ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

THE TORIC RING OF ONE DIMENSIONAL SIMPLICIAL COMPLEXES

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ABSTRACT. Let Δ be a 1-dimensional simplicial complex. Then Δ may be identified with a finite simple graph G. In this article, we investigate the toric ring R_G of G. All graphs G such that R_G is a normal domain are classified. For such a graph, we determine the set $\mathscr{P}_G$ of height one monomial prime ideals of R_G. In the bipartite case, and in the case of whiskered cycles, this set is explicitly described. As a consequence, we determine the canonical class $[\omega_{R_G}]$ and characterize the Gorenstein property of R_G. For a bipartite graph G, we show that R_G is Gorenstein if and only if G is unmixed. For a subclass of non-bipartite graphs G, which includes whiskered cycles, R_G is Gorenstein if and only if G is unmixed and has an odd number of vertices. Finally, it is proved that R_G is a pseudo-Gorenstein ring if G is an odd cycle.

Introduction

Let Δ be a simplicial complex on vertex set $[n] = \{1,2,\dots,n\}$. Typically, in Commutative Algebra, one associates to Δ the Stanley–Reisner ring S/I_Δ, where $S = K[x_1,\dots,x_n]$, K is a field and I_Δ is the Stanley–Reisner ideal of Δ. The theory of Stanley–Reisner ideals has been deeply studied by many researchers. In [8], the authors introduced a different algebraic object attached to Δ, which they called the toric ring of Δ.

Let $S = K[x_1,\dots,x_n]$ be the polynomial ring with coefficients in a field K. For a face $F \in \Delta$, we set $\mathbf{x}_F = \prod_{i\in F} x_i$ if F is non-empty, otherwise we set $\mathbf{x}_\emptyset = 1$. Then, the *toric ring* of Δ is defined to be the K-subalgebra

$$R_\Delta = K[\mathbf{x}_F t : F \in \Delta]$$

of $S[t]$. This algebra is standard graded if we put $\deg(x_1^{a_1}\cdots x_n^{a_n}t^b) = b$, for all monomials $x_1^{a_1}\cdots x_n^{a_n}t^b \in R_\Delta$. This concept was further extended to multicomplexes in [7], where discrete polymatroids were mainly considered. When R_Δ is a normal domain, its divisor class group $\mathrm{Cl}(R_\Delta)$ can be explicitly described in terms of the combinatorics of the pure 1-dimensional skeleton of Δ. This skeleton may be viewed as a graph, which we denote by G_Δ. With such data, one may compute the canonical class $[\omega_{R_\Delta}]$, that is, the class of the canonical module ω_{R_Δ} in $\mathrm{Cl}(R_\Delta)$. Hence, R_Δ is Gorenstein if and only if $[\omega_{R_\Delta}] = 0$. For a Noetherian normal domain R, this is one of the most efficient ways to check the Gorenstein property of R.

Acknowledgment. This paper was written while the first and the third author visited the Faculty of Mathematics of Essen. A. Ficarra was partly supported by the Grant JDC2023-051705-I funded by MICIU/AEI/10.13039/501100011033 and by the FSE+. D.I. Stamate was partly supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P1-1.1-TE-2021-1633, within PNCDI III.
The authors are thankful to the referee for her/his careful reading of the paper and comments that greatly improved the quality of the manuscript.
2020 *Mathematics Subject Classification.* Primary 13A02; 13P10; Secondary 05E40.
Key words and phrases. toric rings, simplicial complexes, class group, canonical module.

In this article, we consider the toric ring of a 1-dimensional simplicial complex Δ. In this case, the 1-dimensional facets of Δ are the edges of G_Δ. On the other hand, given a graph G on $[n]$, we may consider the simplicial complex Δ whose facets are the edges of G. Then $G = G_\Delta$. Therefore, we write R_G instead of R_Δ. With this notation, we have $R_G = K[t, x_1t, \dots, x_nt, \{\mathbf{x}_e t\}_{e\in E(G)}]$. We always assume that G has no isolated vertices. To compute the canonical class, one has to determine the set $\mathscr{P}_G$ of height one monomial prime ideals of R_G. This is a very difficult task. On the other hand, for the class of bipartite graphs and for certain non-bipartite graphs, including whiskered cycles, we are able to determine such a set. Then, we succeed in classifying the Gorenstein algebras among these classes.

The outline of the article is as follows. In Section 1, we summarize the main results proved in [8] about the set $\mathscr{P}_\Delta$ of height one monomial prime ideals of R_Δ. When R_Δ is normal, then $\omega_{R_\Delta} = \bigcap_{P\in\mathscr{P}_\Delta} P$. Thus, in principle, one can fairly explicitly compute the canonical module and the canonical class. By the facts (iii) and (v) recalled in Page 3, $\mathscr{P}_\Delta$ always contains the following set of prime ideals $\mathscr{A}_\Delta = \{P_C : C \in \mathscr{C}(G_\Delta)\} \cup \{Q_1, \dots, Q_n\}$. For the precise definitions of the primes P_C and Q_i see Section 1. It is natural to ask when $\mathscr{P}_\Delta = \mathscr{A}_\Delta$. If R_Δ is normal, this is equivalent to the fact that Δ is a flag complex and G_Δ is a perfect graph (Theorem 1.1).

In Section 2, we consider the rings R_G. In order to apply the machinery developed in Section 1, we need to classify the graphs G such that R_G is normal. This is accomplished in Theorem 2.2. Such a result follows by noting that R_G is isomorphic to the extended Rees algebra of the edge ideal $I(G)$ of G, as shown in [3]. Then, by using results in [3, 9, 10, 11], we show that R_G is a normal domain if and only if G has at most one non-bipartite connected component and this component satisfies the so-called odd cycle condition [10]. Next, we investigate the set $\mathscr{P}_G$. It turns out that the monomial ideal $P_0 = (t, x_1t, \dots, x_nt)$ is always a prime ideal of R_G (Proposition 2.1). For a connected graph G, it is proved in Theorem 2.3 that P_0 is a non minimal prime ideal of (t) if and only if G is bipartite. These two facts are further equivalent to the property that $\mathscr{P}_G = \mathscr{A}_G$ (Theorem 2.3(d)). Thus, in the connected bipartite case we know precisely the set $\mathscr{P}_G$. Rephrasing this theorem, we obtain that P_0 is a minimal prime if and only if G is non-bipartite (Corollary 2.4).

Hence, one is led to the problem of characterizing the connected non-bipartite graphs G such that $\mathscr{P}_G = \mathscr{A}_G \cup \{P_0\}$. This problem is addressed in Theorem 3.1. For a connected graph G, we show that if $\mathscr{P}_G = \mathscr{A}_G \cup \{P_0\}$, then G must be non-bipartite and for any induced odd cycle G_0 of G, we have that any vertex in $V(G)\setminus V(G_0)$ is adjacent to some vertex of G_0. We expect that the converse of this statement holds as well. However, at present we have only partial results supporting this expectation. Therefore, we restrict our attention to unicyclic graphs. In this particular case, we obtain that $\mathscr{P}_G = \mathscr{A}_G \cup \{P_0\}$ if and only if G is a whiskered odd cycle (Theorem 3.3).

