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ON $\mathscr{T}$-PARTIAL G-METRIC SPACES AND AN APPLICATION IN DYNAMIC PROGRAMMING

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ABSTRACT. In this paper, a new class of spaces called $\mathscr{T}$-partial G-metric spaces (X,G) is introduced. In this direction, the generated topology satisfies the T_2-separation axiom. Furthermore, related fixed point theorems are given without using neither the compactness of the space X nor the symmetry of G. This results upgrade and extend many theorems in the literature. At the end of this work, an application to dynamic programming is presented to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

Fixed point theory plays a crucial role in determining the existence and the uniqueness of solutions for functional equations in dynamic programming, differential and integral equations, etc. It is initially formulated in the setting of metric spaces and has expanded into more generalized spaces. Among these general spaces, the G-metric space is particularly relevant to our study.
In 2009, Mustafa and Sims [5] introduced the concept of generalized metric spaces (in short G-metric spaces), as follows:

Definition 1.1. A G-metric on a nonempty set X is a mapping $G : X\times X\times X \to \mathbb{R}^+$ satisfies:

(1) $G(x,y,z) = 0$ if $x = y = z$,
(2) $0 < G(x,y,z)$ for all $x,y,z \in X$ with $x \neq y$,
(3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,
(4) $G(x,y,z) = G(p(x,y,z))$, where p is a permutation of x,y,z,
(5) $G(x,y,z) \leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a \in X$.

Example 1.2. ([5]) The previous properties may be easily interpreted in the setting of metric spaces. Let (X,d) be a metric space and define $G : X\times X\times X \to \mathbb{R}^+$ by:

$$G(x,y,z) = d(x,y)+d(x,z)+d(y,z),$$

for all $x,y,z \in X$.
Then (X,G) is a G-metric space. In this case, $G(x,y,z)$ can be interpreted as the perimeter of the triangle of vertices x,y and z.

A G-metric space (X,G) is called symmetric if $G(x,y,y) = G(y,x,x)$, for all $x,y \in X$. It is well known that the function $d^G(x,y) = G(x,y,y)$ generates a Hausdorff topology if and only if G is symmetric. So, to skip symmetry condition, the authors in [5] took two symmetric equivalent functions d^G_m and

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d_s^G on X and proved that G-metric spaces are provided with a Hausdorff topology τ_G. Namely, their definitions are as follows:

$$d_m^G(x,y) = \max\{G(x,y,y); G(y,x,x)\} \tag{1.1}$$

and

$$d_s^G(x,y) = G(x,y,y) + G(y,x,x). \tag{1.2}$$

In 2012, it was showed in [3] that in the symmetric case, many fixed point theorems on G-metric spaces are particular cases of existing fixed point theorems in metric spaces. In our work, we focus the study on the case of non-symmetry.

On the other side, Matthews [4] has introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In partial metric spaces, the self-distance of an arbitrary point need not be equal to zero.

Zand and Nezhad [12] have introduced a new generalized metric space named G_p-metric spaces as a generalization of both partial metric spaces and G-metric spaces. The following is the definition of a G_p-metric space:

Definition 1.3. Let X be nonempty set. A function $G_p : X \times X \times X \to \mathbb{R}^+$ is a G_p-metric on X if the following conditions hold:

(1) $x = y = z$ if $G_p(x,y,z) = G_p(x) = G_p(y) = G_p(z)$,
(2) $G_p(x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $y \neq z$,
(3) $G_p(x,y,z) = G_p(\mathfrak{p}(x,y,z))$, where $\mathfrak{p}$ is a permutation of x, y, and z,
(4) $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ for all $x,y,z,a \in X$.

The pair (X, G_p) is called a G_p-metric space.

We point out that the topology generated by G_p-metrics is not T_2. Aamri and El Moutawakil [1] presented a substantial modification of the Banach contraction principle. They introduced the notion of τ-distance functions in a general topological space (X,τ). The authors in [6] employ this concept to establish a fixed point theorem for contractive mappings in bounded metric spaces. This idea is based on eliminating the need for compactness. To gain a more thorough understanding of this topic, we recommend interested readers to consult the latest research articles [7, 8, 9, 10].

In this article, we will make modifications to G_p-metrics to introduce a new class of spaces called $\mathscr{T}$-partial G-metric spaces. This novel kind of spaces extends G-metric spaces and satisfies the T_2-separation axiom. In this context, a generalization of the main theorem in [6] is obtained by using τ-distances.

Finally, an application to the study of existence and uniqueness of solutions for a class of functional equations arising in dynamic programming is presented under new and weak conditions.

Now, we recall some facts which will be used in the next. Let (X,τ) be a topological space and $p : X \times X \to [0,\infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x,\varepsilon) = \{y \in X : p(x,y) < \varepsilon\}$.

