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# GENERALIZED HYERS-ULAM STABILITY OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORM

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ABSTRACT. The purpose of this present paper is to study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of homogeneous and non-homogeneous second-order linear differential equations applying Laplace transform method. In particular, our results can guarantee stability over unbounded intervals, and in special cases the obtained Hyers-Ulam constants match the best Hyers-Ulam constants. In addition, the results obtained are conditioned on the convergence of the Laplace transform of some function.

## 1. Introduction

In 1940, Ulam [39] gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. It motivated the study of stability problems for various functional equations. Among the problems raised by Ulam, the following question is concerned about the stability of homomorphisms: Let $G_1$ be a group and let $G_2$ be a group endowed with a metric $\rho$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$\rho(f(xy), f(x)\, f(y)) < \delta,$$

for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with

$$\rho(f(x), h(x)) < \varepsilon,$$

for all $x \in G_1$? If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [13] was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In course of time, the theorem formulated by Hyers was generalized by Rassias [33], Aoki [4] and Bourgin [9] for additive mappings (see also [31]).

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$g\left(f, u, u', u'', \cdots, u^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function $v$ such that

$$\left|g\left(f, v, v', v'', \cdots, v^{(n)}\right)\right| \le \varepsilon,$$

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there exists a solution $u$ of $g\left(f,u,u',u'',\cdots,u^{(n)}\right)=0$ such that $|v(t)-u(t)|\leq\kappa(\varepsilon)$ and

$$\lim_{\varepsilon\to 0}\kappa(\varepsilon)=0.$$

If the preceding statement is also true when we replace $\varepsilon$ and $\kappa(\varepsilon)$ by $\phi(t)$ and $\varphi(t)$, where $\phi,\varphi$ are appropriate functions not depending on $u$ and $v$ explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [26, 27]. Thereafter, in 1998, Alsina and Ger [2] investigated the Hyers-Ulam stability of differential equations. They proved the following result.

**Theorem 1.1.** *Assume that a differentiable function $v:I\to\mathbb{R}$ is a solution of the differential inequality $|v'(t)-v(t)|\leq\varepsilon$, where $I$ is an open subinterval of $\mathbb{R}$. Then there exists a solution $u:I\to\mathbb{R}$ of the differential equation $u'(t)=u(t)$ such that for any $t\in I$, we have $|v(t)-u(t)|\leq 3\varepsilon$.*

This result has been generalized by Takahasi [38]. He proved that the Hyers-Ulam stability holds true for the Banach space valued differential equation $u'(t)=\lambda u(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential and difference equations in more general settings [3, 8, 11, 14, 15, 16, 18, 19, 20, 21]. Jung [14] proved a similar result for the differential equation $r(t)u'(t)=u(t)$, where $r(t)$ is a nonzero function. For more recent results about this subject, we can refer to [5, 7, 10, 22, 28, 29, 35].

To the best of our knowledge, stability analysis using the Laplace transform was first studied by Rezaei, Jung and Rassias [34] in 2013. The next year Alqifiary and Jung [1] proved the generalized Hyers-Ulam stability of linear differential equations by using the Laplace transform method (see also [6, 36, 37, 41, 42]). In 2020, Murali, Ponmana Selvan and Park [23] have investigated the Hyers-Ulam stability of the linear differential equations using Fourier transform method (see also [12, 32, 40]). Very recently, Jung, Ponmana Selvan and Murali [17] established the various forms of Hyers-Ulam stability of the first-order linear differential equations with constant coefficients by using Mahgoub integral transform (see also [30]). Then, Murali *et al.* [24] investigated the different forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of second order linear differential equation of the form $u''(t)+\mu^2u(t)=q(t)$ by using Aboodh transform method (see also [25]).

Motivated and connected with the above literature, in this paper, our main intention is to study the Hyers-Ulam stability of the following second order linear differential equations

$$u''(t)+\alpha u'(t)+\beta u(t)=0 \tag{1.1}$$

and

$$u''(t)+\alpha u'(t)+\beta u(t)=q(t) \tag{1.2}$$

for all $t\in\mathbb{R}$, $u(t)\in C^2(\mathbb{R})$ and $q(t)\in C(\mathbb{R})$, using the Laplace transform method. A detailed definition will be given in the next section, the factor 3 of $\varepsilon$ in Theorem 1.1 is called a Hyers-Ulam constant. Note here that many of the Ulam stability analyzes using the various transforms described earlier restrict interval $I$ to a bounded interval. For example, $I=[a,b]$, $-\infty<a<b<\infty$. Alternatively, we can point out the possibility that the Hyers-Ulam constant depends on the width of the interval and diverges when $t\to\infty$. That is, the obtained conclusion is given in the form of $|v(t)-u(t)|\leq\varepsilon L(t)$ for

$t \in [a,\infty)$, but $\lim_{t\to\infty} L(t) = \infty$. Unfortunately, this case is not Ulam stable on $I = [a,\infty)$ in the sense defined in the next section. However, our study shows that the above $L(t)$ can be chosen without depending on $t \in \mathbb{R}$. In other words, the novelty of our paper is that we can take the interval $I$ as the whole of real numbers. If the inequality $|v(t) - u(t)| \le K\varepsilon$ holds for $t \in \mathbb{R}$, then naturally this inequality holds for $t \in [a,b]$ as well, where $K > 0$ is a Hyers-Ulam constant and $-\infty < a < b < \infty$. If there is a minimum Hyers-Ulam constant, it is called the best Hyers-Ulam constant. For example, it is also known that the factor $K = 3$ of $\varepsilon$ in Theorem 1.1 is not best Hyers-Ulam constant for the equation $u'(t) = u(t)$. Its best constant is known to be $K = 1$ (see, [28]). The second novelty of our study is to derive the best Hyers-Ulam constants for equations (1.1) and (1.2). So we get sharper results on $\mathbb{R}$ than the previous studies.