Finally, in the last section we discuss the Gorenstein property of the rings R_G. By combining some of the results from [8] a very general criterion for the Gorensteiness of R_Δ is stated (Theorem 4.3). Then, we apply this result to our rings R_G, in the case that G is bipartite or G is an odd (whiskered) cycle. Finally, we prove that R_G is pseudo-Gorenstein if G is an odd cycle (Proposition 4.7).

1. Generalities about toric rings of simplicial complexes

In the section we summarize some basic facts from [8] about toric rings of simplicial complexes. Let K be a field. Then, the *toric ring of a simplicial complex* Δ on vertex set $[n]$ is defined as the toric ring

$$R_\Delta = K[\mathbf{x}_F t : F \in \Delta] \ \subset \ K[x_1,\dots,x_n,t],$$

where we set $\mathbf{x}_F = \prod_{i\in F} x_i$, if F is nonempty, and $\mathbf{x}_\emptyset = 1$, otherwise.

We denote by G_Δ the graph on vertex set $[n]$ and whose edges are the 1-dimensional faces of Δ. For a graph G, we denote by $\mathscr{C}(G)$ the set of the minimal vertex covers of G. For a subset $C \subseteq [n]$, we set $\Delta_C = \{F \in \Delta : F \subseteq C\}$.

Let $\mathscr{P}_\Delta$ be the set of height one monomial prime ideals of R_Δ. We are interested in this set, because we have $\omega_{R_\Delta} = \bigcap_{P\in\mathscr{P}_\Delta} P$, if R_Δ is a normal ring, see [2, Theorem 6.3.5(b)]. In particular, $[\omega_{R_\Delta}] = \sum_{P\in\mathscr{P}_\Delta}[P]$ in the divisor class group $\mathrm{Cl}(R_\Delta)$ of R_Δ.

Next, we summarize what is known about the set $\mathscr{P}_\Delta$.

(i) Suppose that R_Δ is a normal domain. Let $P_1,\dots,P_r$ be the minimal monomial prime ideals of $(t) \subseteq R_\Delta$. Then the classes $[P_1],\dots,[P_r]$ generate the divisor class group $\mathrm{Cl}(R_\Delta)$ of R_Δ. Furthermore $\mathrm{Cl}(R_\Delta)$ is free of rank $r-1$ [8, Theorem 1.1 and Corollary 1.8].

(ii) Let P be a monomial prime ideal of R_Δ containing t, then the set $C = \{i : x_i t \in P\}$ is a vertex cover of G_Δ [8, Lemma 1.2].

(iii) If $C \subseteq [n]$ is a vertex cover of G_Δ, then the ideal $P_C = (\mathbf{x}_F t : F \in \Delta_C)$ is a prime ideal containing t and it is a minimal prime ideal if and only if $C \in \mathscr{C}(G_\Delta)$ [8, Theorem 1.3 and Proposition 1.4].

(iv) Not all minimal monomial prime ideals of (t) are of the form P_C for some $C \in \mathscr{C}(G_\Delta)$, see [8, Example 1.5].

(v) The set of height one monomial prime ideals of R_Δ not containing t is $\{Q_1,\dots,Q_n\}$, with $Q_i = (\mathbf{x}_F t : F \in \Delta, i \in F)$ [8, Proposition 1.9].

By (iii) and (v), the set $\mathscr{P}_\Delta$ of height one monomial prime ideals of R_Δ contains the set $\{P_C : C \in \mathscr{C}(G_\Delta)\} \cup \{Q_1,\dots,Q_n\}$. In [8, Theorem 1.10] the authors characterized those simplicial complexes such that this set coincides with $\mathscr{P}_\Delta$ and determined the canonical class $[\omega_{R_\Delta}]$ in such a case [8, Theorem 1.13].

We recall that Δ is called *flag* if all its minimal nonfaces are of dimension one. Equivalently, Δ is flag if and only if it is the clique complex of G_Δ.

Theorem 1.1. *Let Δ be a simplicial complex on $[n]$. Then, the following conditions are equivalent.*

(a) *R_Δ is a normal ring and the set of height one monomial prime ideals of R_Δ is the set*

$$\mathscr{P}_\Delta \ = \ \{P_C : C \in \mathscr{C}(G_\Delta)\} \cup \{Q_1,\dots,Q_n\}.$$

(b) *Δ is a flag complex and G_Δ is a perfect graph.*

Furthermore, if any of these equivalent conditions hold, we have

$$[\omega_{R_\Delta}] \ = \ \sum_{C\in\mathscr{C}(G)} (n+1-|C|)[P_C]. \tag{1}$$

2. The bipartite case

Let G be a graph with no isolated vertices. In this section, we consider the algebras R_G.

For a monomial $u = x_1^{a_1} \cdots x_n^{a_n} t^b \in R_G$, we set $\deg_{x_i}(u) = a_i$ for $1 \le i \le n$, and $\deg_t(u) = b$. Moreover, if $e = \{i, j\} \in E(G)$, we set $\mathbf{x}_e = x_i x_j$.

Proposition 2.1. *Let G be any graph on n vertices and let $R = R_G$. Then, the ideal $P_0 = (t, x_1t, x_2t, \dots, x_nt)$ is a monomial prime ideal of R.*

Proof. Since P_0 is a monomial ideal, it is enough to prove that for any two monomials u, v not belonging to P_0, then the product uv is also not in P_0. Since $u, v \notin P_0$ and $R = K[t, \{x_it\}_{i \in V(G)}, \{\mathbf{x}_e t\}_{e \in E(G)}]$, it follows that $uv = \prod_{k=1}^r (\mathbf{x}_{e_k} t)$ for some edges $e_1, \dots, e_r$, not necessarily distinct. Suppose by contradiction that $uv \in P_0$, then t divides uv or x_jt divides uv for some j.

In the first case, $uv = tw$ for a suitable monomial w. In particular, $\deg_t(w) = r - 1$ and $\sum_{i=1}^n \deg_{x_i}(w) = \sum_{i=1}^n \deg_{x_i}(uv) = 2r$. Since $\deg_t(w) = r - 1$, w is a product of $r - 1$ generators of R and we have $\sum_{i=1}^n \deg_{x_i}(w) \le 2(r - 1)$, absurd.

Similarly, in the second case we could write $uv = (x_jt)w$ and $\sum_{i=1}^n \deg_{x_i}(w) = \sum_{i=1}^n \deg_{x_i}(uv) - \deg_{x_j}(x_jt) = 2r - 1$. This is again impossible because w is a product of $r - 1$ generators of R and $\sum_{i=1}^n \deg_{x_i}(w)$ is at most $2(r - 1)$. □

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring. For a graph G, the *edge ideal of G* is the ideal $I(G)$ generated by all monomials $\mathbf{x}_e$ with $e \in E(G)$. Set $I = I(G)$. Recall that the *Rees algebra of I* is the K-algebra

$$S[It] = \bigoplus_{j \ge 0} I^j t^j = K[x_1, \dots, x_n, \{\mathbf{x}_e t\}_{e \in E(G)}] \subset S[t],$$

and the *associated graded ring of I* is defined as $\mathrm{gr}_I(S) = S[It]/IS[It]$.

