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TRANSLATION RESULTS FOR SOME SELECTION GAMES WITH MINIMAL CUSCO MAPS

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ABSTRACT. We establish relationships between various topological selection games involving the space of minimal cusco maps into the real line and the underlying domain. These connections occur across different topologies, including the topology of pointwise convergence and the topology of uniform convergence on compacta. Full and limited-information strategies are investigated. The primary games we consider are Rothberger-like games, generalized point-open games, strong fan-tightness games, Tkachuk's closed discrete selection game, and Gruenhage's W-games. We also comment on the difficulty of generalizing the given results to other classes of functions.

1. Introduction

Minimal upper-semicontinuous compact-valued functions have a rich history. The topic can be traced back to the study of holomorphic functions and cluster sets, see [13]. The phrase *minimal usco* was coined by Christensen [10], where a topological game similar to the Banach-Mazur game was considered. When the codomain is a linear space, the term *cusco map* refers to usco maps which are convex-valued. Usco and cusco maps have been objects of study since they provide insights into the underlying topological properties of the convex subdifferential and the Clarke generalized gradient [2]. In this paper, using some techniques similar to those of Holá and Holý [29, 30], we tie connections between a space X and the space of minimal cusco maps with the topology of uniform convergence on certain kinds of subspaces of X. The results are analogous to those of of [3] and similar in spirit to those appearing in [12, 4, 5]; in particular, most of the results come in the form of selection game equivalences or dualities, which rely on a variety of game-related results from [11, 56, 55, 57, 4, 5]. In many contexts of interest, we see that, pertaining to the properties investigated herein, the spaces of minimal usco maps, minimal cusco maps, and continuous real-valued functions behave similarly.

Consequences of these results include Corollary 50, which captures [30, Cor. 4.5]: a space X is hemicompact if and only if $\mathrm{MC}_k(X)$, the space of minimal cusco maps into $\mathbb{R}$ on X with the topology of uniform convergence on compact subsets, is metrizable. Corollary 50 also shows that X is hemicompact if and only if $\mathrm{MC}_k(X)$ is not discretely selective. Corollary 55 contains the assertion that X is k-Rothberger if and only if $\mathrm{MC}_k(X)$ has strong countable fan-tightness at $\mathbf{0}$, the constant $\{0\}$ function.

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2. Preliminaries

We use the word *space* to mean *topological space*. Any undefined notions and terminologies are as in [17] or [34]. Unless otherwise stated, all spaces considered are assumed to be Hausdorff. When the parent space is understood from context, we use the notation $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ for the interior and closure of A, respectively. If we must specify the topological space X, we use $\mathrm{int}_X(A)$ and $\mathrm{cl}_X(A)$.

Given a function $f : X \to Y$, we denote the graph of f by $\mathrm{gr}(f) = \{\langle x, f(x)\rangle : x \in X\}$. For a set X, we let $\wp(X)$ denote the set of subsets of X and $\wp^+(X) = \wp(X) \setminus \{\emptyset\}$. For sets X and Y, we let

$$\mathrm{Fn}(X,Y) = \bigcup_{A \in \wp^+(X)} Y^A;$$

that is, $\mathrm{Fn}(X,Y)$ is the collection of all Y-valued functions defined on non-empty subsets of X.

When a set X is implicitly serving as the parent space in context, given $A \subseteq X$, we will let $\mathbf{1}_A$ be the indicator function for A. That is, $\mathbf{1}_A : X \to \{0,1\}$ is defined by the rule

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

For any set X, we let $X^{<\omega}$ denote the set of finite sequences of X and $[X]^{<\omega}$ denote the set of finite subsets of X.

For a space X, we let $K(X)$ denote the set of all non-empty compact subsets of X. We let $\mathbb{K}(X)$ denote the set $K(X)$ endowed with the Vietoris topology; that is, the topology with basis consisting of sets of the form

$$[U_1, U_2, \ldots, U_n] = \left\{ K \in \mathbb{K}(X) : K \subseteq \bigcup_{j=1}^{n} U_j \wedge K^n \cap \prod_{j=1}^{n} U_j \neq \emptyset \right\}.$$

For more about this topology, see [43].

A family $\mathscr{A}$ of subsets of a set X is a *bornology* [25] if $X = \bigcup \mathscr{A}$, $A \cup B \in \mathscr{A}$ for all $A, B \in \mathscr{A}$, and, for each $A \in \mathscr{A}$, $B \subseteq A \implies B \in \mathscr{A}$. We will be interested in certain kinds of bases for bornologies, which we refer to as ideals of closed sets since the conditions for being a bornology are similar to those of ideals.

Definition 1. For a space X, we say that a family $\mathscr{A} \subseteq \wp^+(X)$ of closed sets is an *ideal of closed sets* if

- for $A, B \in \mathscr{A}$, $A \cup B \in \mathscr{A}$;
- for $A \in \mathscr{B}$, if $B \subseteq A$ is closed, then $B \in \mathscr{A}$; and
- for every $x \in X$, $\{x\} \in \mathscr{A}$.

Throughout, we will assume that any ideal of closed sets under consideration doesn't contain the entire space X. Two ideals of closed sets of primary interest are

- the collection of non-empty finite subsets of an infinite space X and
- the collection of non-empty compact subsets of a non-compact space X.

2.1. *Selection Games.* Topological games have a long history, much of which can be gathered from Telgársky's survey [54]. In this paper, we will be dealing only with single-selection games of countable length.

Definition 2. Given sets $\mathscr{A}$ and $\mathscr{B}$, we define the *single-selection game* $\mathsf{G}_1(\mathscr{A},\mathscr{B})$ as follows.

- For each $n \in \omega$, One chooses $A_n \in \mathscr{A}$ and Two responds with $x_n \in A_n$.
- Two is declared the winner if $\{x_n : n \in \omega\} \in \mathscr{B}$. Otherwise, One wins.

The study of games naturally inspires questions about the existence of various kinds of strategies. Infinite games and corresponding full-information strategies were both introduced in [19]. Some forms of limited-information strategies came shortly after, like positional (also known as stationary) strategies [15, 51]. For more on stationary and Markov strategies, see [21].

Definition 3. We define strategies of various strength below.

- We use two forms of *full-information* strategies.
  - A *strategy for player One* in $\mathsf{G}_1(\mathscr{A},\mathscr{B})$ is a function $\sigma : (\bigcup \mathscr{A})^{<\omega} \to \mathscr{A}$. A strategy σ for One is called *winning* if whenever $x_n \in \sigma\langle x_k : k < n\rangle$ for all $n \in \omega$, $\{x_n : n \in \omega\} \notin \mathscr{B}$. If player One has a winning strategy, we write $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{A},\mathscr{B})$.
  - A *strategy for player Two* in $\mathsf{G}_1(\mathscr{A},\mathscr{B})$ is a function $\tau : \mathscr{A}^{<\omega} \to \bigcup \mathscr{A}$. A strategy τ for Two is *winning* if whenever $A_n \in \mathscr{A}$ for all $n \in \omega$, $\{\tau(A_0,\ldots,A_n) : n \in \omega\} \in \mathscr{B}$. If player Two has a winning strategy, we write $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{A},\mathscr{B})$.
- We use two forms of *limited-information* strategies.
  - A *predetermined strategy* for One is a strategy which only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two's moves. Formally it is a function $\sigma : \omega \to \mathscr{A}$. If One has a winning predetermined strategy, we write $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{A},\mathscr{B})$.
  - A *Markov strategy* for Two is a strategy which only considers the most recent move of player One and the current turn number. Formally it is a function $\tau : \mathscr{A} \times \omega \to \bigcup \mathscr{A}$. If Two has a winning Markov strategy, we write $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathscr{A},\mathscr{B})$.

Selection games and selection principles are intimately related. For more details on selection principles and relevant references, see [47, 36, 49, 50]. Since this paper will focus on single-selection games of a countable length, we only recall single-selection principles of a countable length.

Definition 4. Let $\mathscr{A}$ and $\mathscr{B}$ be collections. The *single-selection principle* $\mathsf{S}_1(\mathscr{A},\mathscr{B})$ for a space X is the following property. Given any $A \in \mathscr{A}^\omega$, there exists $\vec{x} \in \prod_{n\in\omega} A_n$ so that $\{\vec{x}_n : n \in \omega\} \in \mathscr{B}$.

As mentioned in [11, Prop. 15], $\mathsf{S}_1(\mathscr{A},\mathscr{B})$ holds if and only if $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathsf{G}_1(\mathscr{A},\mathscr{B})$. Hence, we may establish equivalences between certain selection principles by addressing the corresponding selection games.

Definition 5. For a space X, an open cover $\mathscr{U}$ of X is said to be *non-trivial* if $\emptyset \notin \mathscr{U}$ and $X \notin \mathscr{U}$.

Common cover types that appear in selection principle theory are ω-covers and k-covers, whose definitions will be recalled below. These cover types are generalized in the following definition.

Definition 6. Let X be a space and $\mathscr{A}$ be a set of closed subsets of X. We say that a non-trivial cover $\mathscr{U}$ of X is an *$\mathscr{A}$-cover* if, for every $A \in \mathscr{A}$, there exists $U \in \mathscr{U}$ so that $A \subseteq U$.

Though $\mathscr{A}$-covers were used in [4] with the notation $\mathscr{O}(X, \mathscr{A})$ referring to the collection of all such covers, these covers didn't receive the name of $\mathscr{A}$-covers until [6]. Also, $\mathscr{A}$-covers were independently defined as $\omega_{\mathscr{I}}$-covers and studied in [40] where the authors investigate Ramsey-like properties. However, these notions are not new and the essential idea appears as early as 1975 in [53] where Telgásky defines **K**-covers relative to any collection of sets **K**.

Definition 7. For a collection $\mathscr{A}$, we let $\neg\mathscr{A}$ denote the collection of sets which are not in $\mathscr{A}$. We also define the following classes for a space X and a collection $\mathscr{A}$ of closed subsets of X.

