The Order of Finite Generation of SO(3) and Optimization of Rotation Sequences

Danail Brezov

Abstract

We generalize an old result due to Lowenthal [1] and a more recent one due to Hamada [2] on the order of finite generation of the rotation group SO(3) both for fixed and arbitrary compound transformations. Unlike the above cited authors, we consider decompositions into factors with more than two invariant axes and derive a simple estimate for certain subsets of rotations using intuitive geometric proofs. Particular examples of potential interest for the applications are considered with an emphasis on optimization. Possible generalizations are discussed as well, e.g. dimensional induction, the hyperbolic case and screw motion.

Introduction

If $G_{1,2}$ are one-parameter subgroups of a connected Lie group G, the order of finite generation with respect to $G_{1,2}$ is the minimal number N, such that every element of G can be expressed in N factors $g_i \in G_{1,2}$. Lowenthal [1] published in 1971 his famous result on the order of the rotation group in $\mathbb{R}^3$

Theorem 1 *Each* SO(3) *transformation may be decomposed into*

$$N_\gamma = 1 + \left\lceil \frac{\pi}{\gamma} \right\rceil . \tag{1}$$

alternating rotations about two fixed axes with a relative angle $\gamma \in \left(0, \frac{\pi}{2}\right]$.

Here and below we use the notation $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid x \leq n\} = -\lfloor -x \rfloor$, where $\lfloor x \rfloor$ stands for the integer part of x. Note that the only way to ensure order three is to pick orthogonal axes that gives the classical Euler setting. For $\gamma = 60^\circ$, we may decompose into four factors, $\gamma = 45^\circ$ yields five etc. Below we consider the decomposition conditions for the cases of two and three factors and then generalize the corresponding results using induction.

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

Low Order Cases

In the following we use similar notation to [3]. Let $\mathbf{n}, \mathbf{a}_i \in \mathbb{S}^2$ denote respectively the invariant unit vectors of the compound rotation $\mathcal{R}$ and those in the decomposition $\mathcal{R}_i$, with orientation chosen such that the relative angles

$$\gamma_{ij} = \arccos(\mathbf{a}_i \cdot \mathbf{a}_j), \qquad \beta_i = \arccos(\mathbf{a}_i \cdot \mathbf{n})$$

are (positive) acute or right. Similarly, we shall use the notation

$$\tilde{\gamma}_{ij} = \arccos(\mathbf{a}_i \cdot \mathcal{R}\,\mathbf{a}_j)$$

and let ϕ and ϕ_i denote the rotation angles with the so chosen orientation.

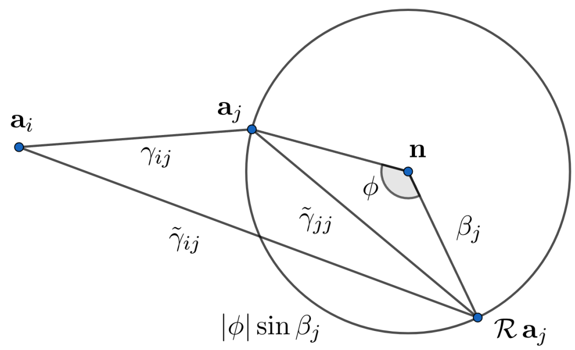

Figure 1: Illustration of Lemma 1: straight lines depict geodesic segments on $\mathbb{S}^2$, while the circle shows the trace of $\mathbf{a}_j$ under the rotation $\mathcal{R}$ by an angle ϕ about $\mathbf{n}$.

Note that in this way $\mathbf{n}$ and $\mathbf{a}_j$ are regarded as points on the closed unit semi-sphere $\bar{\mathbb{S}}^2_+$ and γ_{ij}, β_j, respectively as spherical distances. Below we shall relate lengths of broken geodesics in $\bar{\mathbb{S}}^2_+$ with decomposability conditions and thus derive an estimate for the order of SO(3) beginning with the following

Lemma 1 *With the above notation each* $\mathcal{R}(\mathbf{n}, \phi) \in \mathrm{SO}(3)$ *satisfies*

$$\gamma_{ij} - 2\beta_j \leq \tilde{\gamma}_{ij} \leq \gamma_{ij} + 2\beta_j, \qquad \gamma_{ij} - |\phi|\sin\beta_j \leq \tilde{\gamma}_{ij} \leq \gamma_{ij} + |\phi|\sin\beta_j.$$

Proof. The first estimate says that $2\beta_i$ is the maximal angle, at which $\mathcal{R}$ shifts vectors on the unit sphere (the case of a half-turn). For the second one we introduce spherical coordinates with polar and azimuthal angles: respectively ϕ and β_j and point out that the $\tilde{\gamma}_{jj} \leq |\phi|\sin\beta_j$ since the geodesic distance $\tilde{\gamma}_{jj}$ cannot exceed the length of the corresponding arc (Figure 1). The triangle inequality completes the proof as $\tilde{\gamma}_{ij} \in [\gamma_{ij} - \tilde{\gamma}_{jj}, \gamma_{ij} + \tilde{\gamma}_{jj}]$. □

