b_2}(3) + a_{b_1}(2) a_{b_2}(2) \right]$$

$$+ \delta_{k,3} \sum_{b_1,b_2,b_3}^{3} a_{b_1}(1) a_{b_2}(1) a_{b_3}(2) + \delta_{k,4}\ a_{b_1}(1) a_{b_2}(1) a_{b_3}(1) a_{b_4}(1),$$

and in general, $B_{i,j}^{1,q}$ can be read off the expression for $B_{i,j}^{1,1}$ tabled on page 309 of Comtet (1974). By (16) and (17),

$$C_{2,2} = \sum_{b \le c} a_b(1) a_c(1)\ f\left(e_{b,q} + e_{c,q}\right),$$

$$C_{3,2} = \sum_{b \le c} \left[\sum_{b,c}^{2} a_b(1) a_c(2) \right]\ f\left(e_{b,q} + e_{c,q}\right),$$

$$C_{3,3} = \sum_{b \le c \le d} a_b(1) a_c(1) a_d(1)\ f\left(e_{b,q} + e_{c,q} + e_{d,q}\right).$$

This gives $C_{i,k}$ needed in (13) for $a(2)$, $a(3)$.

5 Examples with $p = q = 2$, $\dot{F} = (f(10), f(01))$

Consider the case $|i| = 2$.

$$a_{20} = \left(r_1^2 I_2 - \dot{F}\right)^{-1}\ C_{20,2}, \tag{18}$$

$$a_{02} = \left(r_2^2 - \dot{F}\right)^{-1}\ C_{02,2}, \tag{19}$$

$$a_{11} = \left(r_1 r_2 I_2 - \dot{F}\right)^{-1}\ C_{11,2}, \tag{20}$$

where

$$C_{20,2} = a_{10,1}^2 f(20) + a_{10,2}^2 f(02) + a_{10,1} a_{10,2} f(11),$$
$$C_{02,2} = a_{01,2}^2 f(20) + a_{01,1}^2 f(02) + a_{01,1} a_{01,2} f(11),$$
$$C_{11,2} = 2 a_{10,1} a_{01,1} f(20) + \left(a_{10,1} a_{01,2} + a_{01,1} a_{10,2}\right) f(11) + 2 a_{10,2} a_{01,2} f(02).$$

Consider the case $|i| = 3$.

$$a_{30} = \left(r_1^3 I_2 - \dot{F}\right)^{-1} \left(C_{30,2} + C_{30,3}\right),$$

$$a_{21} = \left(r_1^2 r_2 I_2 - \dot{F}\right)^{-1} \left(C_{21,2} + C_{21,3}\right), \tag{21}$$

where

$$\begin{aligned}
C_{30,2} &= 2a_{10,1}a_{20,1}f(20) + (a_{10,1}a_{20,2} + a_{10,2}a_{20,1})\, f(11) + 2a_{10,2}a_{20,2}f(02),\\
C_{30,3} &= a_{10,1}^3 f(30) + a_{10,2}^3 f(03) + a_{10,1}^2 a_{10,2} f(21) + a_{10,1}a_{10,2}^2 f(12),\\
C_{21,2} &= 2a_{10,1}a_{11,1}f(20) + (a_{10,1}a_{11,2} + a_{11,1}a_{01,2})\, f(11) + 2a_{10,2}a_{11,2}f(02),\\
C_{21,3} &= 3a_{10,1}^2 a_{01,1} f(30) + \left(a_{10,1}^2 a_{01,2} + 2a_{10,1}a_{01,1}a_{10,2}\right) f(21)\\
&+ \left(a_{10,2}^2 a_{01,1} + 2a_{10,2}a_{01,2}a_{10,1}\right) f(12) + 3a_{10,2}^2 a_{01,2} f(03).
\end{aligned}$$

a_{03} and a_{12} can be written down from a_{30} and a_{21}.

Example 5.1 *Take $F(z) = (G(z_1), z_1 z_2)'$, where $\dot{F} : \mathcal{C} \to \mathcal{C}$ is analytic. For $j = 0, 1, \ldots$, set $g_j = G_{.j}(w_1)/j!$. Then $\dot{F} = (f(10), f(01))$, where $f(10) = (g_1, w_1)'$, $f(01) = (0, w_1)'$. Other $f(j)$ are 0_2 except for $f(11) = (0, 1)'$, $f(j_1 0) = g_{j_1}(1, 0)'$ for $j_1 \geq 2$. The fixed points w are given by $w_1 = G(w_1)$ and $w_2 = w_1 w_2$, that is, $w_1 = 1$ or $w_2 = 0$. The eigenvalues are $r_1 = g_1$ and $r_2 = w_1$.*