- $\mathscr{T}_X$ is the family of all proper non-empty open subsets of X.
- For $x \in X$, $\mathscr{N}_{X,x} = \{U \in \mathscr{T}_X : x \in U\}$.
- For $A \in \wp^+(X)$, $\mathscr{N}_X(A) = \{U \in \mathscr{T}_X : A \subseteq U\}$.
- $\mathscr{N}_X[\mathscr{A}] = \{\mathscr{N}_X(A) : A \in \mathscr{A}\}$,
- CD_X is the set of all closed discrete subsets of X.
- $\mathscr{D}_X$ is the set of all dense subsets of X.
- For $x \in X$, $\Omega_{X,x} = \{A \subseteq X : x \in \mathrm{cl}(A)\}$.
- For $x \in X$, $\Gamma_{X,x}$ is the set of all sequences of X converging to x.
- $\mathscr{O}_X$ is the set of all non-trivial open covers of X.
- $\mathscr{O}_X(\mathscr{A})$ is the set of all $\mathscr{A}$-covers.
- $\Lambda_X(\mathscr{A})$ is the set of all $\mathscr{A}$-covers $\mathscr{U}$ with the property that, for every $A \in \mathscr{A}$, $\{U \in \mathscr{U} : A \subseteq U\}$ is infinite.
- $\Gamma_X(\mathscr{A})$ is the set of all countable $\mathscr{A}$-covers $\mathscr{U}$ with the property that, for every $A \in \mathscr{A}$, $\{U \in \mathscr{U} : A \subseteq U\}$ is co-finite.

Note that, in our notation, $\mathscr{O}_X([X]^{<\omega})$ is the set of all ω-covers of X, which we will denote by Ω_X, and that $\mathscr{O}_X(K(X))$ is the set of all k-covers of X, which we will denote by $\mathscr{K}_X$. We also use $\Gamma_\omega(X)$ to denote $\Gamma_X([X]^{<\omega})$ and $\Gamma_k(X)$ to denote $\Gamma_X(K(X))$.

The notion of ω-covers is commonly attributed to [22], but they were already in use in [41] where they are referred to as *open covers for finite sets*. The isolated notion of k-covers appears as early as [42] in which they are referred to as *open covers for compact subsets.* As mentioned above, these types of covers were studied to some degree as early as [53]. For a focused treatment of k-covers in the realm of selection principles, see [16, 8].

Note that $\mathsf{S}_1(\mathscr{O}_X, \mathscr{O}_X)$ is the Rothberger property and $\mathsf{G}_1(\mathscr{O}_X, \mathscr{O}_X)$ is the Rothberger game. If we let $\mathbb{P}_X = \{\mathscr{N}_{X,x} : x \in X\}$, then $\mathsf{G}_1(\mathbb{P}_X, \neg\mathscr{O})$ is a rephrasing of the point-open game studied by Galvin [20] and Telgársky [53]. The games $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Gamma_{X,x})$ and $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Omega_{X,x})$ are two variants of Gruenhage's W-game (see [24]). We refer to $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Gamma_{X,x})$ as Gruenhage's converging W-game and $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Omega_{X,x})$ as Gruenhage's clustering W-game. The games $\mathsf{G}_1(\mathscr{T}_X, \neg\Omega_{X,x})$ and $\mathsf{G}_1(\mathscr{T}_X, \mathrm{CD}_X)$ were introduced by Tkachuk (see [56, 57]) and tied to Gruenhage's W-games in [57, 12]. The strong countable dense fan-tightness game at x is $\mathsf{G}_1(\mathscr{D}_X, \Omega_{X,x})$ and the strong countable fan-tightness game at x is $\mathsf{G}_1(\Omega_{X,x}, \Omega_{X,x})$ (see [1]).

Lemma 8 (See [5, Lemma 4])**.** For a space X and an ideal of closed sets $\mathscr{A}$ of X, $\mathscr{O}_X(\mathscr{A}) = \Lambda_X(\mathscr{A})$.

In what follows, we say that $\mathscr{G}$ is a *selection game* if there exist classes $\mathscr{A}$, $\mathscr{B}$ so that $\mathscr{G} = \mathsf{G}_1(\mathscr{A}, \mathscr{B})$.

Since we work with full- and limited-information strategies, we reflect this in our definitions of game equivalence and duality.

Definition 9. We say that two selection games $\mathscr{G}$ and $\mathscr{H}$ are *equivalent*, denoted $\mathscr{G} \equiv \mathscr{H}$, if the following hold:

- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathscr{G} \iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathscr{H}$
- $\mathrm{II} \uparrow \mathscr{G} \iff \mathrm{II} \uparrow \mathscr{H}$
- $\mathrm{I} \not\uparrow \mathscr{G} \iff \mathrm{I} \not\uparrow \mathscr{H}$
- $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathscr{G} \iff \mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathscr{H}$

We also use a preorder on selection games.

Definition 10 ([5])**.** Given selection games $\mathscr{G}$ and $\mathscr{H}$, we say that $\mathscr{G} \leqslant_{\mathrm{II}} \mathscr{H}$ if the following implications hold:

- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathscr{G} \implies \mathrm{II} \underset{\text{mark}}{\uparrow} \mathscr{H}$
- $\mathrm{II} \uparrow \mathscr{G} \implies \mathrm{II} \uparrow \mathscr{H}$
- $\mathrm{I} \not\uparrow \mathscr{G} \implies \mathrm{I} \not\uparrow \mathscr{H}$
- $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathscr{G} \implies \mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathscr{H}$

Note that $\leqslant_{\mathrm{II}}$ is transitive and that if $\mathscr{G} \leqslant_{\mathrm{II}} \mathscr{H}$ and $\mathscr{H} \leqslant_{\mathrm{II}} \mathscr{G}$, then $\mathscr{G} \equiv \mathscr{H}$. We use the subscript of II since each implication in the definition of $\leqslant_{\mathrm{II}}$ is related to a transference of winning plays by Two.

Definition 11. We say that two selection games $\mathscr{G}$ and $\mathscr{H}$ are *dual* if the following hold:

- $\mathrm{I} \uparrow \mathscr{G} \iff \mathrm{II} \uparrow \mathscr{H}$
- $\mathrm{II} \uparrow \mathscr{G} \iff \mathrm{I} \uparrow \mathscr{H}$
- $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathscr{G} \iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathscr{H}$
- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathscr{G} \iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathscr{H}$

We note one important way in which equivalence and duality interact.

Lemma 12. Suppose $\mathscr{G}_1, \mathscr{G}_2, \mathscr{H}_1$, and $\mathscr{H}_2$ are selection games so that $\mathscr{G}_1$ is dual to $\mathscr{H}_1$ and $\mathscr{G}_2$ is dual to $\mathscr{H}_2$. Then, if $\mathscr{G}_1 \leqslant_{\mathrm{II}} \mathscr{G}_2$, $\mathscr{H}_2 \leqslant_{\mathrm{II}} \mathscr{H}_1$. Consequently, if $\mathscr{G}_1 \equiv \mathscr{G}_2$, then $\mathscr{H}_1 \equiv \mathscr{H}_2$.

We will use consequences of [11, Cor. 26] to see that a few classes of selection games are dual.

Lemma 13. Let $\mathscr{A}$ be an ideal of closed sets of a space X and $\mathscr{B}$ be a collection.

(i) By [4, Cor. 3.4] and [11, Thm. 38], $\mathsf{G}_1(\mathscr{O}_X(\mathscr{A}), \mathscr{B})$ and $\mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{B})$ are dual. (Note that this is a general form of the duality of the Rothberger game and the point-open game.)

(ii) By [11, Cor. 33], $\mathsf{G}_1(\mathscr{D}_X, \mathscr{B})$ and $\mathsf{G}_1(\mathscr{T}_X, \neg\mathscr{B})$ are dual.
(iii) By [11, Cor. 35], for $x \in X$, $\mathsf{G}_1(\Omega_{X,x}, \mathscr{B})$ and $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\mathscr{B})$ are dual.

We now state the translation theorem we will be using to establish some game equivalences.

Theorem 14 ([5, Thm. 12])**.** Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ be collections. Suppose there are functions $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{B} \to \mathscr{A}$ and $\overrightarrow{T}_{\mathrm{II},n} : (\bigcup \mathscr{A}) \times \mathscr{B} \to \bigcup \mathscr{B}$ for each $n \in \omega$, so that

(i) if $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, then $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$, and
(ii) when $\langle x_n : n \in \omega \rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{I},n}(B_n)$ and $\{x_n : n \in \omega\} \in \mathscr{C}$, then $\left\{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\right\} \in \mathscr{D}$.

Then $\mathsf{G}_1(\mathscr{A}, \mathscr{C}) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{B}, \mathscr{D})$.

This translation theorem essentially automates the work one would do to prove the connections between two games at all the different levels of strategy. For a quick example of how this translation result gets used, when we establish a connection between the open covers of a space X and the continuous real-valued functions on X, we can have $\overleftarrow{T}_{\mathrm{I},n}$ take in a sequence of functions, and output the cover formed by their preimages of a neighborhood around 0, and $\overrightarrow{T}_{\mathrm{II},n}$ would take in an open set and a sequence of continuous functions, and output a function from the sequence whose preimage is that open set.

Similar results to Theorem 14 for longer length games and finite-selection games, even with simplified hypotheses, can be found in [5, 6, 7].

We will also need a separation axiom for some results in the sequel.

Definition 15. Let X be a space and $\mathscr{A}$ be an ideal of closed subsets of X. We say that X is *$\mathscr{A}$-normal* if, given any $A \in \mathscr{A}$ and $U \subseteq X$ open with $A \subseteq U$, there exists an open set V so that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.

Note that, if X is $\mathscr{A}$-normal, then X is regular. If $\mathscr{A} = K(X)$ and X is regular, then X is $\mathscr{A}$-normal.

