SOME RESULTS FROM A GASPER AND RAHMAN'S QUADRATIC SUMMATION

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ABSTRACT. Applying Gasper and Rahman's quadratic summation, we verify two q-supercongruences conjectured by Guo and refine a q-supercongruence of Guo. Moreover, we get some new supercongruences modulo p^2 or p^3, including: for $0 < r < d \leq 2r$ and any prime $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{p-1} (3dk + r) \frac{\left(\frac{r}{d}\right)_k \left(\frac{d-r}{d}\right)_k \left(\frac{r}{2d}\right)_k^2 \left(\frac{1}{2}\right)_k}{k!^4 \left(\frac{d+2r}{2d}\right)_k} \equiv 0 \pmod{p^3},$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the *rising-factorial*.

1. INTRODUCTION

In 2017, employing the p-adic Gamma function and a ${}_7F_6$ summation of Gessel and Stanton [2], He [7] established some supercongruences, including: for primes $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1} (6k+1) \frac{\left(\frac{1}{2}\right)_k^3 \left(\frac{1}{4}\right)_k^2}{k!^5} \equiv 0 \pmod{p^2}. \tag{1.1}$$

Also, He [7] conjectured that (1.1) is true modulo p^3. This conjecture was later proved by Liu [9] through another ${}_7F_6$ summation in [2]. Here and throughout the paper, p is a prime and the *rising-factorial* is given by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)\cdots(a+n-1) \quad \text{for} \quad n \in \mathbb{Z}^+.$$

In addition, we introduce some necessary definitions. Let q be an indeterminate. The *q-integer* is defined as

$$[n] = [n]_q = 1 + q + \cdots + q^{n-1} \quad \text{for} \quad n \in \mathbb{Z}^+.$$

When $|q| < 1$, the *q-shifted factorial* is given by

$$(a;q)_\infty = \prod_{k=0}^{\infty} \left(1 - aq^k\right) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \text{for} \quad n \in \mathbb{Z}.$$

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For brevity, the multiple q-shifted factorial can be directly written as

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n.$$

Moreover, the n-th *cyclotomic polynomial* in q is represented by $\Phi_n(q)$:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(n,k)=1}} \left(q - \zeta^k\right),$$

where ζ is an n-th primitive root of unity.

In recent years, q-supercongruences have attracted many experts' attention and some progress has been made. The reader who has an interest may be referred to [3, 4, 8, 10–14, 16]. Particularly, Wei [15] established a q-analogue of (1.1) modulo p^3 as follows: for any positive integer $n \equiv 3 \pmod 4$, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1} [6k+1] \frac{(q; q^2)_k^2 \, (q, q, q^2; q^4)_k \, q^{2k}}{(q^2; q^2)_k^2 \, (q^4; q^4)_k^3} \equiv 0. \tag{1.2}$$

It is worth mentioning that the following Gasper and Rahman's quadratic summation (see [1, (3.8.12)]) plays an important role in Wei's work: for $|q| < 1$,

$$\begin{aligned}
&\sum_{k \geq 0} \frac{1 - aq^{3k}}{1 - a} \frac{(a, b, q/b; q)_k \, (d, f, a^2q/df; q^2)_k}{(aq/d, aq/f, df/a; q)_k \, (q^2, aq^2/b, abq; q^2)_k} q^k \\
&\quad + \frac{(aq, f/a, b, q/b; q)_\infty \, (d, aq^2/df, fq^2/d, df^2q/a^2; q^2)_\infty}{(a/f, fq/a, aq/d, df/a; q)_\infty \, (aq^2/b, abq, fq/ab, bf/a; q^2)_\infty} \\
&\quad \times \sum_{k \geq 0} \frac{(f, bf/a, fq/ab; q^2)_k q^{2k}}{(q^2, fq^2/d, df^2q/a^2; q^2)_k} \\
&= \frac{(aq, f/a; q)_\infty \, (aq^2/bd, abq/d, bdf/a, dfq/ab; q^2)_\infty}{(aq/d, df/a; q)_\infty \, (aq^2/b, abq, bf/a, fq/ab; q^2)_\infty}.
\end{aligned} \tag{1.3}$$