Definition 1.4. ([1]) The function p is said to be τ-distance if for each $x \in X$ and any neighborhood V of x, there exists $\varepsilon > 0$ such that $B_p(x,\varepsilon) \subset V$.

Definition 1.5. ([1]) Let (X,τ) be a topological space with a τ-distance p.

(1) A sequence $\{x_n\}$ in a Hausdorff topological space X is a p-Cauchy if it satisfies the usual metric condition with respect to p, in other words, if $\lim p(x_n,x_m)=0$.
(2) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x,x_n)=0$.
(3) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n = x$ with respect to τ.
(4) X is said to be p-bounded if $\sup\{p(x,y)/x,y\in X\}<\infty$.

Lemma 1.6. *([1])*
Let (X,τ) be a Hausdorff topological space with a τ-distance p, then
(1) *$p(x,y)=0$ implies $x=y$.*
(2) *Let (x_n) be a sequence in X such that $\lim_{n\to\infty} p(x,x_n)=0$ and $\lim_{n\to\infty} p(y,x_n)=0$, then $x=y$.*

Theorem 1.7. *([1])*
Let (X,τ) be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let $T: X \longrightarrow X$ be a mapping satisfying: there exists $k\in[0,1)$ such that for all $x,y\in X$, we have $p(Tx,Ty)\le kp(x,y)$.
Then T has a unique fixed point.

2. Main results

At the beginning of this section, we introduce a new definition:

Definition 2.1. Let X be nonempty set. A function $G: X\times X\times X\to\mathbb{R}^+$ is a $\mathscr{T}$-partial G-metric on X if the following conditions hold:
(1) $G(x,y,z)=G(x)$ or $G(x,y,z)=G(y)$ or $G(x,y,z)=G(z)$ then $x=y=z$,
(2) $G(x,x,y)\le G(x,y,z)$ for all $y\neq z$,
(3) $G(x)<G(x,y,z)$ for all $x\neq y$,
(4) $G(x,y,z)=G(p(x,y,z))$, where p is a permutation of x,y,z, and
(5) $G(x,y,z)\le G(x,a,a)+G(a,y,z)-\min\{G(x),G(y)\}$ for all $x,y,z,a\in X$, where $G(x)=G(x,x,x)$.

The pair (X,G) is called a $\mathscr{T}$-partial G-metric space.

Clearly, every G-metric space is a $\mathscr{T}$-partial G-metric space with $G(x)=0$ for all $x\in X$. However, the converse of this fact need not hold, as we will present in the following example:

Example 2.2. Let (X,G) be a G-metric space. Then (X,G') is a $\mathscr{T}$-partial G-metric space for $G'(x,y,z)=G(x,y,z)+\varepsilon$, for all $x,y,z\in X$ with $\varepsilon>0$.

The following are related topological notions of a $\mathscr{T}$-partial G-metric space:

Definition 2.3. Let (X,G) be a $\mathscr{T}$-partial G-metric, $x\in X$ and $\varepsilon>0$.
(1) $B_G(x,\varepsilon)=\{y\in X: G(x,y,y)<G(x)+\varepsilon\}$ is called the open ball with center x and radius ε.
(2) A sequence $\{x_n\}$ in X converges to a point $x\in X$ if and only if $\lim_{n,m\to\infty} G(x,x_n,x_m)=G(x)$.
(3) A sequence $\{x_n\}\subset X$ is a Cauchy sequence if $\lim_{m,n\to\infty} G(x_n,x_m,x_m)$ exists and is finite.
(4) X is complete if every Cauchy sequence $\{x_n\}\subset X$ converges to a point $x\in X$.
(5) X is said to be bounded if $\sup\{G(x,y,z)/x,y,z\in X\}<\infty$.

Lemma 2.4. *Let (X,G) be a $\mathscr{T}$-partial G-metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x,y) = e^{G(x,y,y)} - 1. \tag{2.1}$$

Then p is a τ_G-distance on X, where τ_G is the topology induced by G.

Proof. Let (X,τ_G) be the topological space with the topology τ_G and V an arbitrary neighborhood of an arbitrary $x \in X$, then there exists $\varepsilon > 0$ such that $B_G(x,\varepsilon) \subset V$, where $B_G(x,\varepsilon) = \{y \in X, G(x,y,y) < G(x)+\varepsilon\}$ is the open ball in (X,G).
It is easy to see that $B_p(x,e^{\varepsilon}-1) \subset B_G(x,\varepsilon)$, indeed:
Let $y \in B_p(x,e^{\varepsilon}-1)$, then $p(x,y) < e^{\varepsilon}-1$, which implies that $e^{G(x,y,y)} < e^{G(x)+\varepsilon}$. Therefore, $G(x,y,y) < G(x)+\varepsilon$. □