2.2. *Uniform Spaces.*

To introduce the basics of uniform spaces we'll need in this paper, we mostly follow [33, Chapter 6].

We recall the standard notation involved with uniformities. Let X be a set. The diagonal of X is $\Delta_X = \{\langle x, x\rangle : x \in X\}$. For $E \subseteq X^2$, $E^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in E\}$. If $E = E^{-1}$, then E is said to be *symmetric*. If $E, F \subseteq X^2$,

$$E \circ F = \{\langle x, z\rangle : (\exists y \in X)\ \langle x, y\rangle \in F \wedge \langle y, z\rangle \in E\}.$$

For $E \subseteq X^2$, we let $E[x] = \{y \in X : \langle x, y\rangle \in E\}$ and $E[A] = \bigcup_{x\in A} E[x]$.

Definition 16. A *uniformity* on a set X is a set $\mathscr{E} \subseteq \wp^+(X^2)$ which satisfies the following properties:

- For every $E \in \mathscr{E}$, $\Delta_X \subseteq E$.
- For every $E \in \mathscr{E}$, $E^{-1} \in \mathscr{E}$.
- For every $E \in \mathscr{E}$, there exists $F \in \mathscr{E}$ so that $F \circ F \subseteq E$.
- For $E, F \in \mathscr{E}$, $E \cap F \in \mathscr{E}$.
- For $E \in \mathscr{E}$ and $F \subseteq X^2$, if $E \subseteq F$, $F \in \mathscr{E}$.

If, in addition, $\Delta_X = \bigcap \mathscr{E}$, we say that the uniformity $\mathscr{E}$ is *Hausdorff*. By an *entourage* of X, we mean a set $E \in \mathscr{E}$. The pair $(X, \mathscr{E})$ is called a *uniform space*.

Definition 17. For a set X, we say that $\mathscr{B} \subseteq \wp^+(X^2)$ is a *base for a uniformity* if

- for every $B \in \mathscr{B}$, $\Delta_X \subseteq B$;
- for every $B \in \mathscr{B}$, there is some $A \in \mathscr{B}$ so that $A \subseteq B^{-1}$;
- for every $B \in \mathscr{B}$, there is some $A \in \mathscr{B}$ so that $A \circ A \subseteq B$; and
- for $A, B \in \mathscr{B}$, there is some $C \in \mathscr{B}$ so that $C \subseteq A \cap B$.

If the uniformity generated by $\mathscr{B}$ is $\mathscr{E}$, we say that $\mathscr{B}$ is a *base* for $\mathscr{E}$.

If $(X, \mathscr{E})$ is a uniform space, then the uniformity $\mathscr{E}$ generates a topology on X in the following way: $U \subseteq X$ is declared to be open provided that, for every $x \in U$, there is some $E \in \mathscr{E}$ so that $E[x] \subseteq U$. An important result about this topology is

Theorem 18 (see [33])**.** A Hausdorff uniform space $(X, \mathscr{E})$ is metrizable if and only if $\mathscr{E}$ has a countable base.

For a uniform space $(X, \mathscr{E})$, there is a natural way to define a uniformity on $K(X)$ which is directly analogous to the Pompeiu-Hausdorff distance defined in the context of metric spaces.

Definition 19. Let $(X, \mathscr{E})$ be a uniform space and, for $E \in \mathscr{E}$, define

$$hE = \{\langle K, L\rangle \in K(X)^2 : K \subseteq E[L] \wedge L \subseteq E[K]\}.$$

Just as the Pompeiu-Hausdorff distance on compact subsets generates the Vietoris topology, the analogous uniformity also generates the Vietoris topology.

Theorem 20 (see [9, Chapter 2])**.** For a uniform space $(X, \mathscr{E})$, $\mathscr{B} = \{hE : E \in \mathscr{E}\}$ is a base for a uniformity on $K(X)$; the topology generated by the uniform base $\mathscr{B}$ is the Vietoris topology.

For the set of functions from a space X to a uniform space $(Y, \mathscr{E})$, we review the uniformity which generates the topology of uniform convergence on a family of subsets of X. For this review, we mostly follow [33, Chapter 7].

Definition 21. For the set Y^X of functions from a set X to a uniform space $(Y, \mathscr{E})$, we define, for $A \in \wp^+(X)$ and $E \in \mathscr{E}$,

$$\mathbf{U}(A, E) = \{\langle f, g\rangle \in (Y^X)^2 : (\forall x \in A)\ \langle f(x), g(x)\rangle \in E\}.$$

For the set of functions $X \to \mathbb{K}(Y)$, we let $\mathbf{W}(A, E) = \mathbf{U}(A, hE)$.

If $\mathscr{B}$ is a base for a uniformity on Y and $\mathscr{A}$ is an ideal of subsets of X, then $\{\mathbf{U}(A, B) : A \in \mathscr{A}, B \in \mathscr{B}\}$ forms a base for a uniformity on Y^X. The corresponding topology generated by this base for a uniformity is the topology of uniform convergence on $\mathscr{A}$. Consequently, $\{\mathbf{W}(A, B) : A \in \mathscr{A}, B \in \mathscr{B}\}$ is a base for a uniformity on $\mathbb{K}(Y)^X$.

2.3. *Usco and Cusco Mappings.* In this section, we introduce the basic facts of usco and cusco mappings needed for this paper. For a thorough introduction to usco mappings, see [31]. Of primary use are Theorems 31 and 32 which offer convenient characterizations of minimal usco and cusco maps, respectively.

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A *set-valued* function from X to Y is a function $\Phi : X \to \wp(Y)$. These are sometimes also referred to as *multi-functions*.

Definition 22. A set-valued function $\Phi : X \to \wp(Y)$ is said to be *upper semicontinuous* if, for every open $V \subseteq Y$,

$$\Phi^{\leftarrow}(V) := \{x \in X : \Phi(x) \subseteq V\}$$

is open in X. An *usco* map from a space X to Y is a set-valued map Φ from X to Y which is upper semicontinuous and whose range is contained in $\mathbb{K}(Y)$. Let $\mathrm{USCO}(X,Y)$ denote the collection of all usco maps $X \to \mathbb{K}(Y)$.

An usco map $\Phi : X \to \mathbb{K}(Y)$ is said to be *minimal* if its graph is minimal with respect to the $\subseteq$ relation. Let $\mathrm{MU}(X,Y)$ denote the collection of all minimal usco maps $X \to \mathbb{K}(Y)$.

Definition 23. Suppose Y is a Hausdorff linear space. An usco map $\Phi : X \to \mathbb{K}(Y)$ is said to be *cusco* if $\Phi(x)$ is convex for every $x \in X$. A cusco map $\Phi : X \to \mathbb{K}(Y)$ is said to be *minimal* if its graph is minimal with respect to the $\subseteq$ relation. Let $\mathrm{MC}(X,Y)$ denote the collection of all minimal cusco maps $X \to \mathbb{K}(Y)$.

It is clear that any continuous $\Phi : X \to \mathbb{K}(Y)$ is usco and that there are continuous $\Phi : X \to \mathbb{K}(Y)$ which are not minimal. As demonstrated by [3, Ex. 1.31], there are minimal usco maps $\mathbb{R} \to \mathbb{K}(\mathbb{R})$ which are not continuous. We will show, in Example 37, that such an example can be extended to produce a minimal cusco map $\mathbb{R} \to \mathbb{K}(\mathbb{R})$ that is not continuous.

Definition 24. Suppose $\Phi : X \to \wp^+(Y)$. We say that a function $f : X \to Y$ is a *selection* of Φ if $f(x) \in \Phi(x)$ for every $x \in X$. We let $\mathrm{sel}(\Phi)$ be the set of all selections of Φ.

If $D \subseteq X$ is dense and $f : D \to Y$ is so that $f(x) \in \Phi(x)$ for each $x \in D$, we say that f is a *densely defined selection* of Φ.

The notion of subcontinuity was introduced by Fuller [18] which can be extended to so-called densely defined functions in the following way. See also [37].

Definition 25. Suppose $D \subseteq X$ is dense. We say that a function $f : D \to Y$ is *subcontinuous* if, for every $x \in X$ and every net $\langle x_\lambda : \lambda \in \Lambda \rangle$ in D with $x_\lambda \to x$, $\langle f(x_\lambda) : \lambda \in \Lambda \rangle$ has an accumulation point.

The following is a well-known property of usco maps that will be used in this paper.

Lemma 26. If $\Phi \in \mathrm{USCO}(X,Y)$, then any selection of Φ is subcontinuous.

The notion of semi-open sets was introduced by Levine [38].

Definition 27. For a space X, a set $A \subseteq X$ is said to be *semi-open* if $A \subseteq \mathrm{cl}\,\mathrm{int}(A)$.

The notion of quasicontinuity was introduced by Kempisty [35] and surveyed by Neubrunn [44].

Definition 28. A function $f : X \to Y$ is said to be *quasicontinuous* if, for each open $V \subseteq Y$, $f^{-1}(V)$ is semi-open in X.

If $D \subseteq X$ is dense and $f : D \to Y$, we will say that f is quasicontinuous if it is quasicontinuous on D with the subspace topology.

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Definition 29. For $f \in \mathrm{Fn}(X,Y)$, define $\overline{f} : X \to \wp(Y)$ by the rule

$$\overline{f}(x) = \{y \in Y : \langle x,y \rangle \in \mathrm{cl}\ \mathrm{gr}(f)\}.$$

For a linear space Y and $A \subseteq Y$, we will use $\mathrm{conv}\, A$ to denote the convex hull of A.