Next, we considered the decomposition problem beginning with two factors:

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

Lemma 2 *A transformation $\mathcal{R} \in \mathrm{SO}(3)$ is decomposable into a pair of consecutive rotations about $\mathbf{a}_1$ and $\mathbf{a}_2$ (in this order) if and only if $\tilde{\gamma}_{21} = \gamma_{21}$.*

Proof. Necessity is easier to prove since the invariant axis theorem yields

$$(\mathbf{a}_2, \mathcal{R}\,\mathbf{a}_1) = (\mathbf{a}_2, \mathcal{R}_2\mathcal{R}_1\,\mathbf{a}_1) = (\mathcal{R}_2^t\,\mathbf{a}_2, \mathcal{R}_1\,\mathbf{a}_1) = (\mathbf{a}_2,\,\mathbf{a}_1)$$

that is seen as an equality for the cosines of the positive acute or right angles $\tilde{\gamma}_{21}$ and γ_{21}. Next, we note that SO(3) is compact and connected, acting freely on itself via left shifts, so the map $\tilde{\mathcal{R}}_\lambda = \mathcal{R}\mathcal{R}_1^t(\lambda)$ with $\lambda \in \mathbb{S}^1$ satisfies

$$(\mathbf{a}_2, \tilde{\mathcal{R}}_\lambda \mathbf{a}_1) = (\mathbf{a}_2, \mathbf{a}_1)$$

and thus, the λ-orbit of $\mathbf{a}_1$ is a rotation about $\mathbf{a}_2$. But then, $\tilde{\mathcal{R}}_\lambda$, and hence $\mathcal{R}$, can be decomposed into a pair of rotations about $\mathbf{a}_1$ and $\mathbf{a}_2$ (Figure 2). □

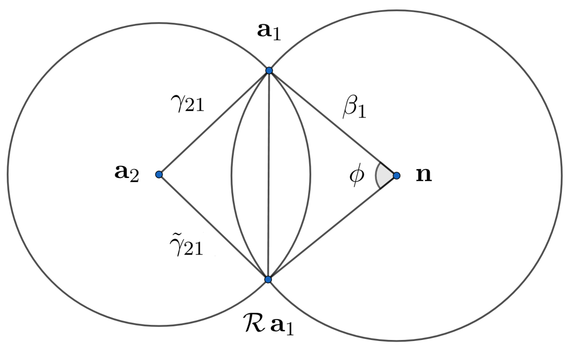

Figure 2: Illustration of Lemma 2, showing the rotation of $\mathbf{a}_1$ about $\mathbf{n}$ is equivalent to a rotation about $\mathbf{a}_2$ provided that $\tilde{\gamma}_{21} = \gamma_{21}$, hence the decomposition $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$.

With this in mind, it is not hard to prove the following

Lemma 3 *The decomposition $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_1$ exists*

- *for an arbitrary angle ϕ if and only if $\beta_1 \leq \gamma_{12}$;*
- *for an arbitrary axis $\mathbf{n}$ if and only if $|\phi| \leq 2\gamma_{12}$.*

Proof. We use the notation $\gamma = \gamma_{12}$ for convenience and conjugate obtaining

$$\mathcal{R}(\mathbf{n}', \phi) = \mathcal{R}_2(\phi_2)\mathcal{R}_1(\phi_1 + \phi_3), \qquad \mathbf{n}' = \mathcal{R}_1(-\phi_3)\,\mathbf{n}$$

with $\beta_1' = \measuredangle(\mathbf{a}_1, \mathbf{n}') = \beta_1$ for an arbitrary angle ϕ_3. The locus of $\mathcal{R}(\mathbf{n}', \phi)\,\mathbf{a}_1$ for any fixed angle $\phi \in \mathbb{S}^1$ is a circle σ centered at $\mathbf{a}_1$ and parameterized with

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

ϕ_3, whose radius obviously does not exceed $2\beta_1$. Therefore, if $\beta_1 \leq \gamma$ this orbit has at least one common point with the γ-orbit of $\mathbf{a}_1$ about $\mathbf{a}_2$, i.e., one can set the value of ϕ_3 in such a way that the angle $\tilde{\gamma}'_{21}$ between $\mathcal{R}(\mathbf{n}', \phi)\,\mathbf{a}_1$ and $\mathbf{a}_2$ equals γ and the above decomposition is guaranteed by Lemma 2. The exact same argument leads to the conclusion that the above ϕ_3-orbit has a common point with the γ-orbit of $\mathbf{a}_1$ about $\mathbf{a}_2$ as long as $|\phi| \leq 2\gamma$ that proves necessity and sufficiency is implied by the invertibility of Lemma 2. $\square$