Lemma 2.5. *Let (X,G) be a bounded $\mathscr{T}$-partial G-metric space, then (X,p) is a bounded topological space with the τ-distance p defined in Lemma 2.4.*

Lemma 2.6. *Let (X,G) be a complete $\mathscr{T}$-partial G-metric space, then (X,τ_G) is a S-complete topological space.*

Proof. Let $\{x_n\}$ be a p-Cauchy sequence, which implies that $\lim_{n,m} p(x_n,x_m) = 0$, and hence $G(x_n,x_m,x_m) \longrightarrow 0$. Therefore, $\{x_n\} \subset (X,G)$ is a Cauchy sequence. Now, since (X,G) is complete, there exists $u \in X$ such that $\lim p(u,x_n) = 0$. □

Proposition 2.7. *A $\mathscr{T}$-partial G-metric on a nonempty X generates a Hausdorff topology τ_G on X with a base of the family of open balls $\{B_G(x,\varepsilon) : x \in X, \varepsilon > 0\}$.*

Proof. Let $x \neq y \in X$. Putting $d_z := G(x,y,z) - \max\{G(x),G(y)\} > 0$, where $z \in X$.
There exists an element $z_0 \in X$ such that:

$$B_G\left(x,\frac{d_{z_0}}{2}\right) \cap B_G\left(y,\frac{d_{z_0}}{2}\right) = \emptyset. \tag{2.2}$$

Indeed: If $a \in B_G(x,\frac{d_z}{2}) \cap B_G(y,\frac{d_z}{2})$ for all $z \in X$, we have

$$\begin{aligned} G(x,y,a) &\leq G(x,a,a) + G(a,y,a) - \min\{G(x),G(y)\} \\ &< G(x) + \frac{d_a}{2} + G(y) + \frac{d_a}{2} - \min\{G(x),G(y)\} \\ &= G(x) + G(y) - \min\{G(x),G(y)\} + G(x,y,a) - \max\{G(x),G(y)\} \\ &= G(x,y,a) \end{aligned} \tag{2.3}$$

It is easy to see that:

$$G(x) + G(y) - \min\{G(x),G(y)\} - \max\{G(x),G(y)\} = 0. \tag{2.4}$$

Therefore, we obtain $G(x,y,a) < G(x,y,a)$, which this is a contradiction.
In addition, we have

$$x \in B_G\left(x,\frac{d_{z_0}}{2}\right), y \in B_G\left(y,\frac{d_{z_0}}{2}\right). \tag{2.5}$$

This finishes the proof. □

The main result of this work is the following:

Theorem 2.8. *Let $T : X \longrightarrow X$ be a mapping of a bounded complete $\mathscr{T}$-partial G-metric space (X,G) such that*

$$\inf_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}>0. \tag{2.6}$$

Then T has a unique fixed point.

Proof. We set $\alpha = \inf_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}$. Hence, for all $x\neq y\in X$, we get

$$G(Tx,Ty,Ty)\leq G(x,y,y)-\alpha, \tag{2.7}$$

which implies that

$$e^{G(Tx,Ty,Ty)}\leq ke^{G(x,y,y)}, \tag{2.8}$$

for all $x\neq y\in X$ where $k=e^{-\alpha}<1$. Also,

$$p(Tx,Ty)\leq kp(x,y), \tag{2.9}$$

for all $x\neq y\in X$ where $k<1$ and p is the function defined in Lemma (2.4).
Finally, using Lemmas 2.7, 2.4, 2.5, 2.6 and Theorem 1.7, we conclude that T has a unique fixed point in X. □

***Example* 2.9.** Consider the set $X=\{2,3,4,5\}$ and the function $G : X\times X\times X\to\mathbb{R}^+$ defined by:

(x,y,z)	$G(x,y,z)$
$(x,y,z)\notin\{(3,4,4),(4,3,3)\}$	$\lvert x-y\rvert+\lvert x-z\rvert+\lvert y-z\rvert+1$
$(3,4,4)$	4
$(4,3,3)$	5

It is straightforward to show that (X,G) is a complete $\mathscr{T}$-partial G-metric space. Define a mapping T as follows:

$$T2=T3=2 \text{ and } T4=T5=3. \tag{2.10}$$

So, we obtain

$G(2,3,3)-G(T2,T3,T3)=2$
$G(2,4,4)-G(T2,T4,T4)=2$
$G(2,5,5)-G(T2,T5,T5)=4$
$G(3,4,4)-G(T3,T4,T4)=1$
$G(4,3,3)-G(T4,T3,T3)=2$
$G(3,5,5)-G(T3,T5,T5)=2$
$G(4,5,5)-G(T4,T5,T5)=2$

Therefore, for all $x\neq y\in X$ we have

$$G(x,y,y)-G(Tx,Ty,Ty)\geq 1. \tag{2.11}$$

In other words:

$$\inf_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}>0. \tag{2.12}$$

Then, all conditions of Theorem 2.8 are satisfied and T has the unique fixed point $2=T2$.