Definition 30. Suppose Y is a Hausdorff locally convex linear space. For $f \in \mathrm{Fn}(X,Y)$, define $\check{f} : X \to \wp(Y)$ by the rule

$$\check{f}(x) = \mathrm{cl}\,\mathrm{conv}\,\overline{f}(x).$$

Theorem 31 (Holá, Holý [26, 28])**.** Suppose Y is regular and that $\Phi : X \to \wp^+(Y)$. Then the following are equivalent:

(i) Φ is minimal usco.
(ii) Every selection f of Φ is subcontinuous, quasicontinuous, and $\Phi = \overline{f}$.
(iii) There exists a selection f of Φ which is subcontinuous, quasicontinuous, and $\Phi = \overline{f}$.
(iv) There exists a densely defined selection f of Φ which is subcontinuous, quasicontinuous, and $\Phi = \overline{f}$.

Important initial contributions to the following characterization of cusco maps are found in [23, 2].

Theorem 32 (Holá, Holý [28])**.** Suppose Y is a Hausdorff locally convex linear space in which the closed convex hull of a compact set is compact and that $\Phi : X \to \wp^+(Y)$. Then the following are equivalent:

(i) Φ is minimal cusco.
(ii) There exists a selection f of Φ which is subcontinuous, quasicontinuous, and $\Phi = \check{f}$.
(iii) There exists a densely defined selection f of Φ which is subcontinuous, quasicontinuous, and $\Phi = \check{f}$.

Remark 33. Based on [28, Thm. 3.2], for each multi-function $\Phi \in \mathrm{MC}(X,\mathbb{R})$, the function $f : X \to \mathbb{R}$ defined by $f(x) = \max \Phi(x)$ is subcontinuous and quasicontinuous.

The following is motivated by Theorem 32.

Definition 34. For any $\Phi \in \mathrm{MC}(X,Y)$, let $\mathrm{sel}_{QS}(\Phi)$ be the collection of selections f of Φ that are quasicontinuous, subcontinuous, and $\check{f} = \Phi$.

As suggested by Theorems 31 and 32, there is a close relationship between minimal usco and minimal cusco maps.

Theorem 35 ([27, Thm. 2.6])**.** Let X be a space and Y be a Hausdorff locally convex linear space in which the closed convex hull of a compact set is compact. Define $\varphi : \mathrm{MU}(X,Y) \to \mathbb{K}(Y)^X$ by the rule $\varphi(F)(x) = \mathrm{cl}\,\mathrm{conv}\,F(x)$. Then φ is a bijection from $\mathrm{MU}(X,Y)$ to $\mathrm{MC}(X,Y)$.

In fact, by [27, Thm. 3.1], when Y is a Banach space and $\mathrm{MU}(X,Y)$ and $\mathrm{MC}(X,Y)$ are both given the topology of either point-wise convergence or uniform convergence on compacta, φ is continuous. When X is locally compact, Y is a Banach space, and the spaces $\mathrm{MU}(X,Y)$ and $\mathrm{MC}(X,Y)$ are given the topology of uniform convergence on compacta, by [27, Thm. 3.2], φ is a homeomorphism. However, it seems unknown whether $\mathrm{MU}(X,Y)$ and $\mathrm{MC}(X,Y)$ are always homeomorphic, though [27, Ex. 3.2] may lead one to conjecture that this is not the case since it is an example where the natural mapping φ is not a homeomorphism.

We will be using the following to construct certain functions.

Lemma 36. Let $f,g : X \to Y$ and $U \in \mathscr{T}_X$ and define $h : X \to Y$ by the rule

$$h(x) = \begin{cases} f(x), & x \in \mathrm{cl}(U); \\ g(x), & x \notin \mathrm{cl}(U). \end{cases}$$

(i) If f and g are subcontinuous, then h is subcontinuous.
(ii) If f is constant, and g is quasicontinuous, then h is quasicontinuous.

Consequently, if f is constant and g is both subcontinuous and quasicontinuous, then h is both subcontinuous and quasicontinuous. Moreover, $\overline{h}$ is minimal usco; under the assumption that Y is a Hausdorff locally convex linear space in which the closed convex hull of a compact set is compact, $\check{h}$ is minimal cusco.

Proof. All except for the fact that $\check{h}$ is minimal cusco when Y is a Hausdorff locally convex linear space in which the closed convex hull of a compact set is compact is proved in [3, Lemma 1.30]. For the remaining remark, simply appeal to Theorem 32. □

Example 37. Consider $\mathrm{MC}(\mathbb{R},\mathbb{R})$. By Lemma 36, $\Phi := \check{\mathbf{1}}_{[0,1]}$ is minimal cusco. However, Φ is not continuous since

$$\{0,1\} = \{x \in \mathbb{R} : \Phi(x) \in [(-0.5,0.75),(0.25,1.5)]\}.$$

Hence, when Y is metrizable, studying the space $\mathrm{MC}(X,Y)$ is, in general, different than studying the space of continuous functions into a metrizable space.

Although the next two results are known, we provide short proofs for the convenience of the reader.

Corollary 38. Suppose X is a space, Y is a Hausdorff locally convex linear space in which the closed convex hull of a compact set is compact, $\Phi,\Psi \in \mathrm{MC}(X,Y)$, $f \in \mathrm{sel}_{QS}(\Phi)$, $g \in \mathrm{sel}_{QS}(\Psi)$, and $f\restriction_U = g\restriction_U$. Then $\Phi\restriction_U = \Psi\restriction_U$.

Proof. Set $\Phi^* = \overline{f}$ and $\Psi^* = \overline{g}$. Then $\Phi^*,\Psi^* \in \mathrm{MU}(X,Y)$ by Theorem 31; so $\Phi^* \subseteq \Phi$ and $\Psi^* \subseteq \Psi$. By [3, Cor. 1.32], $\Phi^*\restriction_U = \Psi^*\restriction_U$. Finally, by Theorem 35, $\Phi\restriction_U = \Psi\restriction_U$. □

Corollary 39. Suppose X is a space and Y is a Hausdorff locally convex linear space in which the closed convex hull of compact set is compact. If $A \subseteq X$ is non-empty, $U,V \in \mathscr{T}_X$ are so that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$, $\Phi \in \mathrm{MC}(X,Y)$, and $f \in \mathrm{sel}_{QS}(\Phi)$ is so that $\Phi = \check{f}$, then, for $y_0 \in Y$, the map $g : X \to Y$ defined by

$$g(x) = \begin{cases} y_0, & x \in \mathrm{cl}(X \setminus \mathrm{cl}(V)); \\ f(x), & \text{otherwise}, \end{cases}$$

has the property that $\Psi := \check{g} \in \mathrm{MC}(X,Y)$, $\langle \Phi,\Psi \rangle \in \mathbf{W}(A,E)$ for any entourage E of Y, and $g[X \setminus U] = \{y_0\}$.

Proof. Note that Lemma 36 implies that g is subcontinuous and quasicontinuous. Then $\Psi := \breve{g} \in \mathrm{MC}(X,Y)$ by Theorem 32.

Since V is open, $\mathrm{cl}(X \setminus \mathrm{cl}(V)) \subseteq X \setminus V$. Then $g(x) = f(x)$ for all $x \in V$. By Corollary 38, we see that $\Phi \restriction_A = \Psi \restriction_A$ as $A \subseteq V$. Also, since $X \setminus U \subseteq X \setminus \mathrm{cl}(V) \subseteq \mathrm{cl}(X \setminus \mathrm{cl}(V))$, we see that $g[X \setminus U] = \{y_0\}$. □

Since every linear space is a topological group, we can use the uniform structure generated by the neighborhoods of the identity; that is, the sets of the form

$$U_R = \{\langle x, y\rangle \in X : xy^{-1} \in U\}$$

where U is a neighborhood of the identity form a base for a uniformity on a topological group X. We can also restrict our attention to a particular basis at the identity to produce this uniform structure. Note that a closed neighborhood of identity generates a closed entourage as viewed as a subset of X^2 with the product topology.

The following can be seen as a modification of [32, Lemma 6.1]. We provide a full proof for the convenience of the reader.

Corollary 40. Let X be a space and Y be a Hausdorff locally convex linear space. Suppose $\Phi, \Psi \in \mathrm{MC}(X,Y)$, E is a closed convex neighborhood of 0_Y, $D \subseteq X$ is dense, and $\langle \Phi(x), \Psi(x)\rangle \in hE_R$ for all $x \in D$. Then $\langle \Phi(x), \Psi(x)\rangle \in hE_R$ for all $x \in X$.

Proof. Define $F : X \to \wp(Y)$ by $F(x) = E_R[\Phi(x)]$ and note that $F(x)$ is convex and closed for each $x \in X$.

We show that the graph of F is closed. Suppose $\langle x, y\rangle \in \mathrm{cl}\,\mathrm{gr}(F)$ and let $\langle \langle x_\lambda, y_\lambda\rangle : \lambda \in \Lambda\rangle$ be a net in $\mathrm{gr}(F)$ so that $\langle x_\lambda, y_\lambda\rangle \to \langle x, y\rangle$. Since $y_\lambda \in E_R[\Phi(x_\lambda)]$, we can let $w_\lambda \in \Phi(x_\lambda)$ be so that $y_\lambda \in E_R[w_\lambda]$. Observe that, since $x_\lambda \to x$ and $w_\lambda \in \Phi(x_\lambda)$ for each $\lambda \in \Lambda$, by Lemma 26, $\langle w_\lambda : \lambda \in \Lambda\rangle$ has an accumulation point $w \in \Phi(x)$. Since $y_\lambda \to y$ and w is an accumulation point of $\langle w_\lambda : \lambda \in \Lambda\rangle$, $\langle w, y\rangle$ is an accumulation point of $\langle\langle w_\lambda, y_\lambda\rangle : \lambda \in \Lambda\rangle$. Moreover, as $\langle w_\lambda, y_\lambda\rangle \in E_R$ for all $\lambda \in \Lambda$ and E_R is closed, we see that $\langle w, y\rangle \in E_R$. Hence, $y \in E_R[w] \subseteq E_R[\Phi(x)] = F(x)$. That is, $\langle x, y\rangle \in \mathrm{gr}(F)$ which establishes that $\mathrm{gr}(F)$ is closed.

