CONTINUOUS AND COMPACTNESS PROPERTIES FOR TEMPERED FRACTIONAL INTEGRALS

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Abstract. Our aim in this paper is to deal with Fréchet-Kolmogorov compactness of Riemann-Liouville tempered fractional integrals on Lebesgue spaces and other related compactness properties associated to the behavior of $\alpha \in (0,1)$. Moreover, we consider some boundedness properties for the ψ-tempered fractional integrals on Lebesgue spaces.

1. Introduction

Integral operators play a key role in several areas of mathematics, such as the theory of Fourier series and Fourier integrals, approximation theory and the theory of linear as well as nonlinear ordinary and partial differential equations.

Integral operators appear naturally with various kernels as fundamental solutions in the construction of the general solutions for a wide range of differential equations [3, 4]. One type of integral operator of particular interest is the so-called tempered fractional integrals which belongs to tempered fractional calculus field. Mathematically, tempered fractional calculus is of interest as the unique intersection between weighted fractional calculus and fractional calculus with analytic kernels [4]. In modeling, tempered fractional calculus it has been used to understand turbulence in geophysical flows and Lévy processes such as Brownian motion. Especially the stochastic applications, such as tempered Lévy flights, have led to a great deal of advanced mathematical research including solving multi-dimensional PDEs by both analytical and numerical methods [1, 7, 10, 15].

In this paper we employ some relatively simple techniques from functional analysis to prove some important compactness results for Riemann-Liouville tempered fractional integrals. The compactness results are of interest because they have a significant role in the proof of existence of solution for nonlinear differential equations. More precisely we consider the following results:

Theorem 1.1. *Let $\alpha \in (0,1)$, $\sigma > 0$ and $p \in [1,\infty)$. Then the Riemann-Liouville tempered fractional integrals $\mathbb{I}^{\alpha,\sigma}_{a^+}, \mathbb{I}^{\alpha,\sigma}_{b^-} : L^p(a,b) \to L^p(a,b)$ are compact.*

Theorem 1.2. *Let $\alpha \in (0,\frac{1}{p})$ and $\sigma > 0$. The Riemann-Liouville tempered fractional integrals $\mathbb{I}^{\alpha,\sigma}_{a^+}, \mathbb{I}^{\alpha,\sigma}_{b^-} : L^p(a,b) \to L^q(a,b)$ are compact for every $q \in [1, p^*_\alpha)$, where $p^*_\alpha = \frac{p}{1-\alpha p}$*

Theorem 1.3. *Let $\alpha = \frac{1}{p}$ and $\sigma > 0$. Then, the Riemann-Liouville tempered fractional integrals $\mathbb{I}^{\alpha,\sigma}_{a^+}, \mathbb{I}^{\alpha,\sigma}_{b^-} : L^p(a,b) \to L^q(a,b)$ are compact for every $q \in [1,\infty)$.*

Finally, in the particular case $p = 2$ we prove that:

Theorem 1.4. *Let $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$. The Riemann-Liouville tempered fractional integrals $\mathbb{I}^{\alpha,\sigma}_{a^+}, \mathbb{I}^{\alpha,\sigma}_{b^-} : L^2(a,b) \to C[a,b]$ are compact.*

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7 Jun 2023 20:15:30 PDT
230607-TorresLedesma Version 1 - Submitted to Rocky Mountain J. Math.

On the other hand, we study some boundedness result considering the ψ-tempered fractional integrals which is a generalization of the Riemann-Liouville tempered fractional integrals. Recently Fahad et al [5] investigated tempered and Hadamard-type fractional calculi together, and the generalization of both which is given by taking the operators with respect to an arbitrary monotonic function. Such a generalization can be thought of as ψ-tempered fractional calculus, and it is a special case both of fractional calculus with analytic kernels with respect to functions and of weighted fractional calculus with respect to functions, for more detail see [3, 4, 8, 11]. To state our main results in this direction, let $\alpha \in (0,1)$, $\sigma > 0$ and let $\psi : [a,b] \to \mathbb{R}$ be a C^1 function such that $\psi' > 0$ on $[a,b]$. Then the tempered fractional integral of order α and index σ with respect to ψ is defined as

$$\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}u(s)ds.$$

Considering this integral we have the following results.

Theorem 1.5. *Let $\alpha \in (0,1)$, $\sigma > 0$ and $p \in [1,\infty]$. Suppose that $\psi' : [a,b] \to \mathbb{R}$ is an increasing function, then, the tempered fractional integral $\mathbb{I}^{\alpha,\sigma}_{a^+,\psi} : L^p(a,b) \to L^p(a,b)$ is bounded. Moreover*

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^p(a,b)} \leq \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)}.$$

Theorem 1.6. *Let $p \geq 1$, $\sigma > 0$, $\alpha \in (0,\frac{1}{p})$ and suppose that ψ' is an increasing function on $[a,b]$, then, the tempered fractional integral $\mathbb{I}^{\alpha,\sigma}_{a^+,\psi} : L^p(a,b) \to L^q(a,b)$ is bounded for every $q \in [1,\frac{p}{1-\alpha p})$*

Theorem 1.7. *Let $\alpha = \frac{1}{p}$ and $\sigma > 0$. Suppose that ψ' is an increasing function on $[a,b]$, then, the tempered fractional integral $\mathbb{I}^{\frac{1}{p},\sigma}_{a^+,\psi} : L^p(a,b) \to L^q(a,b)$ is bounded for every $q \in [1,\infty)$.*

Finally we consider the case $p = 2$ and $\alpha \in (\frac{1}{2},1)$ and we suppose a technical condition on ψ. More precisely we have.