## 2. Preliminaries

Here, we give some definitions of Hyers-Ulam stability and generalized Hyers-Ulam stability of equations (1.1) and (1.2).

**Definition 2.1.** Let $I$ be an interval of $\mathbb{R}$. We say that equation (1.1) has the Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\varepsilon > 0$ and every $v(t) \in C^2(I)$ satisfying the inequality

$$\left|v''(t) + \alpha v'(t) + \beta v(t)\right| \le \varepsilon,$$

for all $t \in I$, there exists a solution $u(t) \in C^2(I)$ satisfies (1.1) such that

$$|v(t) - u(t)| \le K\varepsilon,$$

for all $t \in I$. We call such a $K$ a Hyers-Ulam constant for (1.1).

**Definition 2.2.** Let $I$ be an interval of $\mathbb{R}$, and let $\phi(t)$ be a positive function on $I$. We say that equation (1.1) has the Hyers-Ulam-Rassias stability with respect to $\phi(t)$, if we change both $\varepsilon$'s in Definition 2.1 to $\varepsilon\phi(t)$ and still $K > 0$ exists. We call such a $K$ a Hyers-Ulam-Rassias constant for (1.1).

**Definition 2.3.** Let $I$ be an interval of $\mathbb{R}$. We say that equation (1.2) has the Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\varepsilon > 0$ and every $v(t) \in C^2(I)$ satisfying the inequality

$$\left|v''(t) + \alpha v'(t) + \beta v(t) - q(t)\right| \le \varepsilon,$$

for all $t \in I$, there exists a solution $u(t) \in C^2(I)$ satisfies (1.2) such that

$$|v(t) - u(t)| \le K\varepsilon,$$

for all $t \in I$. We call such a $K$ a Hyers-Ulam constant for (1.2).

**Definition 2.4.** Let $I$ be an interval of $\mathbb{R}$, and let $\phi(t)$ be a positive function on $I$. We say that equation (1.2) has the Hyers-Ulam-Rassias stability with respect to $\phi(t)$, if we change both $\varepsilon$'s in Definition 2.3 to $\varepsilon\phi(t)$ and still $K > 0$ exists. We call such a $K$ a Hyers-Ulam-Rassias constant for (1.2).

*Remark* 2.5. In Definitions 2.2 and 2.4, if $\phi(t) \equiv 1$, then the Hyers-Ulam-Rassias stability with respect to $\phi(t) \equiv 1$ becomes just Hyers-Ulam stability.

If there is a minimum Hyers-Ulam constant, we call it the best Hyers-Ulam constant.

## 3. Main results

The first main result of this paper is as follows.

**Theorem 3.1.** *Let $\varepsilon > 0$ and $\phi(t)$ be a positive function on $\mathbb{R}$. Let $\lambda_1$ and $\lambda_2$ be the roots of $s^2+\alpha s+\beta=0$. Suppose that $v(t)\in C^2(\mathbb{R})$ satisfies*

$$\left|v''(t)+\alpha v'(t)+\beta v(t)\right| \le \varepsilon\phi(t)$$

*for $t\in\mathbb{R}$. Then (i) and (ii) below hold:*

(i) *if $\lambda_1\neq\lambda_2$ and $\mathscr{L}\{\phi(t)\}$ converges absolutely for $\Re(s)\ge\min\{\Re(\lambda_1),\Re(\lambda_2)\}$, then there exists a solution $u(t)\in C^2(\mathbb{R})$ of (1.1) such that*

$$|v(t)-u(t)| \le \frac{\varepsilon}{|\lambda_1-\lambda_2|}\int_0^\infty \phi(t+\sigma)\left|e^{-\lambda_2\sigma}-e^{-\lambda_1\sigma}\right|d\sigma$$

*for $t\in\mathbb{R}$;*

(ii) *if $\lambda_1=\lambda_2$ and $\mathscr{L}\{t\phi(t)\}$ converges absolutely for $\Re(s)\ge\Re(\lambda_1)$, then there exists a solution $u(t)\in C^2(\mathbb{R})$ of (1.1) such that*

$$|v(t)-u(t)| \le \varepsilon\int_0^\infty \phi(t+\sigma)\sigma e^{-\Re(\lambda_1)\sigma}d\sigma$$

*for $t\in\mathbb{R}$.*

*Proof.* Let $\varepsilon>0$ and $\phi(t)>0$ for $t\in\mathbb{R}$. Let $\lambda_1$ and $\lambda_2$ be the roots of $s^2+\alpha s+\beta=0$. Suppose that $v(t)\in C^2(\mathbb{R})$ satisfies

$$\left|v''(t)+\alpha v'(t)+\beta v(t)\right| \le \varepsilon\phi(t) \tag{3.1}$$

for all $t\in\mathbb{R}$. Suppose that the Laplace transform of the function $\phi(t)$ which given by

$$\Phi(s) := \mathscr{L}\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)dt$$

converges absolutely for $\Re(s)\ge\min\{\Re(\lambda_1),\Re(\lambda_2)\}$ if $\lambda_1\neq\lambda_2$; that is,

$$\int_0^\infty \left|e^{-st}\phi(t)\right|dt<\infty \tag{3.2}$$

for $\Re(s)\ge\min\{\Re(\lambda_1),\Re(\lambda_2)\}$ if $\lambda_1\neq\lambda_2$. Moreover, we suppose that the Laplace transform of the function $t\phi(t)$ converges absolutely if $\lambda_1=\lambda_2$; that is,