By Theorem 32, we can let $g : D \to Y$ be subcontinuous and quasicontinuous so that $g(x) \in \Psi(x)$ for each $x \in D$ and $\Psi = \breve{g}$. Since $\mathrm{gr}(F)$ is closed, convex-valued, and $\mathrm{gr}(g) \subseteq \mathrm{gr}(F)$, we see that $\mathrm{cl}\,\mathrm{gr}(g) \subseteq \mathrm{gr}(F)$ and $\breve{g}(x) \subseteq F(x)$. That is, $\Psi(x) \subseteq F(x) = E_R[\Phi(x)]$ for all $x \in X$.

A symmetric argument shows that $\Phi(x) \subseteq E_R[\Psi(x)]$ for all $x \in X$, finishing the proof. □

3. Results

For the remainder of the paper, we will be interested only in real set-valued functions; so we will let $\mathrm{MU}(X) = \mathrm{MU}(X,\mathbb{R})$ and $\mathrm{MC}(X) = \mathrm{MC}(X,\mathbb{R})$. We also use, for $\varepsilon > 0$,

$$\Delta_\varepsilon = \{\langle x, y\rangle \in \mathbb{R}^2 : |x - y| < \varepsilon\}.$$

For $A \subseteq X$, we will use $\mathbf{U}(A,\varepsilon) = \mathbf{U}(A,\Delta_\varepsilon)$ and $\mathbf{W}(A,\varepsilon) = \mathbf{W}(A,\Delta_\varepsilon)$. For $Y \subseteq \mathbb{R}$, let $\mathbb{B}(Y,\varepsilon) = \bigcup_{y\in Y} B(y,\varepsilon)$ and note that

$$\begin{aligned}&\mathbf{W}(A,\varepsilon)\\&= \left\{\langle \Phi,\Psi\rangle \in \mathbb{K}(\mathbb{R})^X : (\forall x \in A)[\Phi(x) \subseteq \mathbb{B}(\Psi(x),\varepsilon) \wedge \Psi(x) \subseteq \mathbb{B}(\Phi(x),\varepsilon)]\right\}.\end{aligned}$$

For $\Phi : X \to \mathbb{K}(Y)$, $A \subseteq X$, and $\varepsilon > 0$, we let $[\Phi;A,\varepsilon] = \mathbf{W}(A,\varepsilon)[\Phi]$.

Then, if $\mathscr{A}$ is an ideal of closed subsets of X, we will use $\mathrm{MU}_{\mathscr{A}}(X)$ (resp. $\mathrm{MC}_{\mathscr{A}}(X)$) to denote the set $\mathrm{MU}(X)$ (resp. $\mathrm{MC}(X)$) with the topology generated by the base for a uniformity $\{\mathbf{W}(A,\varepsilon) : A \in \mathscr{A}, \varepsilon > 0\}$. When $\mathscr{A} = [X]^{<\omega}$, we use $\mathrm{MU}_p(X)$ and $\mathrm{MC}_p(X)$; when $\mathscr{A} = K(X)$, we use $\mathrm{MU}_k(X)$ and $\mathrm{MC}_k(X)$. We will use $\mathbf{0}$ to denote the function that is constantly 0 when dealing with real-valued functions and the function that is constantly $\{0\}$ when dealing with set-valued maps.

Theorem 41. Let X be regular and $\mathscr{A}$ and $\mathscr{B}$ be ideals of closed subsets of X. Then,

(i) $\mathsf{G}_1(\mathscr{O}_X(\mathscr{A}),\Lambda_X(\mathscr{B})) \leqslant_{\mathrm{II}} \mathsf{G}_1(\Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}},\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}})$,

(ii) $\mathsf{G}_1(\Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}},\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_{\mathscr{A}}(X)},\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}})$, and

(iii) if X is $\mathscr{A}$-normal, $\mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_{\mathscr{A}}(X)},\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{O}_X(\mathscr{A}),\Lambda_X(\mathscr{B}))$.

Thus, if X is $\mathscr{A}$-normal, the three games are equivalent.

Proof. We first address (i). Fix some $\mathscr{U}_0 \in \mathscr{O}_X(\mathscr{A})$ and let $W_{\Phi,n} = \Phi^{\leftarrow}[(-2^{-n},2^{-n})]$ for $\Phi \in \mathrm{MC}(X)$ and $n \in \omega$. Let

$$\mathfrak{T}_n = \{\mathscr{F} \in \Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}} : (\exists \Phi \in \mathscr{F})\ W_{\Phi,n} = X\}$$

and $\mathfrak{T}_n^\star = \Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}} \setminus \mathfrak{T}_n$. Define $\overleftarrow{T}_{\mathrm{I},n} : \Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}} \to \mathscr{O}_X(\mathscr{A})$ by the rule

$$\overleftarrow{T}_{\mathrm{I},n}(\mathscr{F}) = \begin{cases}\{W_{\Phi,n} : \Phi \in \mathscr{F}\}, & \mathscr{F} \in \mathfrak{T}_n^\star;\\ \mathscr{U}_0, & \text{otherwise.}\end{cases}$$

To see that $\overleftarrow{T}_{\mathrm{I},n}$ is defined, let $\mathscr{F} \in \mathfrak{T}_n^\star$. Let $A \in \mathscr{A}$ be arbitrary and choose $\Phi \in [\mathbf{0};A,2^{-n}] \cap \mathscr{F}$. It follows that $A \subseteq W_{\Phi,n}$. Hence, $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{F}) \in \mathscr{O}_X(\mathscr{A})$.

We now define

$$\overrightarrow{T}_{\mathrm{II},n} : \mathscr{T}_X \times \Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}} \to \mathrm{MC}(X)$$

in the following way. For each $\langle U,\mathscr{F}\rangle \in \mathscr{T}_X \times \mathfrak{T}_n$, let $\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}) \in \mathscr{F}$ be so that $W_{\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}),n} = X$. For $\langle U,\mathscr{F}\rangle \in \mathscr{T}_X \times \mathfrak{T}_n^\star$ so that $U \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F})$, let $\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}) \in \mathscr{F}$ be so that $U = W_{\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}),n}$. For $\langle U,\mathscr{F}\rangle \in \mathscr{T}_X \times \mathfrak{T}_n^\star$ so that $U \notin \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F})$, let $\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}) = \mathbf{0}$. By construction, if $U \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F})$, then $\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}) \in \mathscr{F}$.

To finish this application of Theorem 14, assume that we have

$$\langle U_n : n \in \omega\rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F}_n)$$

for some sequence $\langle \mathscr{F}_n : n \in \omega\rangle$ of $\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}$ so that $\{U_n : n \in \omega\} \in \Lambda_X(\mathscr{B})$. For each $n \in \omega$, let $\Phi_n = \overrightarrow{T}_{\mathrm{II},n}(U_n,\mathscr{F}_n)$. Now, let $B \in \mathscr{B}$ and $\varepsilon > 0$ be arbitrary. Choose $n \in \omega$ so that $2^{-n} < \varepsilon$ and $B \subseteq U_n$.

If $\mathscr{F}_n \in \mathfrak{T}_n$, then Φ_n has the property that $X = \Phi_n^{\leftarrow}[(-2^{-n}, 2^{-n})]$; hence, $\Phi_n \in [\mathbf{0}; B, \varepsilon]$. Otherwise, $B \subseteq U_n = \Phi_n^{\leftarrow}[(-2^{-n}, 2^{-n})]$ which also implies that $\Phi_n \in [\mathbf{0}; B, \varepsilon]$. Thus, $\{\Phi_n : n \in \omega\} \in \Omega_{\mathrm{MC}_{\mathscr{B}}(X), \mathbf{0}}$.

(ii) holds since $\mathscr{D}_{\mathrm{MC}_{\mathscr{A}}(X)} \subseteq \Omega_{\mathrm{MC}_{\mathscr{A}}(X), \mathbf{0}}$.

Lastly, we address (iii). We define $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{O}_X(\mathscr{A}) \to \mathscr{D}_{\mathrm{MC}_{\mathscr{A}}(X)}$ by the rule

$$\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) = \{\Phi \in \mathrm{MC}(X) : (\exists U \in \mathscr{U})(\exists f \in \mathrm{sel}_{QS}(\Phi))\ f[X \setminus U] = \{1\}\}.$$

To see that $\overleftarrow{T}_{\mathrm{I},n}$ is defined, let $\mathscr{U} \in \mathscr{O}_X(\mathscr{A})$ and consider a basic open set $[\Phi; A, \varepsilon]$. Then let $U \in \mathscr{U}$ be so that $A \subseteq U$ and, by $\mathscr{A}$-normality, let V be open so that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$. Define $f : X \to \mathbb{R}$ by the rule

$$f(x) = \begin{cases} 1, & x \in \mathrm{cl}(X \setminus \mathrm{cl}(V)); \\ \max \Phi(x), & \text{otherwise}. \end{cases}$$

By Remark 33 and Corollary 39, $\check{f} \in [\Phi; A, \varepsilon] \cap \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U})$. So $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) \in \mathscr{D}_{\mathrm{MC}_{\mathscr{A}}(X)}$.

We define $\overrightarrow{T}_{\mathrm{II},n} : \mathrm{MC}(X) \times \mathscr{O}_X(\mathscr{A}) \to \mathscr{T}_X$ in the following way. Fix some $U_0 \in \mathscr{T}_X$. For $\langle \Phi, \mathscr{U} \rangle \in \mathrm{MC}(X) \times \mathscr{O}_X(\mathscr{A})$, if

$$\{U \in \mathscr{U} : (\exists f \in \mathrm{sel}_{QS}(\Phi))\ f[X \setminus U] = \{1\}\} \neq \emptyset,$$

let $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U}) \in \mathscr{U}$ be so that there exists $f \in \mathrm{sel}_{QS}(\Phi)$ with the property that $f[X \setminus \overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U})] = \{1\}$; otherwise, let $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U}) = U_0$. By construction, if $\Phi \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U})$, then $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U}) \in \mathscr{U}$.