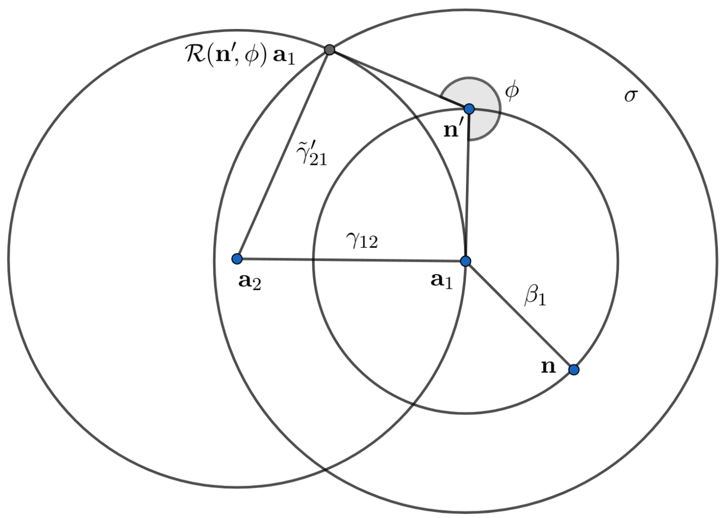

Figure 3: Graphical illustration to the proof of Lemma 3 based on orbit incidence.

Next, we discuss a more general result obtained also in [3] in a different way.

Proposition 1 *The decomposition* $\mathcal{R} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1$ *exists if and only if*

$$|\gamma_{12} - \gamma_{23}| \leq \tilde{\gamma}_{31} \leq \gamma_{12} + \gamma_{23}. \tag{2}$$

Proof. Let us consider a dual system of axes $\{\mathbf{a}'_k\}$ attached to the rotating object called the *body frame*, while the stationary one $\{\mathbf{a}_k\}$ is usually referred to as the *fixed frame* or the *spatial frame*. Obviously, the first rotation axis in the decomposition is the same in the two frames, i.e., $\mathbf{a}'_1 = \mathbf{a}_1$, while the other pairs are related respectively as $\mathbf{a}'_2 = \mathcal{R}'_1\mathbf{a}_2$ and $\mathbf{a}'_3 = \mathcal{R}'_2\mathcal{R}'_1\,\mathbf{a}_3$, where we denote $\mathcal{R}'_k = \mathcal{R}(\mathbf{c}'_k)$. Moreover, suppose that $\mathcal{R}$ can be decomposed in the body frame as $\mathcal{R} = \mathcal{R}'_3\mathcal{R}'_2\mathcal{R}'_1$. Since $\mathbf{a}'_3$ is an invariant vector for $\mathcal{R}'_3$, this yields $\mathbf{a}'_3 = \mathcal{R}\,\mathbf{a}_3$. Then, the matrix entries g'_{ij} and r'_{ij} in the rotating (body) frame are naturally expressed in terms of those in the spatial one as

$$g'_{12} = g_{12}, \qquad g'_{23} = g_{23}, \qquad g'_{13} = r_{13}$$

and the corresponding Gram determinant is given by the expression

$$\mathrm{G}(\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3) = 1 + 2g_{12}\, g_{23}\, r_{13} - g_{12}^2 - g_{23}^2 - r_{13}^2 \geq 0. \tag{3}$$

Next, we claim that the decompositions in the two dual systems of axes coexist (each one implies the other) and are related by the following formula

$$\mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_n = \mathcal{R}'_n\mathcal{R}'_{n-1}\dots\mathcal{R}'_1, \qquad \mathcal{R}'_k = \mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_{k-1}\mathcal{R}_k\mathcal{R}_{k-1}^{-1}\dots\mathcal{R}_2^{-1}\mathcal{R}_1^{-1}$$

that is easy to prove by induction starting with

$$\mathcal{R}'_2\mathcal{R}'_1 = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_1^{-1}\mathcal{R}_1 = \mathcal{R}_1\mathcal{R}_2$$

since we obviously have $\tilde{\mathcal{R}}\,\mathcal{R}(\mathbf{c})\,\tilde{\mathcal{R}}^{-1} = \mathcal{R}(\tilde{\mathcal{R}}\,\mathbf{c})$ and $\mathcal{R}'_1 = \mathcal{R}_1$ by construction. Then, the decomposition $\mathcal{R} = \mathcal{R}'_3\mathcal{R}'_2\mathcal{R}'_1$ is equivalent to $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$, so we need to reorder the vectors in the above Gram determinant, which is the same as replacing r_{13} with r_{31}. Moreover, since $g_{ij} = \cos\gamma_{ij}$ and $r_{31} = \cos\tilde{\gamma}_{31}$, the quadratic inequality $\Delta = \mathrm{G}(\mathbf{a}'_3, \mathbf{a}'_2, \mathbf{a}'_1) \geq 0$ is equivalent to

$$\cos(\gamma_{12} + \gamma_{23}) \leq \cos\tilde{\gamma}_{31} \leq \cos(\gamma_{12} - \gamma_{23}). \tag{4}$$