Recently, by taking suitable parametric substitutions into (1.3), and utilizing the 'creative microscoping' method introduced by Guo and Zudilin [6], Guo [5] gave several generalizations of (1.2), where the modulo $[n]\Phi_n(q)^2$ condition was replaced by the weaker condition modulo $\Phi_n(q)^2$ or $\Phi_n(q)^3$. For example, Guo [5, Theorem 1.2] got the following result: for positive integers n, d, r with $n \equiv -1 \pmod{2d}$ and $r < d \leq 2r$,

$$\sum_{k=0}^{n-1} [3dk + r] \frac{\left(q^r, q^{d-r}; q^d\right)_k \left(q^r, q^r, q^d; q^{2d}\right)_k q^{dk}}{\left(q^d, q^d; q^d\right)_k \left(q^{2d}, q^{2d}, q^{d+2r}; q^{2d}\right)_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.4}$$

Letting $n = p$ to be a prime and $q \to 1$ in (1.4), we arrive at the following result: for $0 < r < d \leq 2r$ and any prime $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{p-1} (3dk + r) \frac{\left(\frac{r}{d}\right)_k \left(\frac{d-r}{d}\right)_k \left(\frac{r}{2d}\right)_k^2 \left(\frac{1}{2}\right)_k}{k!^4 \left(\frac{d+2r}{2d}\right)_k} \equiv 0 \pmod{p^2}, \tag{1.5}$$

which is a generalization of (1.1).

Furthermore, at the end of Guo's paper [5], the following two conjectures were proposed.

Conjecture 1. [5, Conjecture 6.1]. For positive integers n, d, r with $n \equiv -1 \pmod{2d}$ and $r < d$, there holds

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r, q^{d-r}; q^d)_k\,(q^r, q^r, q^d; q^{2d})_k\, q^{dk}}{(q^d, q^d; q^d)_k\,(q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.6}$$

Conjecture 2. [5, Conjecture 6.2]. Let d and r be positive integers such that r is odd and $\gcd(d, r) = 1$. Let n be a positive integer satisfying $n \equiv -r \pmod{2d}$ and $dn > n+r$. Then, we have

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r, q^{d-r}; q^d)_k\,(q^r, q^r, q^d; q^{2d})_k\, q^{dk}}{(q^d, q^d; q^d)_k\,(q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.7}$$

In this paper, we shall confirm the above two conjectures and refine the q-supercongruence (1.4). Our first result is stated as follows.

Theorem 1.1. *Conjecture 1 is true.*

Our second result, an enhanced version of (1.4), can be stated as follows.

Theorem 1.2. *For positive integers n, d, r with $n \equiv -1 \pmod{2d}$ and $r < d \le 2r$, there holds*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r, q^{d-r}; q^d)_k\,(q^r, q^r, q^d; q^{2d})_k\, q^{dk}}{(q^d, q^d; q^d)_k\,(q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.8}$$

Obviously, when $d = 2$ and $r = 1$, (1.8) is a q-analogue of (1.1) modulo p^3. Besides, setting $n = p$ to be a prime and $q \to 1$ in Theorem 1.2, we get a stronger version of congruence (1.5): for $0 < r < d \le 2r$ and any prime $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{p-1}(3dk+r)\,\frac{\left(\frac{r}{d}\right)_k\left(\frac{d-r}{d}\right)_k\left(\frac{r}{2d}\right)_k^2\left(\frac{1}{2}\right)_k}{k!^4\left(\frac{d+2r}{2d}\right)_k} \equiv 0 \pmod{p^3}. \tag{1.9}$$

The last result can be shown as follows.

Theorem 1.3. *Conjecture 2 is true.*

The rest of our paper is arranged as follows. In Section 2, by utilizing Gasper and Rahman's quadratic summation (1.3), the 'creative microscoping' method and the Chinese remainder theorem for coprime polynomials, we first establish a generalized result with two free parameters a and b. Afterwards, we present how Theorems 1.1 and 1.2 can be derived from this parametric form. At last, we prove Theorem 1.3 in Section 3.