***Remark* 2.10.** In the above example, we did not need the symmetry condition, since

$$G(4,3,3)=5\neq 4=G(3,4,4), \tag{2.13}$$

which is a main condition for which $G(x,y,y)$ become a metric.

If we take $G(x)=0$, we obtain:

Corollary 2.11. *Let $T:X\longrightarrow X$ be a mapping of a bounded complete G-metric space (X,G) such that $\inf_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}>0$. Then T has a unique fixed point.*

3. Application

In this section, we investigate the existence and uniqueness of a solution for a specific category of functional equations in the field of dynamic programming. Our study draws inspiration from the works of Belman [2, 11]. To achieve this purpose, suppose that X and Y are Banach spaces, $S\subset X$ is the state space and $D\subset Y$ is the decision space. Let $\rho:S\times D\to S$, $g:S\times D\to\mathbb{R}$ and $\mathscr{G}:S\times D\times\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. $B(S)$ denotes the set of all bounded real-valued functions on S. For $a\in B(S)$, denote $||a||=\sup_{x\in S}|a(x)|$ and define:

$$G(h,k,l)=\sup_{x\in S}\{|h(x)-k(x)|,|h(x)-l(x)|,|k(x)-l(x)|\}+\max\{||h||,||k||,||l||\}, \tag{3.1}$$

where $h,k,l\in B(S)$.
$(B(s),G)$ is a complete $\mathscr{T}$-partial G-metric space.
Consider the following functional equation:

$$f(x)=\sup_{y\in D}\{g(x,y)+\mathscr{G}(x,y,f(\rho(x,y)))\}, \tag{3.2}$$

where g and $\mathscr{G}$ are bounded.
We define $T:B(S)\to B(S)$ by:

$$Tf(x)=\sup_{y\in D}\{g(x,y)+\mathscr{G}(x,y,f(\rho(x,y)))\}. \tag{3.3}$$

In the following, we prove the existence and uniqueness of the solution for the functional (3.2).

Theorem 3.1. *Let $T:B(S)\to B(S)$ be an operator defined by (3.3) and assume the following condition is satisfied:*
There exists $M>0$ such that:

$$|\mathscr{G}(x,y,h(x))-\mathscr{G}(x,y,k(x))|\leq G(h,k,k)-\max\{||Th||,||Tk||\}-M, \tag{3.4}$$

for all $(h,k,x,y)\in B(S)\times B(S)\times S\times D$, where $h(x)\neq k(x)$.
Then the functional equation (3.2) has a unique bounded solution.

Proof. Let ε be an arbitrary positive number, let $x \in S$ and $h,k \in B(S)$, by the definition of T, there exist $y,z \in D$ such that:

$$g(x,z) + \mathscr{G}(x,z,h(\rho(x,z))) \leq T(h(x)) < g(x,y) + \mathscr{G}(x,y,h(\rho(x,y))) + \varepsilon \tag{3.5}$$

and

$$g(x,y) + \mathscr{G}(x,y,k(\rho(x,z))) \leq T(k(x)) < g(x,z) + \mathscr{G}(x,z,k(\rho(x,z))) + \varepsilon. \tag{3.6}$$

It follows that:

$$T(h(x)) - T(k(x)) < |\mathscr{G}(x,y,h(\rho(x,y))) - \mathscr{G}(x,y,k(\rho(x,y)))| + \varepsilon. \tag{3.7}$$

Thus

$$T(h(x)) - T(k(x)) < G(h,k,k) - \max\{||Th||, ||Tk||\} - M + \varepsilon. \tag{3.8}$$

Similarly, we can find

$$T(k(x)) - T(h(x)) < G(h,k,k) - \max\{||Th||, ||Tk||\} - M + \varepsilon. \tag{3.9}$$

In view of (3.8) and (3.9), we obtain

$$|T(h(x)) - T(k(x))| < G(h,k,k) - \max\{||Th||, ||Tk||\} - M + \varepsilon. \tag{3.10}$$

Therefore

$$G(Th,Tk,Tk) < G(h,k,k) - M + \varepsilon. \tag{3.11}$$

Since ε is taken arbitrary, then we obtain

$$G(Th,Tk,Tk) < G(h,k,k) - M, \tag{3.12}$$

for all $h \neq k \in B(S)$.
Equivalently

$$\inf_{h \neq k}\{G(h,k,k) - G(Th,Tk,Tk)\} > 0. \tag{3.13}$$

Finally, by using Theorem 2.6, we conclude that the functional (3.2) has a unique bounded solution. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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