$$\int_0^\infty \left|te^{-st}\phi(t)\right|dt<\infty \tag{3.3}$$

for $\Re(s)\ge\Re(\lambda_1)$ if $\lambda_1=\lambda_2$. Note that (3.3) implies (3.2) because

$$\begin{aligned}\int_0^\infty \left|\phi(t)e^{-\lambda_1 t}\right|dt &= \int_0^1 \left|\phi(t)e^{-\lambda_1 t}\right|dt+\int_1^\infty \left|\phi(t)e^{-\lambda_1 t}\right|dt\\ &\le \int_0^1 \left|\phi(t)e^{-\lambda_1 t}\right|dt+\int_1^\infty |t|\left|\phi(t)e^{-\lambda_1 t}\right|dt\\ &< \int_0^1 \left|\phi(t)e^{-\lambda_1 t}\right|dt+\int_0^\infty \left|\phi(t)te^{-\lambda_1 t}\right|dt<\infty\end{aligned}$$

holds.

Define a function $p : \mathbb{R} \to \mathbb{C}$ by

$$p(t) := \frac{1}{\phi(t)} \left( v''(t) + \alpha v'(t) + \beta v(t) \right) \tag{3.4}$$

for all $t \in \mathbb{R}$. In view of (3.1), we have $|p(t)| \le \varepsilon$ for all $t \in \mathbb{R}$. Using this with (3.2) and (3.4), we have

$$\begin{aligned} \left| \int_0^\infty e^{-st} \left( v''(t) + \alpha v'(t) + \beta v(t) \right) dt \right| &= \left| \int_0^\infty e^{-st} \phi(t) p(t) dt \right| \\ &\le \varepsilon \int_0^\infty \left| e^{-st} \phi(t) \right| dt < \infty \end{aligned}$$

for $\Re(s) > \max\{\Re(\lambda_1), \Re(\lambda_2)\}$. Thus, the Laplace transform $\mathscr{L}\{v''(t) + \alpha v'(t) + \beta v(t)\}$ converges for $s > \max\{\Re(\lambda_1), \Re(\lambda_2)\}$. From the basic theory of the Laplace transform, we find that $\mathscr{L}\{v''(t)\}$, $\mathscr{L}\{v'(t)\}$ and $\mathscr{L}\{v(t)\}$ converge respectively. Needless to say, $\mathscr{L}\{\phi(t)p(t)\}$ also converges from (3.4). Taking Laplace transform from $p(t)\phi(t)$, we have

$$\begin{aligned} \mathscr{L}\{p(t)\phi(t)\} &= \mathscr{L}\{v''(t)\} + \alpha \mathscr{L}\{v'(t)\} + \beta \mathscr{L}\{v(t)\} \\ &= (s^2 + \alpha s + \beta)\mathscr{L}\{v(t)\} - (s + \alpha)v(0) - v'(0), \end{aligned}$$

and thus

$$\mathscr{L}\{v(t)\} = \frac{\mathscr{L}\{p(t)\phi(t)\} + (s + \alpha)v(0) + v'(0)}{s^2 + \alpha s + \beta}$$

for $\Re(s) > \max\{\Re(\lambda_1), \Re(\lambda_2)\}$. Since $\lambda_1$ and $\lambda_2$ are the roots of $s^2 + \alpha s + \beta = 0$, we see that $\lambda_1$ and $\lambda_2$ satisfy $s^2 + \alpha s + \beta = (s - \lambda_1)(s - \lambda_2) = 0$, $\lambda_1 + \lambda_2 = -\alpha$, and $\lambda_1 \lambda_2 = \beta$. Thus, we have

$$\begin{aligned} \mathscr{L}\{v(t)\} &= \frac{\mathscr{L}\{p(t)\phi(t)\} + (s - \lambda_1 - \lambda_2)v(0) + v'(0)}{(s - \lambda_1)(s - \lambda_2)} \\ &= \frac{v(0)}{s - \lambda_1} + \frac{\mathscr{L}\{p(t)\phi(t)\} - \lambda_1 v(0) + v'(0)}{(s - \lambda_1)(s - \lambda_2)} \\ &= v(0)\mathscr{L}\left\{e^{\lambda_1 t}\right\} + \left( \mathscr{L}\{p(t)\phi(t)\} - \lambda_1 v(0) + v'(0) \right) \mathscr{L}\left\{e^{\lambda_1 t}\right\} \mathscr{L}\left\{e^{\lambda_2 t}\right\} \\ &= v(0)\mathscr{L}\left\{e^{\lambda_1 t}\right\} + \left( \mathscr{L}\{p(t)\phi(t)\} - \lambda_1 v(0) + v'(0) \right) \mathscr{L}\left\{e^{\lambda_1 t} * e^{\lambda_2 t}\right\} \\ &= v(0)\mathscr{L}\left\{e^{\lambda_1 t}\right\} + \mathscr{L}\left\{p(t)\phi(t) * \left(e^{\lambda_1 t} * e^{\lambda_2 t}\right)\right\} \\ &\quad + \left( -\lambda_1 v(0) + v'(0) \right) \mathscr{L}\left\{e^{\lambda_1 t} * e^{\lambda_2 t}\right\} \end{aligned}$$

for $t \ge 0$, where the symbol $*$ denotes convolution. Hence we obtain the solution of the equation

$$v''(t) + \alpha v'(t) + \beta v(t) = \phi(t)p(t) \tag{3.5}$$

as

$$v(t) = v(0)e^{\lambda_1 t} + \left( -\lambda_1 v(0) + v'(0) \right) e^{\lambda_1 t} * e^{\lambda_2 t} + p(t)\phi(t) * \left(e^{\lambda_1 t} * e^{\lambda_2 t}\right) \tag{3.6}$$