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2. Proofs of Theorems 1.1 and 1.2

Firstly, we establish a generalized form of Theorems 1.1 and 1.2 with two free parameters a and b.

Theorem 2.1. *Let n, d, r be positive integers with $n \equiv -1 \pmod{2d}$ and $r < d$. Let a, b be indeterminates. Then, modulo $\left(a - q^{(d-r)n}\right)\left(1 - bq^{(2d-r)n}\right)\left(b - q^{(2d-r)n}\right)$,*

$$\sum_{k=0}^{\frac{2dn-rn-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a, q^r, q^{d-r}; q^d\right)_k \left(bq^r, q^r/b, q^d/a^2; q^{2d}\right)_k q^{dk}}{\left(bq^d/a, q^d/ab, aq^r; q^d\right)_k \left(q^{2d}, q^{2d}/a, q^{2r+d}/a; q^{2d}\right)_k}$$
$$\equiv \frac{\left(a - q^{(2d-r)n}\right)\left(ab - 1 - b^2 + bq^{(2d-r)n}\right)}{(b-a)(1-ab)} \frac{\left(q^{d+r}/a; q^d\right)_{(2dn-rn-r)/d}}{\left(aq^r; q^d\right)_{(2dn-rn-r)/d}}$$
$$\times \frac{\left(aq^{2r}, aq^d; q^{2d}\right)_{(2dn-rn-r)/(2d)}}{\left(q^{2d}/a, q^{d+2r}/a; q^{2d}\right)_{(2dn-rn-r)/(2d)}}. \tag{2.1}$$

Proof. Making the substitution $(q, a, b, d, f) \to \left(q^d, q^r/a, q^r, q^{r-(2d-r)n}, q^{r+(2d-r)n}\right)$ into (1.3), we get

$$\sum_{k=0}^{\frac{2dn-rn-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a, q^r, q^{d-r}; q^d\right)_k \left(q^{r-(2d-r)n}, q^{r+(2d-r)n}, q^d/a^2; q^{2d}\right)_k q^{dk}}{\left(q^{d+(2d-r)n}/a, q^{d-(2d-r)n}/a, aq^r; q^d\right)_k \left(q^{2d}, q^{2d}/a, q^{2r+d}/a; q^{2d}\right)_k}$$
$$= \frac{\left(q^{d+r}/a; q^d\right)_{(2dn-rn-r)/d} \left(aq^{2r}, aq^d; q^{2d}\right)_{(2dn-rn-r)/(2d)}}{\left(aq^r; q^d\right)_{(2dn-rn-r)/d} \left(q^{2d}/a, q^{d+2r}/a; q^{2d}\right)_{(2dn-rn-r)/(2d)}}. \tag{2.2}$$

From (2.2), we obtain the following congruence: modulo $\left(1 - bq^{(2d-r)n}\right)\left(b - q^{(2d-r)n}\right)$,

$$\sum_{k=0}^{\frac{2dn-rn-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a, q^r, q^{d-r}; q^d\right)_k \left(bq^r, q^r/b, q^d/a^2; q^{2d}\right)_k q^{dk}}{\left(bq^d/a, q^d/ab, aq^r; q^d\right)_k \left(q^{2d}, q^{2d}/a, q^{2r+d}/a; q^{2d}\right)_k}$$
$$\equiv \frac{\left(q^{d+r}/a; q^d\right)_{(2dn-rn-r)/d} \left(aq^{2r}, aq^d; q^{2d}\right)_{(2dn-rn-r)/(2d)}}{\left(aq^r; q^d\right)_{(2dn-rn-r)/d} \left(q^{2d}/a, q^{d+2r}/a; q^{2d}\right)_{(2dn-rn-r)/(2d)}}. \tag{2.3}$$