Suppose we have

$$\langle \Phi_n : n \in \omega \rangle \in \prod_{n \in \omega} \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}_n)$$

for a sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of $\mathscr{O}_X(\mathscr{A})$ with the property that $\{\Phi_n : n \in \omega\} \in \Omega_{\mathrm{MC}_{\mathscr{B}}(X), \mathbf{0}}$. For each $n \in \omega$, let $U_n = \overrightarrow{T}_{\mathrm{II},n}(\Phi_n, \mathscr{U}_n)$. Since $\mathscr{B}$ is an ideal of sets, we need only show that $\langle U_n : n \in \omega \rangle$ is a $\mathscr{B}$-cover. So let $B \in \mathscr{B}$ be arbitrary and let $n \in \omega$ be so that $\Phi_n \in [\mathbf{0}; B, 1]$. Then we can let $f \in \mathrm{sel}_{QS}(\Phi_n)$ be so that $f[X \setminus U_n] = \{1\}$. Since $\Phi_n \in [\mathbf{0}; B, 1]$, we see that, for each $x \in B$, $f(x) \in \Phi_n(x) \subseteq (-1, 1)$. Hence, $B \cap (X \setminus U_n) = \emptyset$, which is to say that $B \subseteq U_n$. So Theorem 14 applies. □

Corollary 42. Let $\mathscr{A}$ and $\mathscr{B}$ be ideals of closed subsets of X and suppose that X is $\mathscr{A}$-normal. Then

$$\begin{aligned} \mathscr{G} := \mathsf{G}_1(\mathscr{O}_X(\mathscr{A}), \mathscr{O}_X(\mathscr{B})) &\equiv \mathsf{G}_1(\Omega_{\mathrm{MC}_{\mathscr{A}}(X), \mathbf{0}}, \Omega_{\mathrm{MC}_{\mathscr{B}}(X), \mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_{\mathscr{A}}(X)}, \Omega_{\mathrm{MC}_{\mathscr{B}}(X), \mathbf{0}}), \end{aligned}$$

$$\begin{aligned} \mathscr{H} := \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) &\equiv \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X), \mathbf{0}}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X), \mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X), \mathbf{0}}), \end{aligned}$$

and $\mathscr{G}$ is dual to $\mathscr{H}$.

Proof. Apply Theorem 41, Lemmas 8, 12, and 13. □

We offer the following comments relating this result to other structures. In [6, Thm. 31], inspired by Li [39], the game $\mathsf{G}_1(\mathscr{O}_X(\mathscr{A}), \mathscr{O}_X(\mathscr{B}))$ is shown to be equivalent to the selective separability game on certain hyperspaces of X. Along a similar line, [5, Cor. 14] establishes an analogous result to Corollary

42 relative to continuous real-valued functions with the corresponding topology under the assumption that X is functionally $\mathscr{A}$-normal ([3, Def. 1.15]).

Lemma 43. Suppose $\Phi \in \mathrm{USCO}(X,\mathbb{R})$ and that $A \subseteq X$ is sequentially compact. Then $\Phi[A]$ is bounded.

Though the proof is nearly identical to the one of [3, Lemma 2.4], we provide it in full for the convenience of the reader.

Proof. Suppose $\Phi : X \to \mathbb{K}(\mathbb{R})$ is unbounded on $A \subseteq X$ which is sequentially compact. For each $n \in \omega$, let $x_n \in A$ be so that there is some $y \in \Phi(x_n)$ with $|y| \geqslant n$. Then let $y_n \in \Phi(x_n)$ be so that $|y_n| \geqslant n$ and define $f : X \to \mathbb{R}$ to be a selection of Φ so that $f(x_n) = y_n$ for $n \in \omega$. Since A is sequentially compact, we can find $x \in A$ and a subsequence $\langle x_{n_k} : k \in \omega\rangle$ so that $x_{n_k} \to x$. Notice that $\langle f(x_{n_k}) : k \in \omega\rangle$ does not have an accumulation point. Therefore f is not subcontinuous and, by Lemma 26, Φ is not an usco map. $\square$

Theorem 44. Let $\mathscr{A}$ and $\mathscr{B}$ be ideals of closed subsets of X. If X is $\mathscr{A}$-normal and $\mathscr{B}$ consists of sequentially compact sets, then

$$\mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\Lambda_X(\mathscr{B})) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \mathrm{CD}_{\mathrm{MC}_{\mathscr{B}}(X)}).$$

Consequently,

$$\begin{aligned}\mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) &\equiv \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \mathrm{CD}_{\mathrm{MC}_{\mathscr{B}}(X)}).\end{aligned}$$

Proof. Let $\pi_1 : \mathrm{MC}(X) \times \mathscr{A} \times \mathbb{R} \to \mathrm{MC}(X)$, $\pi_2 : \mathrm{MC}(X) \times \mathscr{A} \times \mathbb{R} \to \mathscr{A}$, and $\pi_3 : \mathrm{MC}(X) \times \mathscr{A} \times \mathbb{R} \to \mathbb{R}$ be the standard coordinate projection maps. Define a choice function $\gamma : \mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)} \to \mathrm{MC}(X) \times \mathscr{A} \times \mathbb{R}$ so that

$$[\pi_1(\gamma(W)); \pi_2(\gamma(W)), \pi_3(\gamma(W))] \subseteq W.$$

Let $\Psi_W = \pi_1(\gamma(W))$, $A_W = \pi_2(\gamma(W))$, and $\varepsilon_W = \pi_3(\gamma(W))$. Then we define $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)} \to \mathscr{N}_X[\mathscr{A}]$ by $\overleftarrow{T}_{\mathrm{I},n}(W) = \mathscr{N}_X(A_W)$.

We now define $\overrightarrow{T}_{\mathrm{II},n} : \mathscr{T}_X \times \mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)} \to \mathrm{MC}(X)$ in the following way. For $A \in \mathscr{A}$ and $U \in \mathscr{N}_X(A)$, let $V_{A,U}$ be open so that

$$A \subseteq V_{A,U} \subseteq \mathrm{cl}(V_{A,U}) \subseteq U.$$

For $W \in \mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}$ and $U \in \overleftarrow{T}_{\mathrm{I},n}(W)$, define $f_{W,U,n} : X \to \mathbb{R}$ by the rule

$$f_{W,U,n}(x) = \begin{cases} n, & x \in \mathrm{cl}(X \setminus \mathrm{cl}(V_{A_W,U})); \\ \max \Psi_W(x), & \text{otherwise.} \end{cases}$$

Then we set

$$\overrightarrow{T}_{\mathrm{II},n}(U,W) = \begin{cases} \check{f}_{W,U,n}, & U \in \overleftarrow{T}_{\mathrm{I},n}(W); \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

By Remark 33 and Corollary 39, $\overrightarrow{T}_{\mathrm{II},n}(U,W) \in \mathrm{MC}(X)$ and, if $U \in \overleftarrow{T}_{\mathrm{I},n}(W)$,

$$\overrightarrow{T}_{\mathrm{II},n}(U,W) \in [\Psi_W; A_W, \varepsilon_W] \subseteq W.$$

Suppose we have a sequence

$$\langle U_n : n \in \omega \rangle \in \prod_{n \in \omega} \overleftarrow{T}_{\mathrm{I},n}(W_n)$$

for a sequence $\langle W_n : n \in \omega \rangle$ of $\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}$ so that $\{U_n : n \in \omega\} \notin \Lambda_X(\mathscr{B})$. Let $\Phi_n = \overrightarrow{T}_{\mathrm{II},n}(U_n, W_n)$ for each $n \in \omega$. We can find $N \in \omega$ and $B \in \mathscr{B}$ so that, for every $n \geqslant N$, $B \not\subseteq U_n$. Now, suppose $\Phi \in \mathrm{MC}(X) \setminus \{\Phi_n : n \in \omega\}$ is arbitrary. By Lemma 43, $\Phi[B]$ is bounded, so let $M > \sup|\Phi[B]|$ and $n \geqslant \max\{N, M+1\}$. Now, for $x \in B \setminus U_n$, note that $n \in \Phi_n(x)$ and that, for $y \in \Phi(x)$,

$$y \leqslant \sup|\Phi[B]| < M \leqslant n-1 \implies y - n < -1 \implies |y-n| > 1.$$

In particular, $\Phi_n(x) \not\subseteq \mathbb{B}(\Phi(x), 1)$ which establishes that $\Phi_n \notin [\Phi; B, 1]$. Hence, $\{\Phi_n : n \in \omega\}$ is closed and discrete and Theorem 14 applies.

For what remains, observe that

$$\mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \mathrm{CD}_{\mathrm{MC}_{\mathscr{B}}(X)}) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}})$$

since, if Two can produce a closed discrete set, then Two can avoid clustering around $\mathbf{0}$. Hence, by Corollary 42 we obtain that

$$\begin{aligned}\mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) &= \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\Lambda_X(\mathscr{B}))\\ &\leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \mathrm{CD}_{\mathrm{MC}_{\mathscr{B}}(X)})\\ &\leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_{\mathscr{A}}(X)}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})).\end{aligned}$$

This complete the proof. □

We now offer some relationships related to Gruenhage's W-games.

Proposition 45. Let $\mathscr{A}$ and $\mathscr{B}$ be ideals of closed subsets of X. Then

(i) $\mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}, \neg\Gamma_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}})$ and
(ii) $\mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}, \neg\Gamma_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}) \leqslant_{\mathrm{II}} \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\Gamma_X(\mathscr{B}))$.

Proof. (i) is evident since, if Two can avoid clustering at $\mathbf{0}$, they can surely avoid converging to $\mathbf{0}$.