Whereas, the *extended Rees algebra of $I(G)$* is defined as

$$S[It, t^{-1}] = S[It][t^{-1}] \subset S[t, t^{-1}].$$

We have the isomorphism $\varphi : S[It, t^{-1}] \to R_G$ established by setting $\varphi(t^{-1}) = t$, $\varphi(x_i) = x_it$ for $1 \le i \le n$, and $\varphi(\mathbf{x}_e t) = \mathbf{x}_e t$ for $e \in E(G)$, see [3, Proposition 3.1].

As a first consequence, we classify all graphs G such that R_G is a normal domain. For this purpose, we recall that a connected graph G is said to satisfy the *odd cycle condition* if for any two induced odd cycles C_1 and C_2 of G, either C_1 and C_2 have a common vertex or there exist $i \in V(C_1)$ and $j \in V(C_2)$ such that $\{i, j\} \in E(G)$.

Theorem 2.2. *Let G be any graph. Then R_G is a normal domain if and only if at most one connected component of G is non-bipartite and this connected component satisfies the odd cycle condition.*

Proof. Let $I = I(G)$. Since $R_G \cong S[It, t^{-1}]$, it follows that R_G is normal if and only if $S[It, t^{-1}]$ is normal. By [9, Proposition 2.1.2], the extended Rees algebra $S[It, t^{-1}]$ is normal if and only I is a normal ideal. By [11, Theorem 8.21], I is normal if and only if G has at most one non-bipartite connected component G_i and $I(G_i)$ is a normal ideal. By [3, Theorem 3.3], $I(G_i)$ is normal if and only $S[I(G_i)t]$ is normal if and only if the toric ring $K[I(G_i)]$ is normal. By [10, Corollary 2.3] this is the case if and only if G_i satisfies the odd cycle condition. The assertion follows. □

Next, we want to algebraically characterize the set of height one monomial prime ideals of R_G, for a connected graph G. For this aim, note that

$$\mathrm{gr}_I(S) = \frac{S[It]}{IS[It]} \cong \frac{S[It,t^{-1}]}{t^{-1}S[It,t^{-1}]} \cong \frac{R_G}{(t)R_G}, \tag{2}$$

because t^{-1} is mapped to t under the isomorphism φ. We remark that the first isomorphism holds because $S[It,t^{-1}] = \bigoplus_{j\in\mathbb{Z}} I^j t^j$ (where $I^j = S$ for $j \le 0$) and $t^{-1}S[It,t^{-1}] = \bigoplus_{j<0} I^j t^j$.

Theorem 2.3. *Let G be a connected graph with n vertices. Then, the following conditions are equivalent.*

(a) *The associated graded ring $\mathrm{gr}_{I(G)}(S)$ is reduced.*
(b) *The ideal $(t) \subset R_G$ is radical.*
(c) *G is a bipartite graph.*
(d) *The set*

$$\{P_C : C \in \mathscr{C}(G)\} \cup \{Q_1,\dots,Q_n\}$$

is the set of height one monomial prime ideals of R_G.
(e) *The ideal $P_0 = (t, x_1t,\dots,x_nt) \subset R_G$ is not a minimal prime of (t).*

If any of the above equivalent conditions hold, then R_G is a normal domain.

Proof. We prove the implications (a)$\Longleftrightarrow$(b), (a)$\Longleftrightarrow$(c) and (c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(c).

By equation (2) the equivalence (a)$\Longleftrightarrow$(b) follows. The equivalence (a)$\Longleftrightarrow$(c) is shown in [12, Proposition 14.3.39].

Now, assume (c). Since G is bipartite, it follows that G does not have odd cycles. Thus R_G is a normal domain by Theorem 2.2. In particular, G is triangle-free. Hence G is a flag complex and a perfect graph, because it is bipartite. Thus, statement (d) follows from Theorem 1.1(b)$\Rightarrow$(a). If (d) holds, then P_0 is a monomial prime ideal (Proposition 2.1), but is a not a minimal prime of (t), because P_0 is not of the form P_C for any minimal vertex cover $C \in \mathscr{C}(G)$. Statement (e) follows.

Finally, assume (e) and suppose by contradiction that G is non-bipartite. Then G has at least one induced odd cycle G_1. By Proposition 2.1, P_0 is a monomial prime ideal. By [1, Corollary 4.33], the minimal primes ideals containing (t) are monomial prime ideals. Thus, by hypothesis (e), there exists a proper subset D of $V(G)$ such that $Q = (t, \{x_it\}_{i\in D})$ is a minimal prime of (t) and $Q \subsetneq P_0$. It follows from fact (ii) at page 3 that D is a vertex cover of G. In particular, $D \cap V(G_1)$ is a vertex cover of G_1. Since G_1 is an odd cycle, D must contain two adjacent vertices $i,j \in V(G_1)$. Recall that the *distance* of two vertices $p,q \in V(G)$ is defined to be the number $d(p,q) = r$ if there exists a path from p to q of length r, that is, a sequence of $r+1$ distinct vertices $p = v_0, v_1, \dots, v_{r-1}, v_r = q$ of G such that $\{v_i, v_{i+1}\} \in E(G)$, and no shorter path from p to q exists. If no path between p and q exists, we set $d(p,q) = +\infty$.

Since G is connected and $V(G) \setminus D \neq \emptyset$, the number

$$m \;=\; \min\{d(k,i) : k \in V(G) \setminus D\}$$

exists and is finite.

Let $k \in V(G)\setminus D$ such that $d(k,i) = m$. Then, there exists a path of lenght m, $i = v_0, v_1, \dots, v_{m-1}$, $v_m = k$. By definition of m, it follows that $v_0, v_1, \dots, v_{m-1} \in D$.

If $m \geq 2$, then $\{v_{m-2}, v_{m-1}\}, \{v_{m-1}, v_m\} \in E(G)$. Now, $x_{v_{m-2}}x_{v_{m-1}}t, x_{v_m}t \notin Q$, but

$$(x_{v_{m-2}}x_{v_{m-1}}t)(x_{v_m}t) = (x_{v_{m-2}}t)(x_{v_{m-1}}x_{v_m}t) \in Q$$

because $x_{v_{m-2}}t \in Q$. This is a contradiction.

If $m = 1$, then $v_1 = k$ and $\{i, j\}, \{i, k\} \in E(G)$. We have that $x_ix_jt, x_kt \notin Q$. However, $(x_ix_jt)(x_kt) = (x_jt)(x_ix_kt) \in Q$, because $x_jt \in Q$. Again a contradiction. Therefore, G must be bipartite and (c) follows.