Consider the case $w_1 \neq 1$, $w_2 = 0$. Then $\dot{F} = diag(g_1, w_1)$ and we can take $r_1 = g_1$, $a_0 = (1, 0)'$, or $r_2 = w_1$, $a_0 = (0, 1)'$. That is, $(1, 0)'$ and $(0, 1)'$ are right eigenvectors of $\dot{F}$ with respective eigenvalues g_1 and w_1. For $|i| = 2, 3$, a_i are given by (18)-(21) with $f(j)$ as above. A refined form of this solution is given by Corollary 2.4 and Theorem 2.2 of Withers and Nadarajah (2023a): for $g_1 \neq 0$ or 1, $x_{n+1} = G(x_n)$ has a solution

$$x_n = w_1 + \sum_{i=1}^{\infty} s_i g_1^{1-i} \left(\alpha g_1^n\right)^i,$$

where $s_1 = 1$, and for $i \geq 2$, s_i is given by the recurrence formula

$$s_i = U_i^{-1} s_i' = N_i / D_i,\ s_i' = \sum_{j=2}^{i} \widehat{B}_{i,j}(s)\ v_j,$$

where

$$\begin{aligned}
&U_i = g_1^{i-1} - 1,\ D_i = \prod_{j=2}^{i} U_j,\ v_j = g_1^{j-2} g_j,\\
&N_2 = v_2,\ N_3 = 2v_2^2 + U_2 v_3,\ N_4 = (r+5)v_2^3 + U_2(3r+5)v_2 v_3 + D_3 v_4,
\end{aligned}$$

and $N_5, N_6, s_2, \ldots, s_6$ are given in Withers and Nadarajah (2023a) explicitly. An equivalent result is given by Theorem 3.1 of Withers and Nadarajah (2022). Note that $y_{n+1} = x_n y_n$ implies that

$$y_n = y_0 \prod_{N=0}^{n-1} x_N$$

for $n \geq 1$.

Consider the case $w_1 = 1$, that is, $G(1) = 1$. Then w_2 is arbitrary. Examples of this are $G(x) = x^d$, $G(x) = 1 + b(x+c)^d - b(1+c)^d$, $G(x) = \exp\left[-a(x-1)^d\right]$, $G(x) = 1 + b \ln\left[(x+c)/(1+c)\right]$. The eigenvalues are $r = r_1 = g_1$ and $r = r_2 = 1$.

Consider the case $w_1 = 1$, $r = r_1 = g_1$. A right eigenvector is $(g_1 - 1, w_2)'$.

Consider the case $w_1 = 1$, $r = r_2 = 1$. A right eigenvector is $(0, 1)'$.

Example 5.2 *Take* $F(z) = (z_2 + c_1,\ z_1 z_2 + c_2)'$. *So,* $w_1 = w_2 + c_1$, $w_2 = w_1 w_2 + c_2$, $w_2^2 + (c_1 - 1) w_2 + c_2 = 0$, $w_2 = \left(1 - c_1 \pm \delta^{1/2}\right)/2 = w_{2,1}, w_{2,2}$ *say, where* $\delta = (c_1 - 1)^2 - 4c_2$, *giving two fixed points* $w_i = (w_{2,i} + c_1, w_{2,i})'$ *for* $i = 1, 2$. *The non-zero* $f(j)$ *are* $f(10) = (0, w_2)'$, $f(01) = (1, w_1)'$ *and* $f(11) = (0, 1)'$.

Example 5.3 *An extension of the Mandelbrot equation* $z_{n+1} = z_n^2 + c$ *to* $\mathcal{C}^2$ *is*

$$x_{n+1} = y_n^2 + c_2,\ y_{n+1} = x_n^2 + c_1,$$

that is,

$$F(z) = \begin{pmatrix} z_2^2 + c_2 \\ z_1^2 + c_1 \end{pmatrix},$$

implying that

$$\dot{F} = (f(10), f(01)) = 2 \begin{pmatrix} 0 & w_2 \\ w_1 & 0 \end{pmatrix}$$

and $w_1 = w_2^2 + c_2$, $w_2 = w_1^2 + c_1$, $w_1 - c_2 = \left(w_1^2 + c_1\right)^2$, $w_1^4 + 2c_1 w_1^2 - w_1 + c_1^2 + c_2 = 0$. *Its four roots (and so the four fixed points* w*), can be computed by Section 3.8.3 of Abramowitz and Stegun (1964). For a given fixed point* w*, the eigenvalues are* $r_1, r_2 = \pm 2\nu$, *where* $\nu = (w_1 w_2)^{1/2}$. *The non-zero* $f(j)$ *are* $f(10) = 2w_1 e_2'$, $f(01) = 2w_2 e_1'$, $f(20) = e_2'$, $f(02) = e_1'$.