Similarly, substituting $(q, a, b, d, f) \to \left(q^d, q^{r-(d-r)n}, q^r, bq^r, q^r/b\right)$ into (1.3) and noticing that $\left(q^{r+d-(d-r)n}; q^d\right)_\infty = 0$, $\left(q^{r-(d-r)n}; q^d\right)_k = 0$ for $0 < (dn - rn - r)/d < k$, we have

$$\sum_{k=0}^{\frac{2dn-rn-r}{2d}} \frac{1-q^{3dk+r-(d-r)n}}{1-q^{r-(d-r)n}} \frac{\left(q^{r-(d-r)n}, q^r, q^{d-r}; q^d\right)_k}{\left(bq^{d-(d-r)n}, q^{d-(d-r)n}/b, q^{r+(d-r)n}; q^d\right)_k}$$
$$\times \frac{\left(bq^r, q^r/b, q^{d-2(d-r)n}; q^{2d}\right)_k q^{dk}}{\left(q^{2d}, q^{2d-(d-r)n}, q^{2r+d-(d-r)n}; q^{2d}\right)_k} = 0. \tag{2.4}$$

Consequently, the following result holds, modulo $a-q^{(d-r)n}$,

$$\sum_{k=0}^{\frac{2dn-rn-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a, q^r, q^{d-r}; q^d\right)_k \left(bq^r, q^r/b, q^d/a^2; q^{2d}\right)_k q^{dk}}{\left(bq^d/a, q^d/ab, aq^r; q^d\right)_k \left(q^{2d}, q^{2d}/a, q^{2r+d}/a; q^{2d}\right)_k} \equiv 0. \tag{2.5}$$

Clearly, $a-q^{(d-r)n}$ and $\left(1-bq^{(2d-r)n}\right)\left(b-q^{(2d-r)n}\right)$ are relatively prime polynomials. Thus, combing the following relations:

$$\frac{\left(a-q^{(2d-r)n}\right)\left(ab-1-b^2+bq^{(2d-r)n}\right)}{(b-a)(1-ab)} \equiv 1 \pmod{\left(1-bq^{(2d-r)n}\right)\left(b-q^{(2d-r)n}\right)}, \tag{2.6}$$

$$\frac{\left(1-bq^{(d-r)n}\right)\left(b-q^{(d-r)n}\right)}{(b-a)(1-ab)} \equiv 1 \pmod{a-q^{(d-r)n}}, \tag{2.7}$$

with the Chinese remainder theorem for coprime polynomials, we immediately obtain the desired congruence (2.1) from (2.3) and (2.5). □

Based on Theorem 2.1, we now present the detailed proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. From $r<d$ and $n \equiv -1 \pmod{2d}$, we know that the denominator of the left-hand side of (2.1) is relatively prime to $\Phi_n(q)^2\left(a-q^{(d-r)n}\right)$ when $b=1$. On the other hand, $\left(q^{d+r}/a; q^d\right)_{(2dn-rn-r)/d}$ has the factor $a-q^{(d-r)n}$. Therefore, letting $b=1$ into (2.1), and applying the following relation:

$$\left(a-q^{(2d-r)n}\right)\left(a-2+q^{(2d-r)n}\right)=(a-1)^2-\left(1-q^{(2d-r)n}\right)^2,$$

we have, modulo $\Phi_n(q)^2\left(a-q^{(d-r)n}\right)$,

$$\begin{aligned} &\sum_{k=0}^{\frac{2dn-rn-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a, q^r, q^{d-r}; q^d\right)_k \left(q^r, q^r, q^d/a^2; q^{2d}\right)_k q^{dk}}{\left(q^d/a, q^d/a, aq^r; q^d\right)_k \left(q^{2d}, q^{2d}/a, q^{2r+d}/a; q^{2d}\right)_k} \\ &\equiv \frac{\left(q^{d+r}/a; q^d\right)_{(2dn-rn-r)/d}\left(aq^{2r}, aq^d; q^{2d}\right)_{(2dn-rn-r)/(2d)}}{\left(aq^r; q^d\right)_{(2dn-rn-r)/d}\left(q^{2d}/a, q^{d+2r}/a; q^{2d}\right)_{(2dn-rn-r)/(2d)}}. \end{aligned} \tag{2.8}$$