Finally, under the equivalent conditions (a)-(e), G is connected and bipartite. The normality of R_G follows from Theorem 2.2. □

An immediate consequence of this result is the following corollary.

Corollary 2.4. *Let G be a connected graph with n vertices. Then G is non-bipartite if and only if*

$$(t, x_1t, \ldots, x_nt) \in \mathscr{P}_G.$$

3. The non-bipartite case

By Corollary 2.4, if G is a connected non-bipartite graph on n vertices, we have the inclusion

$$\{P_C : C \in \mathscr{C}(G)\} \cup \{(t, x_1t, \ldots, x_nt)\} \cup \{Q_1, \ldots, Q_n\} \subseteq \mathscr{P}_G. \tag{3}$$

Thus, it would be interesting to characterize those connected graphs such that equality in (3) holds. As a first step, we have the following result.

Theorem 3.1. *Let G be a connected graph on n vertices such that R_G is a normal domain. Consider the following statements.*

(a) *The set*

$$\mathscr{P}_G = \{P_C : C \in \mathscr{C}(G)\} \cup \{(t, x_1t, x_2t, \ldots, x_nt)\} \cup \{Q_1, \ldots, Q_n\}$$

is the set of height one monomial prime ideals of R_G.

(b) *G is non-bipartite and for any induced odd cycle G_0 of G, we have that any vertex in $V(G) \setminus V(G_0)$ is adjacent to some vertex of G_0.*

Then, (a) *implies* (b).

To prove the theorem, we recall some basic facts about semigroups and semigroup algebras. We denote by Δ_G the simplicial complex on $[n]$ whose facets are the edges of the graph G. As is customary, we identify a monomial $x_1^{a_1} \cdots x_n^{a_n} t^b \in R_G$ with its exponent vector $(a_1, \ldots, a_n, b) \in \mathbb{Z}^{n+1}$. Thus, the monomial K-basis of R_G corresponds to the affine semigroup $S \subset \mathbb{Z}^{n+1}$ generated by the lattice points $p_F = \sum_{i \in F} e_i + e_{n+1} \in \mathbb{Z}^{n+1}$, where $F \in \Delta_G$. Here, $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{Z}^{n+1}$.

Following [2], we denote by $\mathbb{Z}S$ the smallest subgroup of $\mathbb{Z}^{n+1}$ containing S and by $\mathbb{R}_+S \subset \mathbb{R}^{n+1}$ the smallest cone containing S. In our case $\mathbb{Z}S = \mathbb{Z}^{n+1}$. Furthermore, $S = \mathbb{Z}^{n+1} \cap \mathbb{R}_+S$ if R_G is normal [2, Proposition 6.1.2].

A *hyperplane* H, defined as the set of solutions of the linear equation $f(x) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = 0$, is called a *supporting hyperplane* of the cone $\mathbb{R}_+S$ if $H \cap \mathbb{R}_+S \neq \emptyset$ and $f(\mathbf{c}) \geq 0$ for all $\mathbf{c} \in \mathbb{R}_+S$. Since any element $\mathbf{c} \in \mathbb{R}_+S$ is a linear combination with non-negative coefficients of the lattice points p_F, with $F \in \Delta_G$, it follows that H is a supporting hyperplane of $\mathbb{R}_+S$, if and only if $f(p_F) \geq 0$ for all $F \in \Delta_G$.

A subset $\mathscr{F}$ of $\mathbb{R}_+S$ is called a *face* of $\mathbb{R}_+S$, if there exists a supporting hyperplane H of $\mathbb{R}_+S$ such that $\mathscr{F} = H \cap \mathbb{R}_+S$. We may assume that the coefficients a_i appearing in $f(x) = 0$ are integers and $\gcd(a_1, \ldots, a_{n+1}) = 1$. If H is the supporting hyperplane of a facet $\mathscr{F}$, the normalized form defining H is unique and we called it the *support form* of $\mathscr{F}$.

Let $P \subset R_G$ be a monomial ideal. By [1, Propositions 2.36 and 4.33] we have that P is a monomial prime ideal if and only if there exists a face $\mathscr{F}$ of the cone $\mathbb{R}_+S$ such that $P = (\mathbf{x}_F t : F \in \Delta_G \setminus \mathscr{F})$. Equivalently, P is a monomial prime ideal, if and only if there exists a supporting hyperplane H of $\mathbb{R}_+S$ such that

$$P = (\mathbf{x}_F t : F \in \Delta_G \text{ and } f(p_F) > 0).$$

Proof of Theorem 3.1. Assume (a) holds. Then, by Corollary 2.4, G is non-bipartite. Hence, G contains at least one induced odd cycle. Suppose for a contradiction that (b) is not satisfied. Then G contains an induced odd cycle G_0 and a vertex $v_0 \in V(G) \setminus V(G_0)$ that is not adjacent to any vertex $v \in V(G_0)$. After a suitable relabeling, we may assume that $v_0 = n$.

We claim that the monomial ideal

$$Q = (t, x_1 t, \ldots, x_{n-1} t, \{x_i x_j t\}_{i \in N_G(n), j \in N_G(i) \setminus \{n\}})$$

is a prime ideal of R_G. Here for a vertex k of G, $N_G(k)$ denotes the set of vertices i such that $\{i, k\}$ is an edge of G.

Let H be the hyperplane defined by the equation $f(x) = 0$ where

$$f(x) = -\sum_{i \notin N_G(n)} x_i - 2x_n + 2x_{n+1}.$$

Let $F \in \Delta_G$. We claim that $f(p_F) > 0$ if $\mathbf{x}_F t \in Q$, and $f(p_F) = 0$ if $\mathbf{x}_F t \notin Q$. This shows that H is a supporting hyperplane of $\mathbb{R}_+S$ where S is the affine semigroup generated by the lattice points p_F, $F \in \Delta_G$, and that $Q = (\mathbf{x}_F t : F \in \Delta_G, f(p_F) > 0)$ is a monomial prime ideal.

If $F = \emptyset$, then $f(p_\emptyset) = 2$. Suppose $F = \{i\}$. If $i < n$, then

$$f(p_{\{i\}}) = \begin{cases} 2 & \text{if } i \in N_G(n), \\ 1 & \text{if } i \notin N_G(n). \end{cases}$$

If $F = \{n\}$, then $f(p_{\{n\}}) = 0$.

Finally, assume $F = \{i, j\} \in E(G)$. If $i = n$, then $j \in N_G(n)$ and $f(p_{\{i,j\}}) = 0$ in this case. Suppose both i and j are different from n. Then,

$$f(p_{\{i,j\}}) = \begin{cases} 2 & \text{if } i, j \in N_G(n), \\ 1 & \text{if } i \in N_G(n), j \notin N_G(n) \text{ or } i \notin N_G(n), j \in N_G(n), \\ 0 & \text{if } i, j \notin N_G(n). \end{cases}$$

Therefore, Q is a prime ideal of R_G containing t. Thus, there exists a minimal monomial prime ideal P such that $(t) \subset P \subseteq Q$. Hence, P is generated by a subset of the generators of Q and contains t. We claim that P is different from P_C, for all $C \in \mathscr{C}(G)$, and different from $(t, x_1 t, \ldots, x_n t)$. This contradicts (a) and shows that (b) holds.