for $t \geq 0$, by using the inverse Laplace transform. Note here that from the definition of the convolution, we have

$$e^{\lambda_1 t} * e^{\lambda_2 t} = \int_0^t e^{\lambda_1 (t-\tau)} e^{\lambda_2 \tau} d\tau$$

and

$$p(t)\phi(t) * \left(e^{\lambda_1 t} * e^{\lambda_2 t}\right) = \int_0^t p(t-\tau)\phi(t-\tau) \int_0^\tau e^{\lambda_1(\tau-\sigma)} e^{\lambda_2 \sigma} d\sigma d\tau$$

for $t \geq 0$. These two functions and $e^{\lambda_1 t}$ can be defined not only for $t \geq 0$, but also for $t < 0$. Similarly, they are twice continuously differentiable on $\mathbb{R}$. Hence, it can be seen that the function $v(t)$ is a solution of (3.5) not only for $t \geq 0$, but also for $t < 0$. That is, $v(t)$ is defined on $\mathbb{R}$, and is a solution of (3.5) on $\mathbb{R}$. Now, define the function

$$v_0(t) := v(0)e^{\lambda_1 t} + \left(-\lambda_1 v(0) + v'(0)\right) e^{\lambda_1 t} * e^{\lambda_2 t} \tag{3.7}$$

on $\mathbb{R}$. Then $v_0(t)$ is a solution of (1.1) because if $p(t) \equiv 0$, then (3.5) becomes (1.1). Note that

$$\begin{aligned} e^{\lambda_1 t} * e^{\lambda_2 t} &= \int_0^t e^{\lambda_1 (t-\tau)} e^{\lambda_2 \tau} d\tau = e^{\lambda_1 t} \int_0^t e^{-(\lambda_1-\lambda_2)\tau} d\tau \\ &= \begin{cases} \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} & \text{if} \quad \lambda_1 \neq \lambda_2, \\ t e^{\lambda_1 t} & \text{if} \quad \lambda_1 = \lambda_2. \end{cases} \end{aligned} \tag{3.8}$$

First, we consider the case $\lambda_1 \neq \lambda_2$. Define the function

$$u_1(t) := c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

on $\mathbb{R}$, where

$$c_1 := \frac{1}{\lambda_1 - \lambda_2} \int_0^\infty p(\tau)\phi(\tau)e^{-\lambda_1 \tau} d\tau \quad \text{and} \quad c_2 := -\frac{1}{\lambda_1 - \lambda_2} \int_0^\infty p(\tau)\phi(\tau)e^{-\lambda_2 \tau} d\tau.$$

Note that $c_1$ and $c_2$ are well-defined, because $|p(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$ and the Laplace transform $\Phi(s) = \mathscr{L}\{\phi(t)\}$ converges absolutely for $\Re(s) \geq \min\{\Re(\lambda_1), \Re(\lambda_2)\}$. Actually, we have

$$\begin{aligned} \left| \int_0^t p(\tau)\phi(\tau)e^{-\lambda_i \tau} d\tau \right| &\leq \int_0^t |p(\tau)| \left| \phi(\tau)e^{-\lambda_i \tau} \right| d\tau \leq \varepsilon \int_0^t \left| \phi(\tau)e^{-\lambda_i \tau} \right| d\tau \\ &\leq \varepsilon \int_0^\infty \left| \phi(\tau)e^{-\lambda_i \tau} \right| d\tau < \infty \end{aligned}$$

for $t \geq 0$ and $i \in \{1, 2\}$. Thus, $c_1$ and $c_2$ are constants, so that $u_1(t)$ is a solution of (1.1) because

$$u_1''(t) + \alpha u_1'(t) + \beta u_1(t) = c_1 \left(\lambda_1^2 + \alpha\lambda_1 + \beta\right) e^{\lambda_1 t} + c_2 \left(\lambda_2^2 + \alpha\lambda_2 + \beta\right) e^{\lambda_2 t} = 0$$

holds. Now we consider the function

$$u(t) := v_0(t) + u_1(t).$$

Then by the principle of superposition, we see that $u(t)$ is a solution of (1.1). The above equalities (3.6), (3.7) and (3.8) with $\lambda_1 \neq \lambda_2$ show that

$$\begin{aligned} v(t)-u(t) = v(t)-v_0(t)-u_1(t) &= p(t)\phi(t) * \left(e^{\lambda_1 t} * e^{\lambda_2 t}\right) - c_1 e^{\lambda_1 t} - c_2 e^{\lambda_2 t} \\ &= p(t)\phi(t) * \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} - c_1 e^{\lambda_1 t} - c_2 e^{\lambda_2 t} \\ &= \frac{1}{\lambda_1-\lambda_2}\int_0^t p(\tau)\phi(\tau)\left(e^{\lambda_1(t-\tau)} - e^{\lambda_2(t-\tau)}\right) d\tau - c_1 e^{\lambda_1 t} - c_2 e^{\lambda_2 t} \\ &= -\frac{e^{\lambda_1 t}}{\lambda_1-\lambda_2}\int_t^\infty p(\tau)\phi(\tau) e^{-\lambda_1 \tau} d\tau + \frac{e^{\lambda_2 t}}{\lambda_1-\lambda_2}\int_t^\infty p(\tau)\phi(\tau)e^{-\lambda_2\tau} d\tau \\ &= \frac{1}{\lambda_1-\lambda_2}\int_t^\infty p(\tau)\phi(\tau)\left(e^{-\lambda_2(\tau-t)} - e^{-\lambda_1(\tau-t)}\right) d\tau \\ &= \frac{1}{\lambda_1-\lambda_2}\int_0^\infty p(t+\sigma)\phi(t+\sigma)\left(e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma}\right) d\sigma \end{aligned}$$