Finally, one may always choose the orientation of $\mathbf{a}_i$ in such a way that $\gamma_{12}, \gamma_{23} \in \left(0, \frac{\pi}{2}\right]$ so that the solution is given namely by formula (2). This proves the necessity of (2). Then, one needs to show that $\Delta \geq 0$ is sufficient for the existence of the corresponding rotating frame $\{\mathbf{a}'_k\}$, or simply point out that the solutions obtained in [4] rely only on the definiteness of Δ. $\square$

One straightforward consequence is the Davenport universality condition

$$\mathcal{R} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 \quad \forall\,\mathcal{R} \in \mathrm{SO}(3) \quad \Leftrightarrow \quad \gamma_{12} = \gamma_{23} = \frac{\pi}{2}\cdot \tag{5}$$

Another one is certainly Lemma 3, which follows directly in the case of coincident first and third axis with the aid of Lemma 1. Note that the non-orthogonal Euler setting $\gamma_{12} = \gamma_{23} = \gamma$ and $\gamma_{13} = 0$ is less restrictive on β_1 compared to the Bryan case, in which all relative angles are equal $\gamma_{ij} \equiv \gamma$. More precisely, the former yields the estimate $\beta_1 \leq \gamma$, while for the latter we have $2\beta_1 \leq \gamma$. We shall see it is a common property of rotational sequences. Note that with the aid of the famous Rodrigues' rotation formula (see [4])

$$\mathcal{R}(\mathbf{n}, \phi) = \cos\phi\,\mathcal{I} + (1 - \cos\phi)\,\mathbf{n}\mathbf{n}^t + \sin\phi\,\mathbf{n}^{\times} \tag{6}$$

we obtain in the case $\gamma_{31} = 0$ from the inequality of Proposition 1

$$\cos\tilde{\gamma}_{11} = \cos^2\beta_1 + \cos\phi\sin^2\beta_1 = (\cos\phi - 1)\sin^2\beta_1 + 1 \geq \cos 2\gamma_{12}$$

hence, the necessary and sufficient condition takes the simple form

$$\sin\frac{|\,\phi\,|}{2}\sin\beta_1 \leq \sin\gamma_{12}. \tag{7}$$

The Induction Step

We shall use the notation $\Sigma_k = \gamma_{12} + \gamma_{23} + \ldots + \gamma_{k-1,k}$ and $\bar{\Sigma}_k = \Sigma_k + \gamma_{k,1}$ respectively for the lengths of the open and closed spherical paths connecting the points $\mathbf{a}_i \in \bar{\mathbb{S}}^2_+$ associated with the rotation axes in the given order. Moreover, we let $\Delta_k^{ij} = \bar{\Sigma}_k - 2\gamma_{ij} \geq 0$ $(k \geq 3)$ represent the path defect given by the triangle inequality on $\mathbb{S}^2$, and omit the subscript if possible. Next, we shall use induction to generalize Proposition 1 for $N = k+1$ factors.

Lemma 4 *The existence of the decomposition $\mathcal{R} = \mathcal{R}_k \ldots \mathcal{R}_2\mathcal{R}_1$, such that $\mathcal{R}_i \in \mathrm{SO}(3)$, implies either the estimate $2\beta_1 \leq \Delta_k^{k,1}$ and/or $|\phi| \sin\beta_1 \leq \Delta_k^{k,1}$.*

Proof. For $k = 3$ the result follows from Lemma 1 and Proposition 1, while for $k > 3$ we proceed by induction noting that $\tilde{\gamma}^{(k+1)}_{k+1,k} = \tilde{\gamma}^{(k)}_{k+1,k}$ due to the invariant axis theorem, while the triangle inequality on the sphere yields $\tilde{\gamma}_{k,1} - \gamma_{k,k+1} \leq \tilde{\gamma}_{k+1,1} \leq \tilde{\gamma}_{k,1} + \gamma_{k,k+1}$, so the result follows by induction. $\square$

Note that typically no $\gamma_{i,i+1}$ exceeds the sum of the rest, e.g. in the case of a closed path, and this condition is both necessary and sufficient as the lower bound for $\tilde{\gamma}_{k,1}$ becomes trivial and the triangle inequality is minimal. Otherwise the precise estimate involves the minimum of Δ_k^{12}, Δ_k^{23} and $\Delta_k^{k,1}$.

Lemma 5 *The existence of the decompositions*

$$\mathcal{R} = \mathcal{R}_1(\phi_1')\mathcal{R}_k(\phi_k)\ldots\mathcal{R}_1(\phi_1), \qquad \mathcal{R} = \mathcal{R}_k(\phi_k)\mathcal{R}_1(\phi_1')\mathcal{R}_{k-1}\ldots\mathcal{R}_1(\phi_1)$$

for an arbitrary $\mathcal{R} \in \mathrm{SO}(3)$ implies that $2\beta_1$ does not exceed the length of the geodesic path connecting the points $\mathbf{a}_i$ on $\mathbb{S}^2_+$ in the corresponding order.