It is clear that P is different from $(t, x_1 t, \ldots, x_n t)$ because $x_n t \notin P$. Now, let $C \in \mathscr{C}(G)$, then $D = C \cap V(G_0)$ is a vertex cover of G_0. Since G_0 is an odd cycle, D must contain two adjacent vertices $i, j \in V(G_0)$. Thus, $x_i x_j t \in P_C$. Since n is not adjacent to any vertex $v \in V(G_0)$, we have that $i,j \notin N_G(n)$. Hence $x_ix_jt \notin Q$ and $x_ix_jt \notin P$, also. Thus, P is different from P_C, for all $C \in \mathscr{C}(G)$, as wanted. □

Due to experimental evidence, we expect that statements (a) and (b) of Theorem 3.1 are indeed equivalent.

Recall that a graph G is called *unicyclic* if G is connected and contains exactly one induced cycle. Note that a unicyclic graph G satisfies the odd cycle condition, and so R_G is a normal domain. Next, we characterize those unicyclic graphs such that equality holds in (3). It turns out that for this class of graphs, the statements (a) and (b) of Theorem 3.1 are equivalent.

For this aim, we introduce the concept of *whiskered cycles*. Hereafter, for convenience and with abuse of notation, we identify the vertices of G with the variables of R_G. Let $k \geq 3$ and $a_1, a_2, \dots, a_k \geq 0$ be non-negative integers. The *whiskered cycle of type* $(a_1, \dots, a_k)$ is the graph $G = C(a_1, \dots, a_k)$ on vertex set

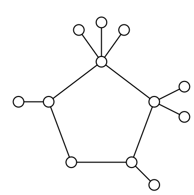
$$V(G) = \{x_1, \dots, x_k\} \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{a_i} \{x_{i,j}\},$$

and with edge set

$$E(G) = \{\{x_1,x_2\},\{x_2,x_3\},\dots,\{x_{k-1},x_k\},\{x_k,x_1\}\} \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{a_i} \{\{x_i, x_{i,j}\}\}.$$

If k is even (odd), G is called a whiskered even (odd) cycle. The vertices $x_{i,j}$ are called the *whiskers* of x_i.

For example, the whiskered cycle $C(3,2,1,0,1)$ is depicted below



The next elementary lemma is required.

Lemma 3.2. *Let $G = C(a_1, \dots, a_k)$ be a whiskered cycle and $C \in \mathscr{C}(G)$ a minimal vertex cover. If $a_i > 0$ for some i, then either $x_i \in C$ or $x_{i,j} \in C$ for all $j = 1, \dots, a_i$.*

Proof. Let $a_i > 0$. Then x_i has at least one whisker. Since C is a minimal vertex cover of G, we must have $C \cap \{x_i, x_{i,j}\} \neq \emptyset$ for all $j = 1, \dots, a_i$. Suppose $x_i \in C$, then $x_{i,j} \notin C$ for all $j = 1, \dots, a_i$, by the minimality of C. Otherwise, if $x_i \notin C$, then $x_{i,j} \in C$ for all $j = 1, \dots, a_i$, because C is a vertex cover of G. □

Hereafter, we regard the set $[0]$ as the empty set.

Let $G = C(a_1, \dots, a_k)$ be a whiskered cycle. Let $j \geq 3$ be a positive integer and let $x_i, x_{i+1}, \dots, x_{i+j}$ be $j+1$ adjacent vertices of the unique induced cycle of G. Here, if $i+p$ exceeds k, for some $1 \leq p \leq j$, we take the remainder modulo k. Then, the *whisker interval* $W(i, i+j)$ is defined as

$$\begin{aligned} W(i,i+j) &= \{x_i, x_{i+1}, \dots, x_{i+j}\} \cup \bigcup_{\ell=i+1}^{i+j-1} \bigcup_{h=1}^{a_\ell} \{x_{\ell,h}\} \\ &= \{x_i, x_{i+1}, \dots, x_{i+j}\} \cup \{\text{whiskers of } x_{i+1}, \dots, x_{i+j-1}\}. \end{aligned}$$

We say that $W(i,i+j)$ is *proper* if $\{x_1,x_2,\dots,x_k\} \not\subseteq W(i,i+j)$.

Note that, if $i_1 \le i_2 \le i_1+j_1-1$ and $i_1+j_1 \le i_2+j_2$, then

$$W(i_1,i_1+j_1)\cup W(i_2,i_2+j_2) = W(i_1,i_2+j_2).$$

We say that $W(i_1,i_1+j_1)$ and $W(i_2,i_2+j_2)$ are *whisker-disjoint*, if

$$|W(i_1,i_1+j_1)\cap W(i_2,i_2+j_2)| \le 1,$$

that is $W(i_1,i_1+j_1)$ and $W(i_2,i_2+j_2)$ intersect at most in one vertex.

It is clear that for any collection of proper whisker intervals $W_1,\dots,W_r$ there exist whisker-disjoint whisker intervals $V_1,\dots,V_t$ such that $W_1\cup\cdots\cup W_r = V_1\cup\cdots\cup V_t$.

Now, we are in the position to state and prove the announced classification.

Theorem 3.3. *Let G be a unicyclic graph on n vertices. Then, the following conditions are equivalent.*

(a) *The set*

$$\mathscr{P}_G = \{P_C : C\in\mathscr{C}(G)\}\cup\{(t,x_1t,x_2t,\dots,x_nt)\}\cup\{Q_1,\dots,Q_n\}$$

is the set of height one monomial prime ideals of R_G.

(b) *G is a whiskered odd cycle.*

Proof. Since G is unicyclic, it follows from Theorem 2.2 that R_G is normal.

The implication (a)⇒(b) follows immediately from Theorem 3.1.

(b)⇒(a). Suppose G is a whiskered odd cycle. Then $G = C(a_1,\dots,a_k)$ for some odd $k\ge 3$ and some non-negative integers $a_1,a_2,\dots,a_k$. Let G_0 be the induced graph of G on vertex set $x_1,\dots,x_k$. Then G_0 is an odd cycle

Let $P\subset R_G$ be a monomial prime ideal containing t and such that $P\not\supseteq P_C$ for all vertex covers C of G. Set $P_0=(t,x_it,x_{i,j}t : i\in[k], j\in[a_i])$. We claim that

$$P_0\subseteq P.$$

The set $D=\{x_i : x_it\in P\}\cup\{x_{i,j} : x_{i,j}t\in P\}$ is a vertex cover of G. We are going to prove that $D=V(G)$. From this, it will follow that $P_0\subseteq P$.