for $t \in \mathbb{R}$. Hence

$$\begin{aligned} |v(t)-u(t)| &\leq \frac{1}{|\lambda_1-\lambda_2|}\left|\int_0^\infty p(t+\sigma)\phi(t+\sigma)\left(e^{-\lambda_2\sigma}-e^{-\lambda_1\sigma}\right)d\sigma\right| \\ &\leq \frac{1}{|\lambda_1-\lambda_2|}\int_0^\infty |p(t+\sigma)|\phi(t+\sigma)\left|e^{-\lambda_2\sigma}-e^{-\lambda_1\sigma}\right| d\sigma \\ &\leq \frac{\varepsilon}{|\lambda_1-\lambda_2|}\int_0^\infty \phi(t+\sigma)\left|e^{-\lambda_2\sigma}-e^{-\lambda_1\sigma}\right| d\sigma \end{aligned}$$

for $t \in \mathbb{R}$.

Next, we consider the case $\lambda_1 = \lambda_2$. Define the function

$$u_2(t) := d_1 t e^{\lambda_1 t} + d_2 e^{\lambda_1 t}$$

on $\mathbb{R}$, where

$$d_1 := \int_0^\infty p(\tau)\phi(\tau)e^{-\lambda_1\tau}d\tau \quad \text{and} \quad d_2 := -\int_0^\infty p(\tau)\phi(\tau)\tau e^{-\lambda_1\tau}d\tau.$$

Note that $d_1$ and $d_2$ are well-defined, because $|p(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$, and (3.3) holds. Actually, we have

$$\left|\int_0^t p(\tau)\phi(\tau)e^{-\lambda_1\tau}d\tau\right| \leq \int_0^t |p(\tau)|\left|\phi(\tau)e^{-\lambda_1\tau}\right| d\tau \leq \varepsilon\int_0^\infty \left|\phi(\tau)e^{-\lambda_1\tau}\right| d\tau < \infty$$

and

$$\left|\int_0^t p(\tau)\phi(\tau)\tau e^{-\lambda_1\tau}d\tau\right| \leq \varepsilon\int_0^\infty \left|\phi(\tau)\tau e^{-\lambda_1\tau}\right| d\tau < \infty$$

for $t \geq 0$. We can easily verify that $u_2(t)$ is a solution of (1.1). Now we consider the function

$$w(t) := v_0(t) + u_2(t).$$

Then by the principle of superposition, we see that $w(t)$ is a solution of (1.1). The above equalities (3.6), (3.7) and (3.8) with $\lambda_1 = \lambda_2$ show that

$$\begin{aligned}
v(t)-w(t) = v(t)-v_0(t)-u_2(t) &= p(t)\phi(t)*\left(e^{\lambda_1 t}*e^{\lambda_2 t}\right)-d_1te^{\lambda_1 t}-d_2e^{\lambda_1 t}\\
&= p(t)\phi(t)*te^{\lambda_1 t}-d_1te^{\lambda_1 t}-d_2e^{\lambda_1 t}\\
&= \int_0^t p(\tau)\phi(\tau)(t-\tau)e^{\lambda_1(t-\tau)}d\tau-d_1te^{\lambda_1 t}-d_2e^{\lambda_1 t}\\
&= te^{\lambda_1 t}\int_0^t p(\tau)\phi(\tau)e^{-\lambda_1\tau}d\tau-e^{\lambda_1 t}\int_0^t p(\tau)\phi(\tau)\tau e^{-\lambda_1\tau}d\tau\\
&\quad -d_1te^{\lambda_1 t}-d_2e^{\lambda_1 t}\\
&= -te^{\lambda_1 t}\int_t^\infty p(\tau)\phi(\tau)e^{-\lambda_1\tau}d\tau+e^{\lambda_1 t}\int_t^\infty p(\tau)\phi(\tau)\tau e^{-\lambda_1\tau}d\tau\\
&= \int_t^\infty p(\tau)\phi(\tau)(\tau-t)e^{-\lambda_1(\tau-t)}d\tau\\
&= \int_0^\infty p(t+\sigma)\phi(t+\sigma)\sigma e^{-\lambda_1\sigma}d\sigma
\end{aligned}$$

for $t\in\mathbb{R}$. Hence

$$\begin{aligned}
|v(t)-w(t)| &\le \left|\int_0^\infty p(t+\sigma)\phi(t+\sigma)\sigma e^{-\lambda_1\sigma}d\sigma\right|\\
&\le \int_0^\infty |p(t+\sigma)|\phi(t+\sigma)\sigma\left|e^{-\lambda_1\sigma}\right|d\sigma\\
&\le \varepsilon\int_0^\infty \phi(t+\sigma)\sigma e^{-\Re(\lambda_1)\sigma}d\sigma
\end{aligned}$$

for $t\in\mathbb{R}$. □

Using the above theorem, we can also establish the following result for equation (1.2).