Theorem 1.8. *Let $\alpha \in (\frac{1}{2},1)$ and $\sigma > 0$. Suppose that ψ is a Lipschitz function on $[a,b]$ that is, there is a positive constant L such that*

$$|\psi(x_1)-\psi(x_2)| \leq L|x_1-x_2|, \quad \forall x_1,x_2 \in [a,b].$$

Moreover suppose that ψ' is an increasing function on $[a,b]$, then the tempered fractional integral $\mathbb{I}^{\alpha,\sigma}_{a^+,\psi} : L^2(a,b) \to H^{\alpha-\frac{1}{2}}[a,b)$ is bounded. Furthermore

$$\lim_{x\to a^+}\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x) = 0.$$

Here $H^{\alpha-\frac{1}{2}}[a,b)$ denotes tha Hölder space of order $\alpha-\frac{1}{2} > 0$.

This paper is organized as follows. In Section 2, the lower gamma incomplete function, Riemann-Liouville tempered fractional integrals are introduced briefly along with some boundedness results of Riemann-Liouville tempered fractional integrals. Section 3 present some compactness results of Riemann-Liouville tempered fractional integrals Our results in this section are based on Arzelá-Ascoli and Fŕchet-Kolmogorov compactness theorems. Finally, in section 4 we deal with the ψ-tempered fractional integral, under some suitable conditions over function ψ we are able to show some boundedness on Lebesgue spaces. We note that our results obtained in section 3 and section 4 are very general which will include some specific cases such as the well-known Riemann-Liouville fractional integrals.

7 Jun 2023 20:15:30 PDT
230607-TorresLedesma Version 1 - Submitted to Rocky Mountain J. Math.

2. Some previous results

For $\alpha > 0$ and $x \geq 0$, the incomplete Gamma function is defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt$$

which is convergen for all $\alpha > 0$. Moreover we have the following estimates

$$e^{-x}\frac{x^\alpha}{\alpha} \leq \gamma(\alpha, x) \leq \frac{x^\alpha}{\alpha} \tag{2.1}$$

and by using integration by parts, we see that

$$\gamma(\alpha+1, x) = \alpha\gamma(\alpha, x) - x^\alpha e^{-x}. \tag{2.2}$$

This equality can be used to extend the definition of $\gamma(\alpha, x)$ to negative, non integer values of α. For example, if $\alpha \in (-1, 0)$ and $x > 0$, then

$$\gamma(\alpha, x) = \frac{1}{\alpha}\gamma(\alpha+1, x) + \frac{1}{\alpha}x^\alpha e^{-x}. \tag{2.3}$$

For more details the reader's can see [6].

Let $\alpha \in (0,1)$ and $\sigma \geq 0$. The left and right Riemann-Liouville tempered fractional integrals of order α are defined as

$$\mathbb{I}_{a^+}^{\alpha,\sigma}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}e^{-\sigma(x-s)}u(s)ds, \quad x > a, \tag{2.4}$$

and

$$\mathbb{I}_{b^-}^{\alpha,\sigma}u(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}e^{-\sigma(s-x)}u(s)ds, \quad x < b, \tag{2.5}$$

respectively.

These integrals have the following L^p boundedness properties

Theorem 2.1. *[12] Let $\alpha \in (0,1)$, $\sigma > 0$, $p \in [1, \infty]$. Then, the tempered fractional integrals of Riemann-Liouville $\mathbb{I}_{a^+}^{\alpha,\sigma}, \mathbb{I}_{b^-}^{\alpha,\sigma} : L^p(a,b) \to L^p(a,b)$ are bounded. Moreover*

$$\|\mathbb{I}_{a^+}^{\alpha,\sigma}u\|_{L^p(a,b)} \leq \frac{\gamma(\alpha, \sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)}. \tag{2.6}$$

and

$$\|\mathbb{I}_{b^-}^{\alpha,\sigma}u\|_{L^p(a,b)} \leq \frac{\gamma(\alpha, \sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)} \tag{2.7}$$

Moreover depending of α we have the following result.

Theorem 2.2. *[12]*

(1) Let $p > 1$, $\alpha \in \left(0, \frac{1}{p}\right)$, $\sigma > 0$ and $p_\alpha^* = \frac{p}{1-\alpha p}$. Then, the Riemann-Liouville tempered fractional integrals $\mathbb{I}_{a^+}^{\alpha,\sigma}, \mathbb{I}_{b^-}^{\alpha,\sigma}$ are bounded from $L^p(a,b)$ into $L^{p_\alpha^*}(a,b)$.