Example 5.4 *An extension of the logistic map* $z_{n+1} = c z_n (1 - z_n)$ *to* $\mathcal{C}^2$ *is*

$$x_{n+1} = c_1 x_n (1 - y_n),\ y_{n+1} = c_2 y_n (1 - x_n),$$

that is,

$$F(z) = \begin{pmatrix} c_1 z_1 (1 - z_2) \\ c_2 z_2 (1 - z_1) \end{pmatrix},$$

implying

$$\dot{F} = (f(10), f(01)) = \begin{pmatrix} c_1 (1 - w_2) & -c_1 w_1 \\ -c_2 w_2 & c_2 (1 - w_1) \end{pmatrix}$$

with four fixed points given by $w_1 = 0$ *or* $1 - c_2^{-1}$ *and* $w_2 = 0$ *or* $1 - c_1^{-1}$. *The eigenvalues of* $\dot{F}$ *are the roots of* $r^2 - rT + D$, *that is,*

$$r_1, r_2 = \left(T \pm \delta^{1/2}\right)/2$$

for

$$\begin{aligned} T &= trace\left(\dot{F}\right) = c_1 (1 - w_2) + c_2 (1 - w_1), \\ \delta &= T^2 - 4D = c_1^2 (1 - w_2)^2 + 2\gamma c_1 c_2 + c_2^2 (1 - w_2)^2,\ \gamma = (1 + w_1)(1 + w_2) - 2, \\ D &= det\ \dot{F} = c_1 c_2 (1 - w_1 - w_2). \end{aligned}$$

The only other non-zero $f(j)$ *is* $f(11) = (-c_1, c_2)'$.

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Appendix

Take $p = 2$ and $q = 3$. In this case, $\dot{F} = (f(100), f(010), f(001))$. By (16) and (17),

$$
\begin{aligned}
& C_{20,2} = a_1(10)^2 f(200) + a_2(10)^2 f(020) + a_3(10)^2 f(002) \\
& + a_1(10)a_2(10)f(110) + a_1(10)a_3(10)f(101) + a_2(10)a_3(10)f(011), \\
& C_{11,2} = 2a_1(10)a_1(01)f(200) + 2a_2(10)a_2(01)f(020) + 2a_3(10)a_3(01)f(002) \\
& + [a_1(10)a_2(01) + a_2(10)a_1(01)]\, f(110) + [a_1(10)a_3(01) + a_3(10)a_1(01)]\, f(101) \\
& + [a_2(10)a_3(01) + a_3(10)a_2(01)]\, f(011).
\end{aligned}
$$

Similarly, we can obtain $a(i)$ for $|i| = 3$.

Take $p = q = 3$. In this case, $\dot{F} = (f(100), f(010), f(001))$. We spell out $C_{i,k}$ needed for $a(i)$, $|i| = 2, 3$. If $|i| = 2$, by (16) and (17),

$$
\begin{aligned}
& i = (200): \ C_{i,2} = a_1(100)^2 \ f(200) + a_2(100)^2 \ f(020) + a_3(100)^2 \ f(002) \\
& + a_1(100)a_2(100) \ f(110) + a_1(100)a_3(100) \ f(101) + a_2(100)a_3(100) \ f(011). \\
& i = 110: \ C_{i,2} = 2a_1(100)a_1(010) \ f(200) + 2a_2(100)a_2(010) \ f(020) \\
& + 2a_3(100)a_3(010) \ f(002) + [a_1(100)a_2(010) + a_2(100)a_1(010)] \ f(110) \\
& + [a_1(100)a_3(010) + a_3(100)a_1(010)] \ f(101) + [a_2(100)a_3(010) + a_3(100)a_2(010)] \ f(011).
\end{aligned}
$$