**Theorem 3.2.** *Let $\varepsilon>0$ and $\phi(t)$ be a positive function on $\mathbb{R}$. Let $\lambda_1$ and $\lambda_2$ be the roots of $s^2+\alpha s+\beta=0$. Suppose that $v(t)\in C^2(\mathbb{R})$ satisfies*

$$\left|v''(t)+\alpha v'(t)+\beta v(t)-q(t)\right|\le\varepsilon\phi(t)$$

*for $t\in\mathbb{R}$. Then (i) and (ii) below hold:*

(i) *if $\lambda_1\neq\lambda_2$ and $\mathscr{L}\{\phi(t)\}$ converges absolutely for $\Re(s)\ge\min\{\Re(\lambda_1),\Re(\lambda_2)\}$, then there exists a solution $u(t)\in C^2(\mathbb{R})$ of (1.2) such that*

$$|v(t)-u(t)|\le\frac{\varepsilon}{|\lambda_1-\lambda_2|}\int_0^\infty\phi(t+\sigma)\left|e^{-\lambda_2\sigma}-e^{-\lambda_1\sigma}\right|d\sigma$$

*for $t\in\mathbb{R}$;*

(ii) *if $\lambda_1=\lambda_2$ and $\mathscr{L}\{t\phi(t)\}$ converges absolutely for $\Re(s)\ge\Re(\lambda_1)$, then there exists a solution $u(t)\in C^2(\mathbb{R})$ of (1.2) such that*

$$|v(t)-u(t)|\le\varepsilon\int_0^\infty\phi(t+\sigma)\sigma e^{-\Re(\lambda_1)\sigma}d\sigma$$

*for* $t \in \mathbb{R}$.

*Proof.* Let $\varepsilon > 0$ and $\phi(t) > 0$ for $t \in \mathbb{R}$. Let $\lambda_1$ and $\lambda_2$ be the roots of $s^2 + \alpha s + \beta = 0$. Suppose that $v(t) \in C^2(\mathbb{R})$ satisfies

$$\left|v''(t) + \alpha v'(t) + \beta v(t) - q(t)\right| \le \varepsilon\phi(t)$$

for all $t \in \mathbb{R}$. Let $u_0(t)$ be a solution of (1.2) for $t \in \mathbb{R}$. Then

$$\begin{aligned}\varepsilon\phi(t) &\ge \left|v''(t) + \alpha v'(t) + \beta v(t) - q(t)\right| \\ &= \left|v''(t) + \alpha v'(t) + \beta v(t) - \left(u_0''(t) + \alpha u_0'(t) + \beta u_0(t)\right)\right| \\ &= \left|(v(t) - u_0(t))'' + \alpha(v(t) - u_0(t))' + \beta(v(t) - u_0(t))\right|\end{aligned}$$

for all $t \in \mathbb{R}$.

First, we consider the case that $\lambda_1 \ne \lambda_2$ and $\mathscr{L}\{\phi(t)\}$ converges absolutely for $\Re(s) \ge \min\{\Re(\lambda_1), \Re(\lambda_2)\}$. Using Theorem 3.1 (i), we see that there exists a solution $w(t)$ of (1.1) such that

$$|(v(t) - u_0(t)) - w(t)| \le \frac{\varepsilon}{|\lambda_1 - \lambda_2|}\int_0^\infty \phi(t+\sigma)\left|e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma}\right| d\sigma$$

for $t \in \mathbb{R}$. Note that

$$|(v(t) - u_0(t)) - w(t)| = |v(t) - (u_0(t) + w(t))|$$

and

$$(u_0(t) + w(t))'' + \alpha(u_0(t) + w(t))' + \beta(u_0(t) + w(t)) = q(t)$$

for $t \in \mathbb{R}$. Hence $u(t) := u_0(t) + w(t)$ is a solution of (1.2) satisfying

$$|v(t) - u(t)| \le \frac{\varepsilon}{|\lambda_1 - \lambda_2|}\int_0^\infty \phi(t+\sigma)\left|e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma}\right| d\sigma$$

for $t \in \mathbb{R}$. This ends the proof of (i).

Next we consider the case that $\lambda_1 = \lambda_2$ and $\mathscr{L}\{t\phi(t)\}$ converges absolutely for $\Re(s) \ge \Re(\lambda_1)$. Using the same method as above, we can see that there exists a solution $u(t)$ of (1.2) such that

$$|v(t) - u(t)| \le \varepsilon\int_0^\infty \phi(t+\sigma)\sigma e^{-\Re(\lambda_1)\sigma} d\sigma$$

for $t \in \mathbb{R}$. The proof is complete. □

## 4. Hyers-Ulam-Rassias stability

In this section, we will establish some stability results.

**Theorem 4.1.** *Let $\varepsilon > 0$ and $\phi(t)$ be a non-increasing positive function on $\mathbb{R}$. Let $\lambda_1$ and $\lambda_2$ be the roots of $s^2 + \alpha s + \beta = 0$. Suppose that $v(t) \in C^2(\mathbb{R})$ satisfies*

$$\left|v''(t) + \alpha v'(t) + \beta v(t) - q(t)\right| \le \varepsilon\phi(t)$$

*for $t \in \mathbb{R}$. Then (i) and (ii) below hold:*

(i) *if* $\lambda_1 \neq \lambda_2$ *and* $\mathscr{L}\{\phi(t)\}$ *converges absolutely for* $\Re(s) \geq \min\{\Re(\lambda_1), \Re(\lambda_2)\}$*, then there exists a solution* $u(t) \in C^2(\mathbb{R})$ *of* (1.2) *such that*

$$|v(t) - u(t)| \leq \frac{\varepsilon\phi(t)}{|\lambda_1 - \lambda_2|} \int_0^\infty \left| e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma} \right| d\sigma$$

*for* $t \in \mathbb{R}$*;*

(ii) *if* $\lambda_1 = \lambda_2$ *and* $\mathscr{L}\{t\phi(t)\}$ *converges absolutely for* $\Re(s) \geq \Re(\lambda_1)$*, then there exists a solution* $u(t) \in C^2(\mathbb{R})$ *of* (1.2) *such that*

$$|v(t) - u(t)| \leq \frac{\varepsilon\phi(t)}{(\Re(\lambda_1))^2}$$

*for* $t \in \mathbb{R}$.