Since D is a vertex cover, there exists a minimal vertex cover C contained in D. By Lemma 3.2, the only adjacent vertices of C can be the vertices of the cycle G_0. In particular, $C_0 = C\cap V(G_0)$ is a (possibly non minimal) vertex cover of G_0.

Since G_0 is an odd cycle, C_0 must contain at least a pair of adjacent vertices x_i,x_j of G_0. Suppose that for all such adjacent vertices $x_i,x_j\in C_0$ we have $x_ix_jt\in P$. Then, P_C would be contained in P, because by Lemma 3.2 the only adjacent vertices of C can be the x_i. But this is against our assumption. Therefore, there exist two adjacent vertices $x_i,x_j\in C$ for which $x_ix_jt\notin P$. Up to relabeling, we may assume $i=2$ and $j=3$. We claim that x_1 and all the whiskers of x_2 and x_3 belong to D.

Suppose that $x_1\notin D$. Then $x_1t\notin P$. Since also $x_2x_3t\notin P$, the product $(x_1t)(x_2x_3t)$ should not be in P. However, $(x_1t)(x_2x_3t)=(x_1x_2t)(x_3t)\in P$, which is a contradiction. Therefore, $x_1\in D$. Similarly, suppose that $x_{2,j}\notin D$ for some j. Then $x_{2,j}t\notin P$. Since also $x_2x_3t\notin P$, the product $(x_{2,j}t)(x_2x_3t)$ should not be in P. However, $(x_{2,j}t)(x_2x_3t)=(x_2x_{2,j}t)(x_3t)\in P$, a contradiction. Therefore, $x_{2,j}\in D$. Similarly $x_{3,\ell}\in D$ and our claim follows. We distinguish two cases now.

CASE 1. Suppose $k=3$. By the previous discussion, $x_1,x_2,x_3,x_{2,j},x_{3,\ell}\in D$, for all $j\in[a_2]$

and $\ell \in [a_3]$. It remains to prove that the whiskers of x_1 belong to D. Indeed, the vertex cover $C_1 = \{x_1, x_2, \text{whiskers of } x_3\}$ is contained in D. Since $P_{C_1} \not\subseteq P$, we must have $x_1x_2t \notin P$. By the argument used before, we obtain that all whiskers of x_1 belong to D. Hence, $D = V(G)$ and so P contains P_0, as wanted.

CASE 2. Suppose $k > 3$. By the argument above, we have also that $x_4 \in D$. Hence,

$$W(1,4) = \{x_1,x_2,x_3,x_4\} \cup \bigcup_{i=2,3} \bigcup_{h \in [a_i]} \{x_{i,h}\} \subseteq D.$$

Now, we recursively determine vertex covers $C_i \subset D$ in order to obtain each time new whisker intervals that belong to D, and in the end to have that $D = V(G)$.

Let

$$C_1 = \big(C \setminus \{x_2, \text{whiskers of } x_1 \text{ and } x_4\}\big) \cup \{x_1, x_4\} \cup \{\text{whiskers of } x_2\}.$$

It is clear that C_1 is a cover of G. Since, by assumption, P does not contain P_{C_1}, it follows that P does not contain x_ix_jt, for some adjacent vertices $x_i, x_j \in C_1$. Since $x_2 \notin C_1$, it follows that x_ix_jt is different from x_2x_3t. Thus, $j = i-1$ and $i \in \{4, \dots, k\}$ or $j = 1$ and $i = k$. Let p and q be the adjacent vertices of $i-1$ and i, different from i and $i-1$. Then $p = i-2$ and $q = i+1$. Here we take the remainder modulo k, if these numbers exceed k. Arguing as before,

$$W(i-2, i+1) \subseteq D.$$

After repeating this argument as many times as possible, if $D = V(G)$ then we are finished. Otherwise, at a given step of this procedure, we have that there exist integers $i_1, j_1, \dots, i_r, j_r$, with $j_1, \dots, j_r \geq 3$ such that

$$W(i_1, i_1+j_1) \cup W(i_2, i_2+j_2) \cup \cdots \cup W(i_r, i_r+j_r) \subseteq D, \tag{4}$$

and these whisker intervals are proper and mutually whisker-disjoint.

Now, starting from the vertex cover C, we construct another vertex cover C' of G contained in D, having the following properties:

(i) The only adjacent vertices of C' belong to the cycle G_0.

(ii) if $x_i, x_j \in C'$ are adjacent vertices that belong to a whisker interval above, say $W(i_a, i_a + j_a)$, then either $\{i,j\} = \{i_a, i_a+1\}$ or $\{i,j\} = \{i_a + j_a - 1, i_a + j_a\}$.

The vertex cover C' having the properties (i) and (ii) is constructed as follows. Let $W(i, i+j)$ be a whisker interval in (4). We distinguish two cases: j even, say $j = 2\ell$, and j odd, say $j = 2\ell+1$.

If $j = 2\ell$, we add to C the vertices

$$x_i, x_{i+1}, x_{i+3}, x_{i+5}, \dots, x_{i+2\ell-3}, x_{i+2\ell-1}, x_{i+2\ell}$$

and remove all the corresponding whiskers, and moreover, we remove the vertices

$$x_{i+2}, x_{i+4}, \dots, x_{i+2\ell-2}$$

and add all the corresponding whiskers. We call C' the resulting set.

Whereas, if $j = 2\ell+1$, we add to C the vertices

$$x_i, x_{i+1}, x_{i+3}, x_{i+5}, \dots, x_{i+2\ell-3}, x_{i+2\ell-1}, x_{i+2\ell+1}$$

and remove all the corresponding whiskers, and moreover, we remove the vertices

$$x_{i+2}, x_{i+4}, \dots, x_{i+2\ell-2}, x_{i+2\ell}$$

and add all the corresponding whiskers. We call C' the resulting set.

When we have more than one whisker interval, we repeat the operations above for all whisker intervals, and call C' the set obtained in this way. Such a set is well defined, because our whisker intervals are proper and mutually whisker-disjoint. It is clear that C' is a vertex cover of G satisfying the properties (i) and (ii).

Now, we argue as follows. Since P does not contain $P_{C'}$ by assumption, and since G_0 is an odd cycle, by (i) there exists two adjacent vertices $x_i, x_{i+1} \in C'$ such that $x_i x_{i+1} t \notin P$. If $\{i, i+1\} \subseteq W(i_a, i_a + j_a)$, by (ii) either $\{i, i+1\} = \{i_a, i_a + 1\}$ or $\{i, i+1\} = \{i_a + j_a - 1, i_a + j_a\}$. Say, $\{i, i+1\} = \{i_a, i_a + 1\}$, then arguing as before, we have that

$$W(i-1, i+2) \subseteq D.$$

Otherwise, if $\{i, i+1\}$ is not contained in any of the whisker intervals constructed up to this point, then $W(i-1, i+2) \subseteq D$. In both cases, we can enlarge the set of the whisker intervals contained in D. Therefore, after a finite number of steps, we obtain either $D = V(G)$ or a non-proper whisker interval is contained in D. In this latter case, up to relabeling we may assume that $W(1, k) \subseteq D$. So, we only need to argue that the whiskers of x_1 and x_k are in D.