If $|i| = 3$ then

$$i = (300): \ C_{i,2} = 2a_1(100)a_1(200)\ f(200) + 2a_2(100)a_2(200)\ f(020)$$
$$+ 2a_3(100)a_3(200)\ f(002) + [a_1(100)a_2(200) + a_2(100)a_1(200)]\ f(110)$$
$$+ [a_1(100)a_3(200) + a_3(100)a_1(200)]\ f(101)$$
$$+ [a_2(100)a_3(200) + a_3(100)a_2(200)]\ f(011).$$
$$i = (300): \ C_{i2} = a_1(100)^3\ f(300) + a_2(100)^3\ f(030) + a_3(100)^3\ f(003)$$
$$+ a_1(100)^2 a_2(100)\ f(210) + a_1(100)a_2(100)^2\ f(120) + a_1(100)^2 a_3(100)\ f(201)$$
$$+ a_1(100)a_3(100)^2\ f(102) + a_2(100)^2 a_3(100)\ f(021) + a_2(100)a_3(100)^2\ f(012)$$
$$+ a_1(100)a_2(100)a_3(100)\ f(111).$$
$$i = (210): \ C_{i2} = 2a_1(200)a_1(010)\ f(200) + 2a_2(200)a_2(010)\ f(020)$$
$$+ 2a_3(200)a_3(010)\ f(002)$$
$$+ [a_1(200)a_2(010) + a_1(100)a_2(110) + a_2(200)a_1(010) + a_2(100)a_1(110)]\ f(110)$$
$$+ [a_1(200)a_3(010) + a_1(100)a_3(110) + a_3(200)a_1(010) + a_3(100)a_1(110)]\ f(101)$$
$$+ [a_2(200)a_3(010) + a_2(100)a_3(110) + a_3(200)a_2(010) + a_3(100)a_2(110)]\ f(011).$$
$$i = (210): \ C_{i,3} = 3a_1(100)^2 a_1(010)\ f(300) + 3a_2(100)^2 a_2(010)\ f(030)$$
$$+ 3a_3(100)^2 a_3(010)\ f(003) + \left[a_1(100)^2 a_2(010) + 2a_1(100)a_2(100)a_1(010)\right]\ f(210)$$
$$+ \left[a_1(100)^2 a_3(010) + 2a_1(100)a_3(100)a_1(010)\right]\ f(201) + \left[a_2(100)^2 a_1(010) + 2a_2(100)a_1(100)a_2(010)\right]\ f(120)$$
$$+ \left[a_3(100)^2 a_1(010) + 2a_3(100)a_1(100)a_3(010)\right]\ f(102)$$
$$+ \left[a_2(100)^2 a_3(010) + 2a_2(100)a_3(100)a_2(010)\right]\ f(021)$$
$$+ \left[a_3(100)^2 a_2(010) + 2a_3(100)a_2(100)a_3(010)\right]\ f(012)$$
$$+ [a_1(100)a_2(100)a_3(010) + a_2(100)a_3(100)a_1(010) + a_3(100)a_1(100)a_2(010)]\ f(111).$$
$$i = (111): \ C_{i,2} = 2\left[a_1(100)a_1(011) + a_1(010)a_1(101) + a_1(001)a_1(110)\right]\ f(200)$$
$$+ 2\left[a_2(100)a_2(011) + a_2(010)a_2(101) + a_2(001)a_2(110)\right]\ f(020)$$
$$+ 2\left[a_3(100)a_3(011) + a_3(010)a_3(101) + a_3(001)a_3(110)\right]\ f(002)$$
$$+ [a_1(100)a_2(011) + a_2(100)a_1(011) + a_1(010)a_2(101) + a_2(010)a_1(101) + a_1(001)a_2(110) + a_2(001)a_1(110)]\ f(110)$$
$$+ [a_1(100)a_3(011) + a_3(100)a_1(011) + a_1(010)a_3(101) + a_3(010)a_1(101) + a_1(001)a_3(110) + a_3(001)a_1(110)]\ f(101)$$
$$+ [a_2(100)a_3(011) + a_3(100)a_2(011) + a_2(010)a_3(101) + a_3(010)a_2(101) + a_2(001)a_3(110) + a_3(001)a_2(110)]\ f(011).$$
$$i = (111): \ C_{i,3} = 6a_1(100)a_1(010)a_1(001)\ f(300) + 6a_2(100)a_2(010)a_3(001)\ f(030) + 6a_3(100)a_3(010)a_3(001)\ f(003)$$
$$+ 2\left[a_1(100)a_1(010)a_2(001) + a_1(100)a_1(001)a_2(010) + a_1(010)a_1(001)a_2(100)\right]\ f(210)$$
$$+ 2\left[a_1(100)a_1(010)a_3(001) + a_1(100)a_1(001)a_3(010) + a_1(010)a_1(001)a_3(100)\right]\ f(201)$$
$$+ 2\left[a_1(100)a_3(010)a_3(001) + a_1(100)a_3(001)a_3(010) + a_1(010)a_3(001)a_3(100)\right]\ f(102)$$
$$+ 2\left[a_2(100)a_2(010)a_3(001) + a_2(100)a_2(001)a_3(010) + a_2(010)a_2(001)a_3(100)\right]\ f(021)$$
$$+ 2\left[a_2(100)a_3(010)a_3(001) + a_2(100)a_3(001)a_3(010) + a_2(010)a_3(001)a_3(100)\right]\ f(012)$$
$$+ [a_1(100)a_2(010)a_3(001) + a_1(100)a_3(010)a_2(001) + a_2(100)a_1(010)a_3(001) + a_2(100)a_3(010)a_1(001) + a_3(100)a_1(010)a_2(001) + a_3(100)a_2(010)a_1(001)]\ f(111).$$