*Proof.* The assumption of this theorem differs from that of Theorem 3.2 in that the function $\phi(t)$ is assumed to be non-increasing. Hence we see that

$$\begin{aligned} |v(t) - u(t)| &\leq \frac{\varepsilon}{|\lambda_1 - \lambda_2|} \int_0^\infty \phi(t+\sigma) \left| e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma} \right| d\sigma \\ &\leq \frac{\varepsilon\phi(t)}{|\lambda_1 - \lambda_2|} \int_0^\infty \left| e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma} \right| d\sigma \end{aligned}$$

for the case (i). Moreover we have

$$\begin{aligned} |v(t) - u(t)| &\leq \varepsilon \int_0^\infty \phi(t+\sigma)\sigma e^{-\Re(\lambda_1)\sigma} d\sigma \\ &\leq \varepsilon\phi(t) \int_0^\infty \sigma e^{-\Re(\lambda_1)\sigma} d\sigma = \frac{\varepsilon\phi(t)}{(\Re(\lambda_1))^2} \end{aligned}$$

for the case (ii). The proof is complete. □

This theorem can be rewritten as the following result.

**Corollary 4.2.** *Let* $\phi(t)$ *be a non-increasing positive function on* $\mathbb{R}$*. Let* $\lambda_1$ *and* $\lambda_2$ *be the roots of* $s^2 + \alpha s + \beta = 0$*. Then (i) and (ii) below hold:*

(i) *if* $\lambda_1 \neq \lambda_2$ *and* $\mathscr{L}\{\phi(t)\}$ *converges absolutely for* $\Re(s) \geq \min\{\Re(\lambda_1), \Re(\lambda_2)\}$*, then* (1.2) *has Hyers-Ulam-Rassias stability with respect to* $\phi(t)$ *with Hyers-Ulam-Rassias constant*

$$K = \frac{1}{|\lambda_1 - \lambda_2|} \int_0^\infty \left| e^{-\lambda_2\sigma} - e^{-\lambda_1\sigma} \right| d\sigma;$$

(ii) *if* $\lambda_1 = \lambda_2$ *and* $\mathscr{L}\{t\phi(t)\}$ *converges absolutely for* $\Re(s) \geq \Re(\lambda_1)$*, then* (1.2) *has Hyers-Ulam-Rassias stability with respect to* $\phi(t)$ *with Hyers-Ulam-Rassias constant*

$$K = \frac{1}{(\Re(\lambda_1))^2}.$$

If the signs of $\Re(\lambda_1)$ and $\Re(\lambda_2)$ are both positive, then we have following simple result.

**Corollary 4.3.** *Let* $\phi(t)$ *be a non-increasing positive function on* $\mathbb{R}$*. Let* $\lambda_1$ *and* $\lambda_2$ *be the roots of* $s^2 + \alpha s + \beta = 0$*. Then (i) and (ii) below hold:*

(i) *if* $\lambda_1 \neq \lambda_2$ *and* $\Re(\lambda_1) > 0$ *and* $\Re(\lambda_2) > 0$, *then* (1.2) *has Hyers-Ulam-Rassias stability with respect to* $\phi(t)$ *with Hyers-Ulam-Rassias constant*

$$K = \frac{1}{|\lambda_1 - \lambda_2|} \int_0^\infty \left| e^{-\lambda_2 \sigma} - e^{-\lambda_1 \sigma} \right| d\sigma;$$

(ii) *if* $\lambda_1 = \lambda_2$ *and* $\Re(\lambda_1) > 0$, *then* (1.2) *has Hyers-Ulam-Rassias stability with respect to* $\phi(t)$ *with Hyers-Ulam-Rassias constant*

$$K = \frac{1}{(\Re(\lambda_1))^2}.$$

*Proof.* Suppose that $\phi(t)$ is a non-increasing positive function on $\mathbb{R}$. First we will show case (i). Suppose that $\lambda_1 \neq \lambda_2$, $\Re(\lambda_1) > 0$ and $\Re(\lambda_2) > 0$. Then

$$\begin{aligned}\int_0^\infty \left|\phi(t)e^{-st}\right| dt &\le \int_0^\infty \phi(t) e^{-\min\{\Re(\lambda_1),\Re(\lambda_2)\}t} dt \\ &\le \phi(0) \int_0^\infty e^{-\min\{\Re(\lambda_1),\Re(\lambda_2)\}t} dt = \frac{\phi(0)}{\min\{\Re(\lambda_1),\Re(\lambda_2)\}} < \infty\end{aligned}$$

for $\Re(s) \ge \min\{\Re(\lambda_1),\Re(\lambda_2)\}$, and so that $\lambda_1 \neq \lambda_2$ and $\mathscr{L}\{\phi(t)\}$ converges absolutely for $\Re(s) \ge \min\{\Re(\lambda_1),\Re(\lambda_2)\}$. Hence all conditions of Corollary 4.2 (i) are satisfied.

Next we will show case (ii). Suppose that $\lambda_1 = \lambda_2$ and $\Re(\lambda_1) > 0$. Then

$$\begin{aligned}\int_0^\infty \left|t\phi(t)e^{-st}\right| dt &\le \int_0^\infty t\phi(t) e^{-\min\{\Re(\lambda_1),\Re(\lambda_2)\}t} dt \\ &\le \phi(0) \int_0^\infty t e^{-\min\{\Re(\lambda_1),\Re(\lambda_2)\}t} dt = \frac{\phi(0)}{(\Re(\lambda_1))^2} < \infty\end{aligned}$$

for $\Re(s) \ge \Re(\lambda_1)$, and so that $\lambda_1 = \lambda_2$ and $\mathscr{L}\{t\phi(t)\}$ converges absolutely for $\Re(s) \ge \Re(\lambda_1)$. Hence all conditions of Corollary 4.2 (ii) are satisfied. Hence by Corollary 4.2, we have Corollary 4.3. □

If $\Re(\lambda_1)$ is negative and $\Re(\lambda_2) \ge \Re(\lambda_1)$, then we can choose $\phi(t)$ as $e^{(\Re(\lambda_1)-\delta)t}$ for any $\delta > 0$ and get the following result.