Proof. For the first decomposition we simply apply Lemma (4) taking into account that $\Delta_k^{11} = \bar{\Sigma}_k$ and the lower bound for $\tilde{\gamma}_{11}$ is trivial. To show the second one we express $\mathcal{R} = \mathcal{R}_1\mathcal{R}_k'\mathcal{R}_{k-1}\ldots\mathcal{R}_1$ where $\mathcal{R}_k'$ is an appropriate conjugation of $\mathcal{R}_k$ with $\mathcal{R}_1$, for which the statement of Lemma 4 asserts that

$$2\beta_1 \leq \gamma_{12} + \gamma_{23} + \ldots + \gamma'_{k-1,k} + \gamma'_{k,1}$$

with $\gamma'_{k,1} = \gamma_{k,1}$ and by the triangle inequality $\gamma'_{k-1,k} \leq \gamma_{k-1,1} + \gamma_{k,1}$. $\square$

Theorem 2 *With the above notation let $\bar{\gamma} = \frac{1}{k}\bar{\Sigma}_k$ be the mean spherical distance of the path. Then, an arbitrary $\mathcal{R} \in \mathrm{SO}(3)$ can be decomposed into*

$$N_{\bar{\gamma}}(\beta) \leq 1 + \left\lceil \frac{2\beta}{\bar{\gamma}} \right\rceil \tag{8}$$

rotations about the $\mathbf{a}_i$'s where $\beta = \min\beta_i$ (axes may need to be reordered).

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

Proof. The result is a straightforward consequence of Lemma 5 with a proper permutation of the axes, choosing the initial one to be the 'closest' to $\mathbf{n}$. $\square$

Now, let us consider the case, in which the angle of the compound rotation is under control, but we have no information about its axis $\mathbf{n}$ starting with

$$\mathcal{R} = \mathcal{R}_1(\phi_1')\mathcal{R}_k \dots \mathcal{R}_2(\phi_2)\mathcal{R}_1(\phi_1)$$

where the condition for the decomposition given by Lemma 5 yields

$$\cos\tilde{\gamma}_{11} = (\mathbf{a}_1, \mathcal{R}\mathbf{a}_1) \geq \cos\bar{\Sigma}_k. \tag{9}$$

Using Rodrigues' rotation formula (6) like in equation (7), from the above scalar product we obtain with the optimal choice of a first axis the condition

$$\sin\frac{|\,\phi\,|}{2}\sin\beta \leq \sin\frac{\bar{\Sigma}_k}{2}, \qquad \beta = \min\beta_i. \tag{10}$$

Note that in the latter estimate we assume $2\beta > \bar{\Sigma}_k \leq \pi$, since otherwise no restriction on the angle is necessary. Moreover, it is not hard to show that

$$\tilde{\gamma}_{11} = 2\arcsin\left|\,\sin\frac{\phi}{2}\sin\beta\,\right| \leq |\,\phi\,|\sin\beta \tag{11}$$

which is also a good approximation for small values of $|\,\phi\,|$, so the condition

$$|\,\phi\,|\sin\beta \leq \bar{\Sigma}_k$$

combined with Theorem 2, provides the (not necessarily minimal) estimate

$$N_{\bar{\gamma}}(\beta, \phi) \leq 1 + \left\lceil \frac{\min\left(|\,\phi\,|\sin\beta,\, 2\beta\right)}{\bar{\gamma}} \right\rceil \tag{12}$$

where the second option is considered only if $|\,\phi\,| > 2$. In particular, for unrestricted angle ϕ this brings us back to (8) as $\pi\sin\beta \geq 2\beta \in (0, \pi]$ and if there is no information about the invariant axis $\mathbf{n}$, we finally end up with

Theorem 3 *With β, $\bar{\gamma}$ and ϕ as before, one may decompose $\mathcal{R}(\mathbf{n}, \phi)$ into*

$$N_{\bar{\gamma}}(\phi) \leq 1 + \left\lceil \frac{|\,\phi\,|}{\bar{\gamma}} \right\rceil. \tag{13}$$

rotations about the $\mathbf{a}_i$'s for an arbitrary invariant axis $\mathbf{n} \in \mathbb{S}^2$.

Relations to Classical Results

In this section we see how the problems we focus on correspond wit some classical results in spherical geometry, such as the theorems of Rodrigues-Hamilton and Donkin. Let us begin with the setup: if the rotation axes are associated with points z_i on the unit sphere, then they form a polygon with side lengths $\gamma_{jk} = \arccos g_{jk}$ and let $\tilde{\gamma}_{jk} = \arccos \tilde{g}_{jk}$ with $\tilde{g}_{jk}$ denoting the co-factor of g_{jk} in the Gram determinant $g = \omega^2$. Then, the spherical cosine theorem determines the vertex angles α_k in the case of three axes, namely

$$\cos\alpha_i = -\frac{\tilde{g}_{jk}}{\sqrt{\omega^2 + \tilde{g}_{jk}^2}}, \qquad \sin\alpha_i = \frac{\omega}{\sqrt{\omega^2 + \tilde{g}_{jk}^2}} \tag{14}$$