(ii) Fix $U_0 \in \mathscr{T}_X$ and define $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{N}_X[\mathscr{A}] \to \mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}$ by $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{N}_X(A)) = [\mathbf{0}; A, 2^{-n}]$. Then define $\overrightarrow{T}_{\mathrm{II},n} : \mathrm{MC}(X) \times \mathscr{N}_X[\mathscr{A}] \to \mathscr{T}_X$ by $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{N}_X(A)) = \Phi^{\leftarrow}[(-2^{-n}, 2^{-n})]$. Note that, if $\Phi \in [\mathbf{0}; A, 2^{-n}] = \overleftarrow{T}_{\mathrm{I},n}(\mathscr{N}_X(A))$, then $A \subseteq \Phi^{\leftarrow}[(-2^{-n}, 2^{-n})]$, which establishes that $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{N}_X(A)) \in \mathscr{N}_X(A)$.

Suppose we have

$$\langle \Phi_n : n \in \omega \rangle \in \prod_{n \in \omega} \overleftarrow{T}_{\mathrm{I},n}(\mathscr{N}_X(A_n))$$

for a sequence $\langle A_n : n \in \omega \rangle$ of $\mathscr{A}$ so that $\langle \Phi_n : n \in \omega \rangle \notin \Gamma_{\mathrm{MC}_{\mathscr{B}}(X),\mathbf{0}}$. Then we can find $B \in \mathscr{B}$, $\varepsilon > 0$, and $N \in \omega$ so that $2^{-N} < \varepsilon$ and, for all $n \geqslant N$, $\Phi_n \notin [\mathbf{0}; B, \varepsilon]$.

To finish this application of Theorem 14, we need to show that

$$B \not\subseteq \overrightarrow{T}_{\mathrm{II},n}(\Phi_n, \mathscr{N}_X(A_n))$$

for all $n \geqslant N$. So let $n \geqslant N$ and note that, since $\Phi_n \notin [\mathbf{0}; B, \varepsilon]$, there is some $x \in B$ and $y \in \Phi_n(x)$ so that $|y| \geqslant \varepsilon > 2^{-N} \geqslant 2^{-n}$. That is, $\Phi_n(x) \not\subseteq (-2^{-n}, 2^{-n})$ and so $x \notin \overrightarrow{T}_{\mathrm{II},n}(\Phi_n, \mathscr{N}_X(A_n))$. This finishes the proof. □

Though particular applications of Corollary 42 and Theorem 44 abound, we record a few that capture the general spirit using ideals of usual interest after recalling some other facts and some names for particular selection principles.

Definition 46. We identify some particular selection principles by name.

- $\mathsf{S}_1(\Omega_{X,x}, \Omega_{X,x})$ is known as the *strong countable fan-tightness property for X at x*.
- $\mathsf{S}_1(\mathscr{D}_X, \Omega_{X,x})$ is known as the *strong countable dense fan-tightness property for X at x*.
- $\mathsf{S}_1(\mathscr{T}_X, \mathrm{CD}_X)$ is known as the *discretely selective property for X*.
- We refer to $\mathsf{S}_1(\Omega_X, \Omega_X)$ as the *ω-Rothberger property* and $\mathsf{S}_1(\mathscr{K}_X, \mathscr{K}_X)$ as the *k-Rothberger property*.

Definition 47. For a partially ordered set $(\mathbb{P}, \leqslant)$ and collections $\mathscr{A}, \mathscr{B} \subseteq \mathbb{P}$ so that, for every $B \in \mathscr{B}$, there exists some $A \in \mathscr{A}$ with $B \subseteq A$, we define the *cofinality of $\mathscr{A}$ relative to $\mathscr{B}$* by

$$\mathrm{cof}(\mathscr{A}; \mathscr{B}, \leqslant) = \min\{\kappa \in \mathrm{CARD} : (\exists \mathscr{F} \in [\mathscr{A}]^\kappa)(\forall B \in \mathscr{B})(\exists A \in \mathscr{F})\ B \subseteq A\}$$

where CARD is the class of cardinals and $[\mathscr{A}]^\kappa$ is the set of κ-sized subsets of $\mathscr{A}$.

Lemma 48. Let $\mathscr{A}, \mathscr{B} \subseteq \wp^+(X)$ for a space X.

As long as X is T_1,

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) \iff \mathrm{cof}(\mathscr{A}; \mathscr{B}, \subseteq) \leqslant \omega.$$

(See [22, 55], and [5, Lemma 23].)

If $\mathscr{A}$ consists of G_δ sets,

$$\begin{aligned} \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) &\iff \mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) \\ &\iff \mathrm{cof}(\mathscr{A}; \mathscr{B}, \subseteq) \leqslant \omega. \end{aligned}$$

(See [20, 53] and [5, Lemma 24].)

Observe that Lemma 48 informs us that, for a T_1 space X,

- $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathbb{P}_X, \neg\mathscr{O}_X)$ if and only if X is countable,
- $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathscr{O}_X)$ if and only if X is σ-compact, and
- $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathscr{K}_X)$ if and only if X is hemicompact.

Corollary 49. For an ideal $\mathscr{A}$ of closed subsets of a T_1 space X, $\mathrm{cof}(\mathscr{A}; \mathscr{A}, \subseteq) \leqslant \omega$ if and only if $\mathrm{MC}_{\mathscr{A}}(X)$ is metrizable.

Proof. If $\{A_n : n \in \omega\} \subseteq \mathscr{A}$ is so that, for every $A \in \mathscr{A}$, there is an $n \in \omega$ with $A \subseteq A_n$, notice that the family $\{\mathbf{W}(A_n, 2^{-m}) : n, m \in \omega\}$ is a countable base for the uniformity on $\mathrm{MC}_{\mathscr{A}}(X)$; so Theorem 18 demonstrates that $\mathrm{MC}_{\mathscr{A}}(X)$ is metrizable.

Now, suppose $\mathrm{MC}_{\mathscr{A}}(X)$ is metrizable, which implies that $\mathrm{MC}_{\mathscr{A}}(X)$ is first-countable. Using a descending countable basis at $\mathbf{0}$, we see that

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}, \neg\Gamma_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}),$$

and, in particular,

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_{\mathscr{A}}(X),\mathbf{0}}).$$

By Theorem 44, we see that

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{A})).$$

So, by Lemma 48, $\mathrm{cof}(\mathscr{A};\mathscr{A},\subseteq) \leqslant \omega$. □

As a particular consequence of this, we see that

Corollary 50. For any regular space X, the following are equivalent.

(i) X is countable.
(ii) $\mathrm{MC}_p(X)$ is metrizable.
(iii) $\mathrm{MC}_p(X)$ is not discretely selective.
(iv) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\Omega_X, \Omega_X)$.
(v) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\Omega_{\mathrm{MC}_p(X),\mathbf{0}}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$.
(vi) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_p(X)}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$.

Also, the following are equivalent.

(i) X is hemicompact.
(ii) $\mathbb{K}(X)$ is hemicompact. (See [4, Thm. 3.22].)
(iii) $\mathrm{MC}_k(X)$ is metrizable. (See [30, Cor. 4.5].)
(iv) $\mathrm{MC}_k(X)$ is not discretely selective.
(v) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X)$.
(vi) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\Omega_{\mathrm{MC}_k(X),\mathbf{0}}, \Omega_{\mathrm{MC}_k(X),\mathbf{0}})$.
(vii) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_k(X)}, \Omega_{\mathrm{MC}_k(X),\mathbf{0}})$.

Theorem 51 (See [5, Cor. 11] and [55])**.** Let $\mathscr{A}$ and $\mathscr{B}$ be ideals of closed subsets of X. Then

$$\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\Gamma_X(\mathscr{B}))$$

and

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\mathscr{O}_X(\mathscr{B})) \iff \mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathscr{A}], \neg\Gamma_X(\mathscr{B}))$$

Corollary 52. For any regular space X, the following are equivalent.

(i) $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_X, \Omega_X)$.
(ii) $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_{\mathrm{MC}_p(X),\mathbf{0}}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$.
(iii) $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_p(X)}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$.
(iv) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_p(X)}, \mathrm{CD}_{\mathrm{MC}_p(X)})$.

(v) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Omega_X)$.
(vi) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Gamma_\omega(X))$.

Also, the following are equivalent.

(i) $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X)$.
(ii) $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_{\mathrm{MC}_k(X),\mathbf{0}}, \Omega_{\mathrm{MC}_k(X),\mathbf{0}})$.
(iii) $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{D}_{\mathrm{MC}_k(X)}, \Omega_{\mathrm{MC}_k(X),\mathbf{0}})$.
(iv) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_k(X)}, \mathrm{CD}_{\mathrm{MC}_k(X)})$.
(v) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathscr{K}_X)$.
(vi) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\Gamma_k(X))$.

In general, Corollaries 50 and 52 are strictly separate, as the following example demonstrates.

Example 53. Let X be the one-point Lindelöfication of ω_1 with the discrete topology, an instance of a Fortissimo space [52, Space 25] (see also [14]). In [4, Ex. 3.24], it is shown that X has the property that $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X)$, but $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X)$. Since the compact subsets of X are finite, we see also that $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_X, \Omega_X)$, but $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\Omega_X, \Omega_X)$.

However, according to Theorem 54, if Two can win against predetermined strategies in some Rothberger-like games, Two can actually win against full-information strategies in those games.

Theorem 54. Let X be any space.

(i) By Pawlikowski [45],
$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{O}_X, \mathscr{O}_X) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{O}_X, \mathscr{O}_X).$$
(ii) By Scheepers [48] (see also [7, Cor. 4.12]),
$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\Omega_X, \Omega_X) \iff \mathrm{I} \uparrow \mathsf{G}_1(\Omega_X, \Omega_X).$$
(iii) By [7, Thm. 4.21],
$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X).$$

Corollary 55. For any regular space X, the following are equivalent.