**Corollary 4.4.** *Let* $\delta > 0$. *Let* $\lambda_1$ *and* $\lambda_2$ *be the roots of* $s^2 + \alpha s + \beta = 0$. *Then (i) and (ii) below hold:*

(i) *if* $\lambda_1 \neq \lambda_2$, $\Re(\lambda_1) < 0$ *and* $\Re(\lambda_2) \ge \Re(\lambda_1)$, *then* (1.2) *has Hyers-Ulam-Rassias stability with respect to* $\phi(t) = e^{(\Re(\lambda_1)-\delta)t}$ *with Hyers-Ulam-Rassias constant*

$$K = \frac{1}{|\lambda_1 - \lambda_2|} \int_0^\infty \left| e^{-\lambda_2 \sigma} - e^{-\lambda_1 \sigma} \right| d\sigma;$$

(ii) *if* $\lambda_1 = \lambda_2$ *and* $\Re(\lambda_1) < 0$, *then* (1.2) *has Hyers-Ulam-Rassias stability with respect to* $\phi(t) = e^{(\Re(\lambda_1)-\delta)t}$ *with Hyers-Ulam-Rassias constant*

$$K = \frac{1}{(\Re(\lambda_1))^2}.$$

*Proof.* Let $\delta > 0$. Suppose that $\Re(\lambda_1)$ is negative. Then $\phi(t) = e^{(\Re(\lambda_1)-\delta)t}$ is a non-increasing positive function on $\mathbb{R}$. First we will show case (i). Suppose that $\lambda_1 \neq \lambda_2$ and $\Re(\lambda_2) \geq \Re(\lambda_1)$. Then

$$\int_0^\infty \left|\phi(t)e^{-st}\right| dt \leq \int_0^\infty e^{(\Re(\lambda_1)-\delta)t} e^{-\Re(\lambda_1)t} dt = \int_0^\infty e^{-\delta t} dt < \infty$$

for $\Re(s) \geq \Re(\lambda_1) = \min\{\Re(\lambda_1), \Re(\lambda_2)\}$, and so that $\lambda_1 \neq \lambda_2$ and $\mathscr{L}\{\phi(t)\}$ converges absolutely for $\Re(s) \geq \min\{\Re(\lambda_1), \Re(\lambda_2)\}$. Hence all conditions of Corollary 4.2 (i) are satisfied.

Next we will show case (ii). Suppose that $\lambda_1 = \lambda_2$. Then

$$\int_0^\infty \left|t\phi(t)e^{-st}\right| dt \leq \int_0^\infty t e^{(\Re(\lambda_1)-\delta)t} e^{-\Re(\lambda_1)t} dt = \int_0^\infty t e^{-\delta t} dt < \infty$$

for $\Re(s) \geq \Re(\lambda_1)$, and so that $\lambda_1 = \lambda_2$ and $\mathscr{L}\{t\phi(t)\}$ converges absolutely for $\Re(s) \geq \Re(\lambda_1)$. Hence all conditions of Corollary 4.2 (ii) are satisfied. Hence by Corollary 4.2, we have Corollary 4.4. □

In Corollary 4.3, if we choose $\phi(t) \equiv 1$, then we obtain the following Hyers-Ulam stability result.

**Corollary 4.5.** *Let $\lambda_1$ and $\lambda_2$ be the roots of $s^2 + \alpha s + \beta = 0$. Then (i) and (ii) below hold:*

(i) *if $\lambda_1 \neq \lambda_2$ and $\Re(\lambda_1) > 0$ and $\Re(\lambda_2) > 0$, then* (1.2) *has Hyers-Ulam stability with Hyers-Ulam constant*

$$K = \frac{1}{|\lambda_1 - \lambda_2|} \int_0^\infty \left| e^{-\lambda_2 \sigma} - e^{-\lambda_1 \sigma} \right| d\sigma;$$

(ii) *if $\lambda_1 = \lambda_2$ and $\Re(\lambda_1) > 0$, then* (1.2) *has Hyers-Ulam stability with Hyers-Ulam constant*

$$K = \frac{1}{(\Re(\lambda_1))^2}.$$

*Remark* 4.6. In 2020, Baias and Popa [5] studied the Hyers-Ulam stability and the minimum Hyers-Ulam constant for equation (1.1). Note that using their results, given Hyers-Ulam constants in Corollary 4.5 are the best Hyers-Ulam constants. This fact shows that our results are sharp.

## 5. Conclusions

This study explicitly evaluates the error between the approximate solution and exact solution of second order differential equations using the Laplace transform method. In recent years, approaches to the (generalized) Hyers-Ulam stability using the Laplace transform, Fourier transform, Mahgoub transform, Aboodh transform, etc. have been studied. However, most of them are limited to analyzes on finite intervals or the Ulam constant depends on the interval width. On the other hand, this study realized stability analysis on unbounded intervals. In addition, this study gives sharp results on error. The decisive reason is that the minimum Hyers-Ulam constants can be derived in special cases. That is, we derived the minimum error between the approximate solution and the true solution on $\mathbb{R}$. Investigating the error between the approximate and exact solutions can be expected to contribute to computer science.

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