Since $\gcd(n,2d)=1$, the smallest positive integer k such that $\left(q^m; q^{2d}\right)_k \equiv 0 \pmod{\Phi_n(q)}$ is $(2d-m)(n+1)/(2d)$ for m in the range $0<m<2d$. When $2r<d$, we get $0<(d-2r)(n+1)/(2d)<(2dn-rn-r)/(2d)$, which means that $\left(q^{d+2r}; q^{2d}\right)_k$ has the factor $1-q^{(d-2r)n}$ for k in the range of $(d-2r)(n+1)/(2d) \le k \le (2dn-rn-r)/(2d)$. Therefore, when $a=1$, the nominator of the right-hand side of (2.8) is surely divisible by $\Phi_n(q)^3$ and the denominator of the left-hand side of (2.8) may have the factor $\Phi_n(q)$. Consequently, taking $a=1$ into (2.8), we get

$$\sum_{k=0}^{\frac{2dn-rn-r}{2d}} [3dk+r] \frac{\left(q^r, q^{d-r}; q^d\right)_k \left(q^r, q^r, q^d; q^{2d}\right)_k q^{dk}}{\left(q^d, q^d; q^d\right)_k \left(q^{2d}, q^{2d}, q^{d+2r}; q^{2d}\right)_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{2.9}$$

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Finally, for $(2dn-rn-r)/(2d) < k \le n-1$, $\left(q^r;q^d\right)_k$ has the factor $1-q^{dn-rn}$ and $\left(q^r;q^{2d}\right)_k$ has the factor $1-q^{2dn-rn}$, which means that the k-th term of the left-hand side of (2.9) is divisible by $\Phi_n(q)^2$ too. Then, we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. Clearly, the condition of Theorem 1.2 is a special case of Theorem 1.1. Therefore, following the same line of the proof of Theorem 1.1, we can deduce that (2.8) remains valid under the condition of Theorem 1.2. And when $d \le 2r$, we have $(d-2r)(n+1)/(2d) \le 0$. Thus, the smallest positive integer k satisfying $\left(q^{d+2r};q^{2d}\right)_k \equiv 0 \pmod{\Phi_n(q)}$ is $n+(d-2r)(n+1)/(2d)$. Since $r<d$, we have $n+(d-2r)(n+1)/(2d) > (2dn-rn-r)/(2d)$. Consequently, $\left(q^{d+2r};q^{2d}\right)_{(2dn-rn-r)/(2d)}$ is coprime with $\Phi_n(q)$. Then, when $a=1$, the right-hand side of (2.8) is divisible by $\Phi_n(q)^3$ and the denominator of reduced form of the left-hand side of (2.8) is coprime with $\Phi_n(q)$, which means that the desired congruence is true. □

3. Proof of Theorem 1.3

In this section, we start with the following parametric generalization of Theorem 1.3.

Theorem 3.1. *Let d and r be positive integers such that r is odd and $\gcd(d,r)=1$. Let n be a positive integer satisfying $n \equiv -r \pmod{2d}$ and $dn > n+r$. Let a,b be indeterminates. Then, modulo $\left(a-q^{(d-1)n}\right)\left(1-bq^{(2d-1)n}\right)\left(b-q^{(2d-1)n}\right)$,*

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a,q^r,q^{d-r};q^d\right)_k \left(bq^r,q^r/b,q^d/a^2;q^{2d}\right)_k q^{dk}}{\left(bq^d/a,q^d/ab,aq^r;q^d\right)_k \left(q^{2d},q^{2d}/a,q^{2r+d}/a;q^{2d}\right)_k}$$
$$\equiv \frac{\left(a-q^{(2d-1)n}\right)\left(ab-1-b^2+bq^{(2d-1)n}\right)}{(b-a)(1-ab)} \frac{\left(q^{d+r}/a;q^d\right)_{(2dn-n-r)/d}}{\left(aq^r;q^d\right)_{(2dn-n-r)/d}}$$
$$\times \frac{\left(aq^{2r},aq^d;q^{2d}\right)_{(2dn-n-r)/(2d)}}{\left(q^{2d}/a,q^{d+2r}/a;q^{2d}\right)_{(2dn-n-r)/(2d)}}. \tag{3.1}$$