(i) X is ω-Rothberger.
(ii) $X^{<\omega}$ is Rothberger, where $X^{<\omega}$ is the disjoint union of X^n for all $n \geqslant 1$. (See [46] and [7, Cor. 3.11].)
(iii) $\mathscr{P}_{\mathrm{fin}}(X)$ is Rothberger, where $\mathscr{P}_{\mathrm{fin}}(X)$ is the set $[X]^{<\omega}$ with the subspace topology inherited from $\mathbb{K}(X)$. (See [7, Cor. 4.11].)
(iv) $\mathrm{I} \not\uparrow \mathsf{G}_1(\Omega_X, \Omega_X)$.
(v) $\mathrm{MC}_p(X)$ has strong countable fan-tightness at $\mathbf{0}$.
(vi) $\mathrm{MC}_p(X)$ has strong countable dense fan-tightness at $\mathbf{0}$.
(vii) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Omega_X)$.
(viii) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Omega_X)$.

(ix) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_p(X)}, \mathrm{CD}_{\mathrm{MC}_p(X)})$.
(x) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_p(X)}, \mathrm{CD}_{\mathrm{MC}_p(X)})$.
(xi) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_p(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_p(X),\mathbf{0}})$.
(xii) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_p(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_p(X),\mathbf{0}})$.

Also, the following are equivalent.

(i) X is k-Rothberger.
(ii) $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathscr{K}_X, \mathscr{K}_X)$.
(iii) $\mathrm{MC}_k(X)$ has strong countable fan-tightness at $\mathbf{0}$.
(iv) $\mathrm{MC}_k(X)$ has strong countable dense fan-tightness at $\mathbf{0}$.
(v) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathscr{K}_X)$.
(vi) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathscr{K}_X)$.
(vii) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_k(X)}, \mathrm{CD}_{\mathrm{MC}_k(X)})$.
(viii) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{T}_{\mathrm{MC}_k(X)}, \mathrm{CD}_{\mathrm{MC}_k(X)})$.
(ix) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_k(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_k(X),\mathbf{0}})$.
(x) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_{\mathrm{MC}_k(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_k(X),\mathbf{0}})$.

We end this section with a couple examples that demonstrate the difference between these two classes of results.

Example 56. The reals $\mathbb{R}$ are hemicompact (and thus k-Rothberger) but not ω-Rothberger. Indeed, since every ω-Rothberger space is Rothberger by [46] and $\mathbb{R}$ is not Rothberger, $\mathbb{R}$ is not ω-Rothberger. In particular, $\mathrm{MC}_k(\mathbb{R})$ is metrizable whereas $\mathrm{MC}_p(\mathbb{R})$ does not even have countable strong fan-tightness at $\mathbf{0}$.

Example 57. The rationals $\mathbb{Q}$ are countable (and thus ω-Rothberger) but, by [8, Prop. 5], not k-Rothberger. In particular, $\mathrm{MC}_p(\mathbb{Q})$ is metrizable whereas $\mathrm{MC}_k(\mathbb{Q})$ does not have countable strong fan-tightness at $\mathbf{0}$.

4. Obstacles to generalization

In this section, we discuss difficulties that arise in trying to extend Theorems 31 and 32 to classes of usco maps other than minimal usco and cusco maps.

First, notice that not every selection of a minimal cusco map is quasicontinuous and subcontinuous, as suggested by Theorem 32. Since every usco map contains a minimal usco map, every usco map has some quasicontinuous and subcontinuous selection. However, the key to making sure that those selections from a minimal cusco map Φ bring one back to Φ via closure and convex hull, as in Theorem 32, is convexity. Without an analogous structure in place, there is no clear way to make sure a correspondence of this type holds for other classes of usco maps.

Inspired by the operation of the convex hull, one my think similar additions may generate interesting examples. However, adding points to the vertical sections of a minimal usco map may create a graph which is not closed.

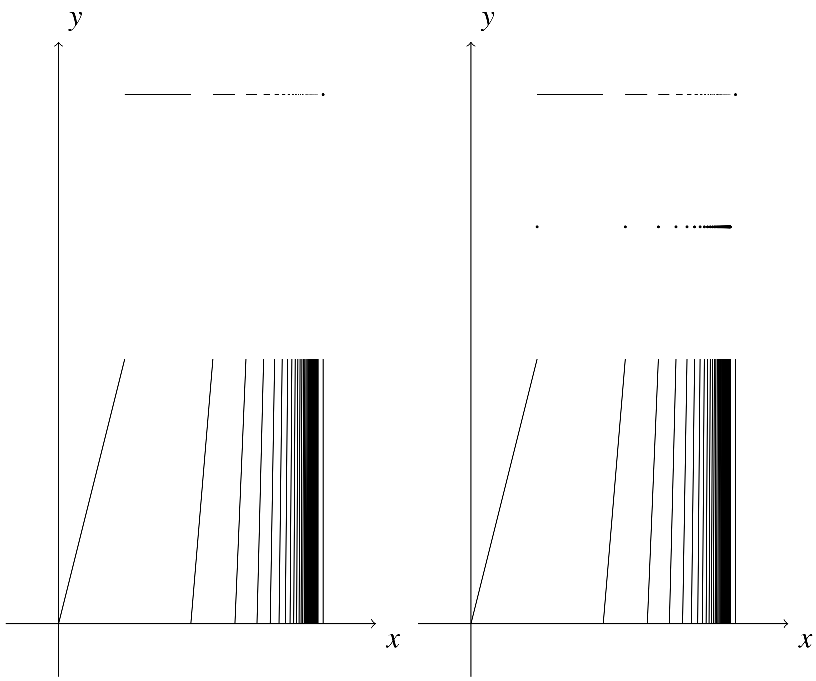

FIGURE 1. Adding midpoints

Example 58. Let $a_n = \frac{n}{n+1}$, $b_n = \frac{n+1}{n+2}$, and $\mathrm{mid}_n = \frac{a_n+b_n}{2}$. Define the funtion $f : (0,1) \to [0,2]$ by

$$f(x) = \begin{cases} \frac{2}{b_n-a_n}(x-a_n) & x \in (a_n, \mathrm{mid}_n) \\ 2 & x \in [\mathrm{mid}_n, b_n] \end{cases}$$

Then f is quasicontinuous and subcontinuous by Lemma 36. Thus, $\overline{f}$ is minimal usco. However,

$$G = \overline{f} \cup \left\{ \left(x, \frac{\max \overline{f}(x) + \min \overline{f}(x)}{2} \right) : x \in [0,1] \right\},$$

the map created by adding the midpoint of each vertical section, is not usco. Indeed, let $\varepsilon > 0$ be small enough so that $\frac{3}{2} \notin W := (-\varepsilon, 1+\varepsilon) \cup (2-\varepsilon, 2+\varepsilon)$. Then $1 \in G^{\leftarrow}(W)$, but for every N, there is an $x \in (1 - \frac{1}{N}, 1]$ so that $\frac{3}{2} \in G(x)$; thus $x \notin G^{\leftarrow}(W)$, meaning this preimage is not open, and therefore G is not usco. The graphs of $\overline{f}$ and G are in Figure 1.

Note, however, by [10], we can take the closure $\overline{G}$ of the G defined in Example 58 to create an usco map since the resulting graph will be contained in the graph of a cusco map. Unfortunately, it's not clear what kind of structural facts could be used to ensure that quasicontinuous and subcontinuous selections generate the same given map. For example, one could add the singleton $\{0\}$ to every section when mapping into the reals and, in the end, there would be maps with distinct selections that are quasicontinuous and subcontinuous but that don't generate the original map with the given procedure. In the convex setting, one can use half-spaces to separate compact convex sets from convex sets to

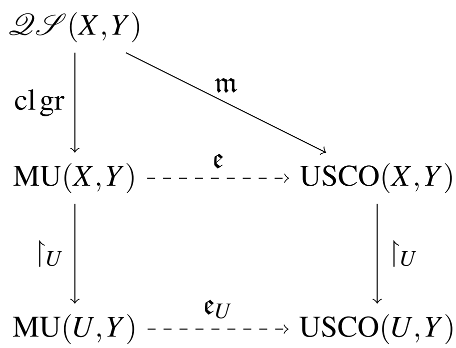

FIGURE 2. Commuting diagram

eventually arrive at Theorem 32. These tools offered by the convex setting do not adapt to the operation of adding midpoints as described above.

More broadly, we would like to find maps that complete the commuting diagram in Figure 2, but is not clear at this time if anything other than minimal usco (where $\mathfrak{m}$ is the closure and $\mathfrak{e}$ is the identity) and minimal cusco (where $\mathfrak{m}$ is the point-wise convex hull of the closure and $\mathfrak{e}$ is the point-wise convex hull) maps work.

5. Questions

We end with a few questions.

Question 1. As mentioned after Corollary 42, many of the equivalences here can be expanded to include $\mathrm{MU}_{\mathscr{A}}(X)$ and, under the additional assumption that X is functionally $\mathscr{A}$-normal, even the appropriately topologized ring $C_{\mathscr{A}}(X)$ of continuous real-valued function from X. Is there a more general theory, perhaps relative to the set of quasicontinuous and subcontinuous real-valued functions, that unifies all of these results?

Question 2. In general, one could define an operator $\mathscr{C} : K(Y) \to K(Y)$ to apply to the outputs of minimal usco maps. In the cusco case, $\mathscr{C}$ is the convex hull. For what kind of operators $\mathscr{C}$ do we obtain analogues to Theorems 31 and 32?

Question 3. Are there other maps $\mathfrak{m}$ as in Figure 2 that make the diagram commute?

Question 4. Can results similar to Theorems 41 and 44 be established relative to $\Omega_{\mathrm{MC}_{\mathscr{A}}(X),\Phi}$ for any $\Phi \in \mathrm{MC}(X)$?

Question 5. How many of the equivalences and dualities of this paper can be established for games of longer length and for finite-selection games?

Question 6. How much of this theory can be recovered when we study $\mathrm{MC}(X,Y)$ for $Y \neq \mathbb{R}$, for example, when Y is $[0,1]$, $Y = \mathbb{S}^1$ (the circle group), or Y is a general Hausdorff locally convex linear space?

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