and the decomposition of unity $\mathcal{I} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1$ derived in [4] with scalar parameters $\tau_k = \dfrac{\omega}{\tilde{g}_{ij}}$ is actually a verification of the famous Rodrigues-Hamilton theorem (see [6]). Moreover, we have the dual statement, known as Donkin's theorem, where the axes of rotation are related to the poles of the given triangle, which has α_k as its side lengths and g_{ij} as the corresponding vertex angles. Note also that the two-axes decompositions discussed in [4] may be regarded similarly as a manifestation of the Rodrigues-Hamilton theorem, with the $\mathbf{a}_3$ replaced by the compound rotation invariant vector $\mathbf{n}$

$$\tau_1 = \frac{\tilde{\zeta}_3}{g_{12}\zeta_1 - \zeta_2}, \qquad \tau_2 = \frac{\tilde{\zeta}_3}{g_{12}\zeta_2 - \zeta_1}, \qquad \tau = \frac{\tilde{\zeta}_3}{g_{12} - \zeta_1\zeta_2} \tag{15}$$

where we denote $\zeta_i = \mathbf{n}\cdot\mathbf{a}_i$ and $\tilde{\zeta}_i = \mathbf{n}\cdot\tilde{\mathbf{a}}_i$, respectively. Note that the first two equalities above provide the scalar parameters for the decomposition

$$\mathcal{R}(\tau_2\mathbf{a}_2)\mathcal{R}(\tau_1\mathbf{a}_1) = \mathcal{R}(\tau\mathbf{n})$$

while the third one can be interpreted as a necessary and sufficient condition. Similarly, the classical Euler type decomposition (for non-orthogonal axes)

$$\mathcal{R}(\phi,\mathbf{n}) = \mathcal{R}(\phi_3,\mathbf{a}_1)\mathcal{R}(\phi_2,\mathbf{a}_2)\mathcal{R}(\phi_1,\mathbf{a}_1) = \mathcal{R}(\phi_2,\mathbf{a}_2')\mathcal{R}(\phi_1+\phi_3,\mathbf{a}_1)$$

where $\mathbf{a}_2' = \mathcal{R}(\phi_3,\mathbf{a}_1)\,\mathbf{a}_2$ may be obtained in this way as

$$\phi_1+\phi_3 = 2\arctan\frac{\tilde{\zeta}_3'}{g_{12}\zeta_1 - \zeta_2'}, \quad \tau_2 = \frac{\tilde{\zeta}_3'}{g_{12}\zeta_2' - \zeta_1}, \quad \tau = \frac{\tilde{\zeta}_3'}{g_{12} - \zeta_1\zeta_2'} \tag{16}$$

with the notation $\zeta_i' = \mathbf{n}\cdot\mathbf{a}_i'$ and $\tilde{\zeta}_i' = \mathbf{n}\cdot\tilde{\mathbf{a}}_i'$ where $\tilde{\mathbf{a}}_3' = \mathbf{a}_1\times\mathbf{a}_2'$. This problem, however, allows for a simpler treatment [4]. Note that in the case of gimbal lock $\mathbf{a}_3 = \mathcal{R}\mathbf{a}_1$ the above holds for the unprimed quantities ζ_i, $\tilde{\zeta}_i$.

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

Let us note that in the case of two axes considered by Lowenthal and Hamada the above estimate for the order becomes exact and in particular, if there is no additional information about $\mathbf{n}$ or ϕ, they all reduce to formula (1). More generally, for equal relative angles γ_{ij} the estimate is independent of the choice of path. However, this is possible only for two or three axes with $g_{ij} \geq 0$. In other cases one can minimize the length of the rotation sequence by maximizing the one of the corresponding spherical path $\bar{\Sigma}_k$ connecting the $\mathbf{a}_i$'s. One straightforward way to do so is by choosing $\gamma = \max \gamma_{ij}$ and proceeding with only two axes, but in some cases this maximum may not be unique, for instance if the axes determine a proper spherical polygon, we can also choose the maximal billiard orbit with fixed number of reflections. The first axis $\mathbf{a}_1$, on the other hand, should be chosen closest to $\mathbf{n}$ so that β_1 is minimal. Typically one may need to make a compromise between minimizing β_1 and maximizing γ as formula (12) suggests. In practice, however, we may not have control over any of these parameters. Besides, one typically strives to minimize not the number of factors in a rotation sequence, but the energy cost, measured as geodesic length, so more factors can be better.