Proof. Letting $q \to q^d$, $a=q^r/a$, $b=q^r$, $d=q^{r-(2d-1)n}$ and $f=q^{r+(2d-1)n}$ in (1.3), we have

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a,q^r,q^{d-r};q^d\right)_k \left(q^{r-(2d-1)n},q^{r+(2d-1)n},q^d/a^2;q^{2d}\right)_k q^{dk}}{\left(q^{d+(2d-1)n}/a,q^{d-(2d-1)n}/a,aq^r;q^d\right)_k \left(q^{2d},q^{2d}/a,q^{2r+d}/a;q^{2d}\right)_k}$$
$$= \frac{\left(q^{d+r}/a;q^d\right)_{(2dn-n-r)/d} \left(aq^{2r},aq^d;q^{2d}\right)_{(2dn-n-r)/(2d)}}{\left(aq^r;q^d\right)_{(2dn-n-r)/d} \left(q^{2d}/a,q^{d+2r}/a;q^{2d}\right)_{(2dn-n-r)/(2d)}}. \tag{3.2}$$

From (3.2), we obtain the following congruence: modulo $\left(1-bq^{(2d-1)n}\right)\left(b-q^{(2d-1)n}\right)$,

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}} \frac{1-q^{3dk+r}/a}{1-q^r/a} \frac{\left(q^r/a,q^r,q^{d-r};q^d\right)_k \left(bq^r,q^r/b,q^d/a^2;q^{2d}\right)_k q^{dk}}{\left(bq^d/a,q^d/ab,aq^r;q^d\right)_k \left(q^{2d},q^{2d}/a,q^{2r+d}/a;q^{2d}\right)_k}$$

$$\equiv \frac{\left(q^{d+r}/a;q^d\right)_{(2dn-n-r)/d}\left(aq^{2r},aq^d;q^{2d}\right)_{(2dn-n-r)/(2d)}}{\left(aq^r;q^d\right)_{(2dn-n-r)/d}\left(q^{2d}/a,q^{d+2r}/a;q^{2d}\right)_{(2dn-n-r)/(2d)}}. \tag{3.3}$$

Similarly, setting $q\to q^d$, $a=q^{r-(d-1)n}$, $b=q^r$, $d=bq^r$, $f=q^r/b$ into (1.3) and noticing that $\left(q^{r+d-(d-1)n};q^d\right)_\infty=0$, $\left(q^{r-(d-1)n};q^d\right)_k=0$ for $0<(dn-n-r)/d<k$, we have

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}}\frac{1-q^{3dk+r-(d-1)n}}{1-q^{r-(d-1)n}}\frac{\left(q^{r-(d-1)n},q^r,q^{d-r};q^d\right)_k}{\left(bq^{d-(d-1)n},q^{d-(d-1)n}/b,q^{r+(d-1)n};q^d\right)_k}$$
$$\times\frac{\left(bq^r,q^r/b,q^{d-2(d-1)n};q^{2d}\right)_k q^{dk}}{\left(q^{2d},q^{2d-(d-1)n},q^{2r+d-(d-1)n};q^{2d}\right)_k}=0. \tag{3.4}$$

Therefore, the following result holds, modulo $a-q^{(d-1)n}$,

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}}\frac{1-q^{3dk+r}/a}{1-q^r/a}\frac{\left(q^r/a,q^r,q^{d-r};q^d\right)_k\left(bq^r,q^r/b,q^d/a^2;q^{2d}\right)_k q^{dk}}{\left(bq^d/a,q^d/ab,aq^r;q^d\right)_k\left(q^{2d},q^{2d}/a,q^{2r+d}/a;q^{2d}\right)_k}\equiv 0. \tag{3.5}$$

Note that $a-q^{(d-1)n}$ and $\left(1-bq^{(2d-1)n}\right)\left(b-q^{(2d-1)n}\right)$ are relatively prime polynomials. Then, based on the $r=1$ case of relations (2.6) and (2.7) and using the Chinese remainder theorem for coprime polynomials, we derive the desired congruence from (3.3) and (3.5). □

Now, with the help of Theorem 3.1, we give a proof of Theorem 1.3.