Since $W(1, k) \subseteq D$, the vertex cover

$$C_2 = \{x_1, x_k\} \cup \{x_3, x_5, \dots, x_{k-2}\} \cup \{\text{whiskers of } x_2, x_4, \dots, x_{k-1}\}$$

is contained in D. Since P does not contain P_{C_2}, we must have that $x_1 x_k t \notin P$. By the similar argument used before, $W(k-1, 2) \subseteq D$. Therefore $D = V(G)$.

Since $D = V(G)$, it follows that $P_0 \subseteq P$. Therefore, any minimal monomial prime ideal P of (t) different from P_C for all $C \in \mathscr{C}(G)$, must contain P_0. Thus $P = P_0$ by Corollary 2.4. Hence, the set of height one monomial prime ideals containing t is given by $\{P_C : C \in \mathscr{C}(G)\} \cup \{P_0\}$ and (a) follows. ☐

4. The Gorenstein property

In this last section, we discuss the Gorenstein property for the toric ring of a simplicial complex Δ. Summarizing some of the results of [8], we have the following

Lemma 4.1. *Assume that R_Δ is normal and let $P_1, \dots, P_r$ be the height one monomial prime ideals containing t and $Q_1, \dots, Q_n$ the height one monomial prime ideals not containing t. Furthermore, let*

$$f_i = \sum_{j=1}^{n+1} c_{i,j} x_j$$

be the support forms associated to P_i, $i = 1, \dots, r$. Then,

(a) $\mathrm{Cl}(R_\Delta)$ *is generated by* $[P_1], \dots, [P_r]$ *with unique relation* $\sum_{i=1}^{r} c_{i,n+1} [P_i] = 0$.
(b) *For all* $j = 1, \dots, n$, $[Q_j] = -\sum_{i=1}^{r} c_{i,j} [P_i]$.
(c) $[\omega_{R_\Delta}] = \sum_{i=1}^{r} [P_i] + \sum_{j=1}^{n} [Q_j]$.

Substituting the expressions for $[Q_j]$ given in (b) into the formula for $[\omega_{R_\Delta}]$ given in (c), we obtain

$$[\omega_{R_\Delta}] \; = \; \sum_{i=1}^{r} [P_i] - \sum_{j=1}^{n} \sum_{i=1}^{r} c_{i,j} [P_i] \; = \; \sum_{i=1}^{r} \big(1 - \sum_{j=1}^{n} c_{i,j}\big) [P_i].$$

Hence, we have proved that

Corollary 4.2. $[\omega_{R_\Delta}] = \sum_{i=1}^{r}(1-\sum_{j=1}^{n} c_{i,j})[P_i]$.

Theorem 4.3. *The following conditions are equivalent*

(a) R_Δ *is Gorenstein.*
(b) *There exists an integer a such that* $1-\sum_{j=1}^{n} c_{i,j} = ac_{i,n+1}$ *for all* $i=1,\dots,r$.

Proof. Observe that R_Δ is Gorenstein if and only if $[\omega_{R_\Delta}]=0$. By Lemma 4.1(a) and Corollary 4.2, this is the case if and only if there exists an integer a such that $1-\sum_{j=1}^{n} c_{i,j} = ac_{i,n+1}$ for all $i=1,\dots,r$. □

Now, we will apply Theorem 4.3 to the algebras R_G which we discussed before.
In the bipartite case, we recover the next result from [3, Corollary 4.3].

Proposition 4.4. *Let G be a connected bipartite graph on n vertices. Then R_G is Gorenstein if and only if G is unmixed.*

Proof. By Theorem 2.3(d), R_G is normal and the height one monomial prime ideals containing t are of the form P_C, $C\in\mathscr{C}(G)$. In the proof of [8, Theorem 1.3], it is shown that the support form associated to P_C is

$$f_C(x) = -\sum_{i\notin C} x_i + x_{n+1}. \tag{5}$$

Let $\mathscr{C}(G)=\{C_1,\dots,C_r\}$ and $P_i=P_{C_i}$. Then, by Theorem 4.3 and formula (5) it follows that R_G is Gorenstein if and only if there exists an integer a such that $1+(n-|C_i|)=a$ for all $i=1,\dots,r$. This yields the conclusion. □

Next, we consider non-bipartite graphs.

Proposition 4.5. *Let G be a connected non-bipartite graph with n vertices satisfying the odd cycle condition. Let $\mathscr{C}(G)=\{C_1,\dots,C_r\}$, $P_i=P_{C_i}$, for $i=1,\dots,r$, and $P_0=(t,x_1t,x_2t,\dots,x_nt)$. Assume that the set of height one monomial prime ideals containing t is $\{P_0,P_1,\dots,P_r\}$. Then*

(a) $[\omega_{R_G}] = (1+n)[P_0]+\sum_{i=1}^{r}(1+n-|C_i|)[P_i]$.
(b) R_G *is Gorenstein if and only if n is odd and G is unmixed.*

Proof. One can easily see that the support form of P_0 is $f_0(x) = -\sum_{i=1}^{n} x_i + 2x_{n+1}$. Part (a) follows from Corollary 4.2. By using the support forms f_0 and f_{C_i}, it follows from Theorem 4.3(b) that R_G is Gorenstein if and only if there exists an integer a such that $1+n=2a$ and $1+n-|C_i|=a$ for all i. This implies that R_G is Gorenstein if and only if n is odd and G is unmixed. □

Finally, we consider the case in which G is a k-cycle, which we denote by C_k.

Corollary 4.6. R_{C_k} *is Gorenstein if and only if* $k\in\{3,4,5,7\}$.

Proof. By Theorem 2.2, R_{C_k} is normal. We claim that R_{C_k} is Gorenstein if and only C_k is unmixed. If k is even, C_k is bipartite and the claim follows from Proposition 4.4. If k is odd, the claim follows from Theorem 3.3 and Proposition 4.5.

It can be easily seen that C_k is unmixed if $k\in\{3,4,5,7\}$. Otherwise, if $k=6$ or $k>7$ then C_k is not unmixed, as we show next.

Let $k>7$ odd, say $k=2\ell+1$. Then, $\ell\geq 4$ and

$$\{1,2,4,6,8,10,\dots,2\ell-2,2\ell\},\quad \{1,2,4,5,7,8,10,\dots,2\ell-2,2\ell\}$$

are minimal vertex covers of C_k of size $\ell+1$ and $\ell+2$.