The case $N = 2$ is quite unusual as it corresponds to a spherical geodesic joining $\mathbf{a}_1$ and $\mathbf{a}_2$ so its length cannot exceed $\frac{\pi}{2}$ (the orientation is irrelevant). Moreover, we only have two parameters in this setting so the decomposition works on a zero measure set in SO(3). If we go back the same way to $\mathbf{a}_1$, however, the closed geodesic path may reach the desired length $\gamma = \pi$ which gives the classical Euler decomposition, allowing for the decomposition of each rotation. Yet, there are singularities, e.g. if we take a half-turn in a plane determined by $\mathbf{a}_1$ and $\mathbf{a}_2$, two of the angles in the solution would be coupled, namely $\phi_1 - \phi_3 = \pi$, $\phi_2 = \pi$. This is referred to as 'gimbal lock' (see [4]), a well known engineering problem, especially in spacecraft navigation. Another way to end up with a gimbal lock is to direct the three unit vectors $\mathbf{a}_i$ along the edges of a tetrahedron (starting at one vertex) and consider a rotation about the axis of symmetry (their vector sum) that brings $\mathbf{a}_1$ to $\mathbf{a}_3$, which yields this time $\phi_1 + \phi_3 = \phi_2 = -2\arctan\frac{\sqrt{2}}{2} \approx -70.53^\circ$. Note that here the condition $2\beta_1 \leq \gamma$ is satisfied so we can factorize regardless of the angle. It would not be the case, however, if we choose for example $\mathbf{n}$ orthogonal to $\mathbf{a}_1$ and in plane of symmetry for $\mathbf{a}_2$ and $\mathbf{a}_3$. Then $|\phi|$ cannot exceed $\frac{2\pi}{3}$ as formula (3) indicates. Note that in the gimbal lock setting, defined as $\mathcal{R}\mathbf{a}_1 = \pm\mathbf{a}_3$, we can decompose as $\mathcal{R} = \mathcal{R}_2\mathcal{R}_2$ setting $\phi_3 = 0$ or use it to minimize the overall length of the spherical path $|\phi_1| + |\phi_2| + |\phi_3|$.

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

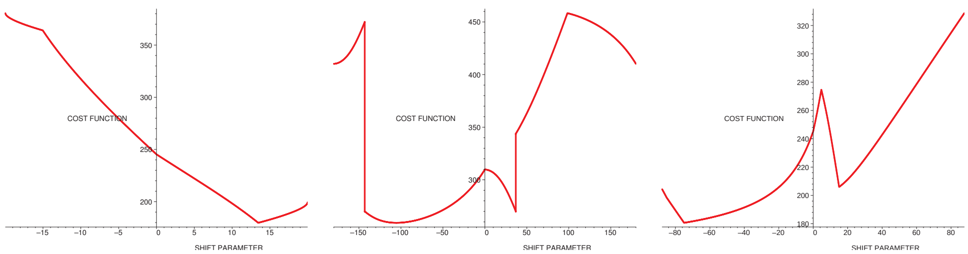

Figure 4: Plot of $\mathcal{E}_\gamma(\lambda)$ for different versions of (17): (a) the $XYXZ$ decomposition of a rotation by an angle $\phi = -120°$ about the vector $(3,4,5)^t$, (b) the $ZXZX$ decomposition of a half-turn about $(5,4,3)^t$ and (c) the same in the $XYZX$ setting.

We borrow the next example form [5] (with utter consent of the author). Consider a decomposition into four factors with one repeated invariant axis

$$\mathcal{R} = \mathcal{R}_3(\phi_3)\mathcal{R}_1(\lambda)\mathcal{R}_2(\phi_2)\mathcal{R}_1(\phi_1), \quad \mathcal{R} = \mathcal{R}_i(\lambda)\mathcal{R}_3(\phi_3)\mathcal{R}_2(\phi_2)\mathcal{R}_1(\phi_1) \quad (17)$$

where $i = 1, 2$. The additional parameter λ may be used for optimization of rotational sequences by introducing the cost function (total geodesic length)

$$\mathcal{E}_\gamma(\lambda) = |\lambda| + \sum_{i=1}^{3} |\phi_i(\lambda)|. \quad (18)$$

One might expect the minimum to be reached typically at $\lambda = 0$, but our numerical tests reveal quite a different picture (see Figure 4). Not only this is not the case, but in one of the exmples we have a local maximum at that point. The cost function has an explicit form and can easily be minimized that allows higher efficiency. The reader may see all technical details in [5].

Similar arguments clearly hold for the spin cover SU(2) as well and may be applied to spin systems, and in particular qubits used in quantum computation, with the proper definition of relative angles in that case. Namely, we use the Killing form to define the dot product as $q_1 \cdot q_2 = \frac{1}{2}\mathrm{tr}(q_1 q_2)$ where $q_{1,2}$ can be thought of as unit quaternions. Their normalized imaginary parts play the role of directional vectors and the effective rotation angle is easily derived from its magnitude and the trace. In SU(2) however this interpretation is a bit artificial, thus we may think instead of decompositions into Abelian subgroups. The lift $\mathrm{SO}(3) \xrightarrow{\pi^{-1}} \mathrm{SU}(2)$ to the spin cover is actually a lift from $\mathbb{RP}^3$ to $\mathbb{S}^3$ preserving the local distance functions γ_{ij}, so the above results are transferable. We refer to [4] for technical details and [7, 8] for the relevance to quantum mechanics, in particular to qubits using spin systems.