Proof of Theorem 1.3. From $\gcd(d,r)=1$ and $n\equiv -r \pmod{2d}$, we get $\gcd(n,2d)=1$. Thus, the denominator of left-hand side of (3.1) is relatively prime to $\Phi_n(q)^2\left(a-q^{(d-1)n}\right)$ if $b=1$. Meanwhile, $\left(q^{d+r}/a;q^d\right)_{(2dn-n-r)/d}$ is divisible by $a-q^{(d-1)n}$. Therefore, setting $b=1$ into (3.1), and invoking the following relation:

$$\left(a-q^{(2d-1)n}\right)\left(a-2+q^{(2d-1)n}\right)=(a-1)^2-\left(1-q^{(2d-1)n}\right)^2,$$

we obtain, modulo $\Phi_n(q)^2\left(a-q^{(d-1)n}\right)$,

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}}\frac{1-q^{3dk+r}/a}{1-q^r/a}\frac{\left(q^r/a,q^r,q^{d-r};q^d\right)_k\left(q^r,q^r,q^d/a^2;q^{2d}\right)_k q^{dk}}{\left(q^d/a,q^d/a,aq^r;q^d\right)_k\left(q^{2d},q^{2d}/a,q^{2r+d}/a;q^{2d}\right)_k}$$
$$\equiv \frac{\left(q^{d+r}/a;q^d\right)_{(2dn-n-r)/d}\left(aq^{2r},aq^d;q^{2d}\right)_{(2dn-n-r)/(2d)}}{\left(aq^r;q^d\right)_{(2dn-n-r)/d}\left(q^{2d}/a,q^{d+2r}/a;q^{2d}\right)_{(2dn-n-r)/(2d)}}. \tag{3.6}$$

Moreover, from the conditions of this theorem, we can also get $n+r\geq 2d$ and $d\geq 2$. Consequently, the smallest positive integer k such that $\left(q^{d-r};q^d\right)_k\equiv 0 \pmod{\Phi_n(q)}$ is $(n+r)/d$. On the other hand, when $d\geq 3$, the smallest positive integer k satisfying $\left(q^{d+2r};q^{2d}\right)_k\equiv 0 \pmod{\Phi_n(q)}$ is $(dn-2n+d-2r)/(2d)$ and there is $0<(n+r)/d\leq (dn-2n+d-2r)/(2d)<(2dn-n-r)/(2d)$. Thus, when $a=1$, the nominator of the right-hand side of (3.6) is surely divisible by $\Phi_n(q)^3$ and the denominator of the left-hand side of (3.6) may have the factor $\Phi_n(q)$. As a result, letting $a=1$ into (3.6), we get

$$\sum_{k=0}^{\frac{2dn-n-r}{2d}} [3dk+r] \frac{\left(q^r, q^{d-r}; q^d\right)_k \left(q^r, q^r, q^d; q^{2d}\right)_k q^{dk}}{(q^d, q^d; q^d)_k \, (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{3.7}$$

Furthermore, when $(2dn-n-r)/(2d) < k \leq n-1$, $1-q^{dn-n}$ is a factor of $\left(q^r; q^d\right)_k$ and $\left(q^r; q^{2d}\right)_k$ contains the factor $1-q^{2dn-n}$, which means that the k-th term of the left-hand side of (3.7) is still divisible by $\Phi_n(q)^2$. Then, the proof of Theorem 1.3 is completed. □

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