Let $k\geq 6$ even, say $k=2\ell$. If $k=6$, then $\{1,3,5\}$ and $\{1,2,4,5\}$ are minimal vertex covers of C_6 of different size. Suppose $\ell\geq 4$, then

$$\{1,3,5,7,9,\dots,2\ell-3,2\ell-1\},\quad \{1,2,4,5,7,9,\dots,2\ell-3,2\ell-1\}$$

are minimal vertex covers of C_k of size ℓ and $\ell+1$. □

Let R be a standard graded Cohen–Macaulay K-algebra with canonical module ω_R. Following [4], we say that R is *pseudo-Gorenstein* if $\dim_K(\omega_R)_a=1$, where $a=\min\{i:(\omega_R)_i\neq 0\}$.

Let G be a graph such that R_G is a normal domain. By a theorem of Hochster, R_G is a Cohen–Macaulay K-algebra. Furthermore, R_G is standard graded with the grading given by $\deg(x_1^{a_1}\cdots x_n^{a_n}t^b)=b$, for all monomials $x_1^{a_1}\cdots x_n^{a_n}t^b\in R_G$.

Proposition 4.7. *Let G be an odd cycle. Then R_G is pseudo-Gorenstein.*

Proof. Let k be the number of vertices of G. Then $k=2\ell+1$ for some $\ell\geq 1$. Set $P_0=(t,x_1t,\dots,x_kt)$. By Theorem 3.3, the set of height one monomial prime ideals of R_G is given by $\{P_C : C\in\mathscr{C}(G)\}\cup\{P_0,Q_1,\dots,Q_k\}$, and moreover

$$\omega_{R_G}=\Big(\bigcap_{C\in\mathscr{C}(G)}P_C\Big)\cap P_0\cap Q_1\cap\cdots\cap Q_k.$$

By [1, Corollary 4.33], ω_{R_G} and $\bigcap_{i=1}^k Q_i$ are monomial ideals. Let $u\in\bigcap_{i=1}^k Q_i$ be a monomial. Note that for each i, the monomial generators of Q_i have multidegree $\geq e_i+e_{k+1}$. Hence, the multidegree of u is $\geq e_1+\cdots+e_k+e_{k+1}$. Thus, $u=u_1u_2\cdots u_b=x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}t^b$, where $u_1,u_2,\dots,u_b$ are b, not necessarily distinct, generators of R_G, and $a_1,a_2,\dots,a_k\geq 1$. Note that

$$k\leq\sum_{i=1}^k\deg_{x_i}(u)=\sum_{i=1}^k\sum_{j=1}^b\deg_{x_i}(u_j)=\sum_{j=1}^b\sum_{i=1}^k\deg_{x_i}(u_j)\leq 2b. \tag{6}$$

Thus $2b\geq k$. Hence $b\geq\ell+1$ and the initial degree of $\bigcap_{i=1}^k Q_i$ is $\ell+1$.

We claim that the only monomials of degree $\ell+1$ belonging to $\bigcap_{i=1}^k Q_i$ are

$$w_0=(x_1x_2\cdots x_k)t^{\ell+1},\quad w_i=(x_1\cdots x_{i-1}x_i^2x_{i+1}\cdots x_k)t^{\ell+1},\quad i=1,\dots,k. \tag{7}$$

Indeed, for all $j=1,\dots,k$, we can write

$$w_0=(x_jt)(x_{j+1}x_{j+2}t)\cdots(x_{j+k-2}x_{j+k-1}t)\in Q_j, \tag{8}$$

where $j+p$ is understood to be q, where $j+p\equiv q$ modulo k and $1\leq q\leq k$. Thus $w_0\in\bigcap_{i=1}^k Q_i$. Similarly, we can write

$$w_i=(x_{i-1}x_it)(x_ix_{i+1}t)(x_{i+2}x_{i+3}t)\cdots(x_{i+2(\ell-1)}x_{i+2(\ell-1)+1}t)$$

with the same convention as before for the indices. Hence, we see that $w_i\in Q_j$ for all j, because $j=i+p$, for some $-1\leq p\leq 2\ell-1$, and $x_{j-1}x_jt, x_jx_{j+1}t\in Q_j$.

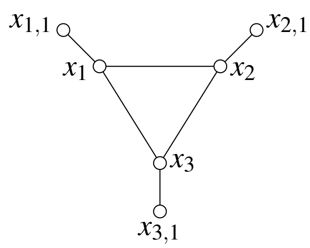
Conversely, let $u=u_1u_2\cdots u_{\ell+1}\in\bigcap_{i=1}^k Q_i$ where $u_1,u_2,\dots,u_{\ell+1}$ are $\ell+1$ generators of R_G. Note that at most one of the u_i can be of the form x_jt and the remaining monomials u_p are of the form x_ix_jt, otherwise $\sum_{i=1}^k\deg_{x_i}(u)<k$, contradicting (6). Therefore, we have $\sum_{i=1}^k\deg_{x_i}(u)\in\{2\ell+1,2\ell+2\}=\{k,k+1\}$. Since we must have $\deg_{x_i}(u)\geq 1$ for all $i=1,\dots,k$, we see that the only monomials of degree $\ell+1$ belonging to $\bigcap_{i=1}^k Q_i$ are those listed in (7).

Next, we show that $w_0 \in P_0 \cap (\bigcap_{C\in\mathscr{C}(G)} P_C)$ and $w_i \notin P_0$ for all $i = 1,\dots,k$. Indeed, let $C \in \mathscr{C}(G)$, then $x_j \in C$ for some j. Thus $x_j t \in P_C$ and by (8) it follows that $w_0 \in P_C$, as well. This same argument shows that $w_0 \in P_0$, and so $w_0 \in P_0 \cap (\bigcap_{C\in\mathscr{C}(G)} P_C)$.

Now let $i \in \{1,\dots,k\}$. For any factorization $w_i = v_1v_2\cdots v_{\ell+1}$ of w_i into a product of generators $v_p \in R_G$, we have $\sum_{j=1}^{k} \deg_{x_j}(v_p) = 2$ for all p. This shows that $w_i \notin P_0$.

Therefore, the only monomial of degree $\ell+1$ belonging to ω_{R_G} is w_0. Since ω_{R_G} is a monomial ideal, its initial degree is larger or equal to the initial degree of $\bigcap_{i=1}^{k} Q_i$. Hence, $\min\{i : (\omega_{R_G})_i \neq 0\} = \ell+1$ and $\dim_K(\omega_{R_G})_{\ell+1} = 1$, that is, R_G is pseudo-Gorenstein. □

Example 4.8. Let $G = C(1,1,1)$ be the whiskered triangle depicted below.



Note that G is unmixed, but it has an even number of vertices. Thus, by Proposition 4.5 it follows that R_G is not Gorenstein. Indeed, by using *Macaulay2* [5], we checked that the canonical module of R_G is

$$\omega_{R_G} = (x_1x_2x_3x_{1,1}x_{2,1}x_{3,1}t^4, x_1^2x_2^2x_3^2x_{1,1}x_{2,1}x_{3,1}t^5).$$

On the other hand, R_G is pseudo-Gorenstein. In general however the algebra R_G of a whisker cycle G need not to be pseudo-Gorenstein. The algebra $R_{C(1,1,2)}$ gives such an example.

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