19 Nov 2023 23:12:15 PST
230621-Brezov Version 2 - Submitted to Rocky Mountain J. Math.

Final Remarks

At the end, let us discuss possible generalizations, starting with dimensional induction. The SO(4) setting is pretty straightforward due to the local product structure of the group. Roughly speaking, we apply the above results individually to each copy of SO(3), measuring the geodesic distance on a product of unit spheres $\mathbb{S}^2 \times \mathbb{S}^2$ and imposing our restrictions on pairs of angles. In particular, for plane rotations we have an action of SO(3) in $\mathbb{R}^4$ and our estimates apply directly upon projecting onto the invariant plane. The method is given in [9] and works in any dimension via Plücker embedding. For generic orthogonal transformations in $\mathbb{R}^n$ the problem is not well studied.

Our next remark concerns rigid motions in $\mathbb{E}^3$ represented via screws (see [10, 11] for details) modeled using unit dual extension to the underlying algebra $\mathbb{R} \to \mathbb{R}[\varepsilon]$, incorporating translations as nilpotent elements ($\varepsilon^2 = 0$). We introduce the dual angle $\underline{\varphi} = \varphi + \varepsilon d$ and axis vector $\underline{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{m} \in \mathbb{S}^2[\varepsilon]$ (i.e., $\mathbf{n}^2 = 1$ and $\mathbf{m} \perp \mathbf{n}$) using the screw displacement $d = \mathbf{n} \cdot \mathbf{p}$ (with $\mathbf{p}$ denoting the translation vector) and moment $\mathbf{m}$, which provide the Plücker coordinates of the screw axis $\underline{\mathbf{n}}$ given by Mozzi-Chasles theorem stating that every rigid motion in $\mathbb{E}^3$ is a screw motion, i.e., rotation and translation with a common axis. Dual extensions allow for invoking the transfer principle in generalizing results proven for rotations to screw motions by simply replacing the real objects with their dual versions: $\underline{\mathbf{n}}$, $\underline{\varphi}$, the unit sphere $\mathbb{S}^2[\varepsilon]$, etc.

Finally, it is worth mentioning the isometry group of $\mathbb{R}^{2,1}$ whose spin cover $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$ plays a major role in geometry and physics. Despite its similar decomposition properties [4] we need to acknowledge that the different types of mappings (elliptic, hyperbolic and parabolic) make things more complicated, e.g. by introducing new singularities. For the classical order Lowenthal proves a simple result [12]. It would be interesting to extend our argument for SO(3) to that setting and see how it invokes the geometry of the Poincaré disc, which then easily transfers to $\mathrm{SO}(2, 2)$. This however goes beyond the scope of the present study, so we leave it for future investigation.

Acknowledgment

I am deeply grateful to Professor Mitsuru Hamada at Tamagawa University for unintentionally bringing this quite interesting problem to my attention.

References

[1] Lowenthal F., *Uniform Finite Generation of the Rotation Group*, Rocky Mountain J. Math. **1** (1971) 575-586.

[2] Hamada M., *The Minimum Number of Rotations About Two Axes for Constructing an Arbitrarily Fixed Rotation*, R. Soc. Open Sci. **1** (2014) 140-145.

[3] Brezov D., Mladenova C. and Mladenov I., *On the Necessary and Sufficient Condition for the Decomposition of Orthogonal Transformations*, AIP Conf. Proc. **1960** (2015) 050003-1-050003-6.

[4] Brezov D., Mladenova C. and Mladenov I., *A Decoupled Solution to the Generalized Euler Decomposition Problem in* $\mathbb{R}^3$ *and* $\mathbb{R}^{2,1}$, J. Geom. Symmetry Phys. **33** (2014) 47-78.

[5] Brezov D., *Optimization and Gimbal Lock Control via Shifted Decomposition of Rotations*, J. Appl. & Comput. Math. (2018) 7:410.

[6] Bhat S. and Crasta N., *Closed Rotation Sequences*, Discrete Comput. Geom. **53** (2015) 366-396.

[7] Wharton K.B. and Koch D., *Unit quaternions and the Bloch sphere*, J. Phys. A: Math. Theor. **48** (2015) 235302.

[8] Kyoseva E. and Vitanov N., *Arbitrarily accurate passband composite pulses for dynamical suppression of amplitude noise*, Phys. Rev. A. **88** 063410 (2013).

[9] Brezov D., *Higher-Dimensional Representations of* SL_2 *and its Real Forms via Plücker Embedding*, Adv. Appl. Clifford Algebras **27** (2017) 2375-2392.

[10] Condurache D., *A Davenport Dual Angles Approach for Minimal Parameterization of the Rigid Body Displacement and Motion*, Mechanism and Machine Theory **140** (2019) 104-122.

[11] Brezov D., *Factorization and Generalized Roots of Dual Complex Matrices with Rodrigues' Formula*, Adv. Appl. Clifford Algebras **30**:29 (2020).

[12] Lowenthal F., *Uniform Finite Generation of the Isometry Groups of Euclidean and Non-Euclidean Geometry*, Can. J. Math. **23**:2 (1971) 364-373.