(2) Let $\alpha \in (0,1)$, $\sigma > 0$ and $p = \frac{1}{\alpha}$. Then the tempered fractional integrals of Riemann-Liouville $\mathbb{I}_{a^+}^{\alpha,\sigma}, \mathbb{I}_{b^-}^{\alpha,\sigma}$ are bounded from $L^{\frac{1}{\alpha}}(a,b)$ into $L^q(a,b)$ for every $q \in [1, \infty)$.

(3) Let $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$. Then, for each $u \in L^2(a,b)$, $\mathbb{I}_{a^+}^{\alpha,\sigma}u \in H^{\alpha-\frac{1}{2}}(a,b)$ and

$$\lim_{x\to a^+}\mathbb{I}_{a^+}^{\alpha,\sigma}u(x) = 0,$$

where $H^{\alpha-\frac{1}{2}}(a,b)$ denotes the Hölder space of order $\alpha - \frac{1}{2} > 0$.

3. Compactness property of Riemann-Liouville tempered fractional integrals

In this section we deal with the compactness property of Riemann-Liouville tempered fractional integrals. We start our analysis with the Fréchet-Kolmogorov compactness on L^p, more precisely we are going to prove Theorem 1.1.

Proof of Theorem 1.1. Lep $p = 1$, $B_1 = \{u \in L^1(a,b) : \|u\|_{L^1(a,b)} \leq 1\}$ and $h > 0$. For $u \in B_1$ we have

$$\begin{aligned}
&\int_{a+h}^{b-h} |\mathbb{I}_{a^+}^{\alpha,\sigma} u(x+h) - \mathbb{I}_{a^+}^{\alpha,\sigma} u(x)| dx \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} \int_a^x \left| (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} - (x-s)^{\alpha-1} e^{-\sigma(x-s)} \right| |u(s)| ds dx \\
&+ \frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} \int_x^{x+h} (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} |u(s)| ds dx
\end{aligned}$$

By doing the change of variable $t = s - x$ and using Fubini's theorem we have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} \int_x^{x+h} (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} |u(s)| ds dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^h (h-t)^{\alpha-1} e^{-\sigma(h-t)} \int_{a+h}^{b-h} |u(t+h)| dx dt \\
&\leq \frac{1}{\Gamma(\alpha)} \|u\|_{L^1(a,b)} \int_0^h (h-t)^{\alpha-1} e^{-\sigma(h-t)} dt = \frac{1}{\sigma^\alpha \Gamma(\alpha)} \gamma(\alpha, \sigma h) \|u\|_{L^1(a,b)}
\end{aligned} \tag{3.1}$$

By other side, note that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} \int_a^x \left| (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} - (x-s)^{\alpha-1} e^{-\sigma(x-s)} \right| |u(s)| ds dx \\
&= \frac{1}{\Gamma(\alpha)} \left(\int_{a+h}^{b-h} \int_a^{a+h} + \int_{a+h}^{b-h} \int_{a+h}^x \right) \left| (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} - (x-s)^{\alpha-1} e^{-\sigma(x-s)} \right| |u(s)| ds dx.
\end{aligned}$$

Let

$$\begin{aligned}
\Sigma_1 &= \int_{a+h}^{b-h} \int_a^{a+h} \left| (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} - (x-s)^{\alpha-1} e^{-\sigma(x-s)} \right| |u(s)| ds dx \\
\Sigma_2 &= \int_{a+h}^{b-h} \int_{a+h}^x \left| (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} - (x-s)^{\alpha-1} e^{-\sigma(x-s)} \right| |u(s)| ds dx.
\end{aligned}$$

To Σ_2 we have

$$\begin{aligned}
\Sigma_2 &= \frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} \int_s^{b-h} \left((x-s)^{\alpha-1} e^{-\sigma(x-s)} - (x+h-s)^{\alpha-1} e^{-\sigma(x+h-s)} \right) dx |u(s)| ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} |u(s)| \left(\frac{1}{\sigma^\alpha} \gamma(\alpha, \sigma h) - \frac{1}{\sigma^\alpha} [\gamma(\alpha, \sigma(b-s)) - \gamma(\alpha, \sigma(b-s-h))] \right) ds \\
&\leq \frac{1}{\sigma^\alpha \Gamma(\alpha)} \gamma(\alpha, \sigma h) \|u\|_{L^1(a,b)}.
\end{aligned}$$

In the same way, Fubini's theorem yields that

$$\begin{aligned}\Sigma_1 &= \frac{1}{\Gamma(\alpha)}\int_a^{a+h}\int_{a+h}^{b-h}\left((x-s)^{\alpha-1}e^{-\sigma(x-s)}-(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}\right)dx|u(s)|ds\\
&= \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_a^{a+h}[\gamma(\alpha,\sigma(b-h-s))-\gamma(\alpha,\sigma(a+h-s))-\gamma(\alpha,\sigma(b-s))+\gamma(\alpha,\sigma(a+2h-s))]|u(s)|ds\\
&\leq \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_a^{a+h}[\gamma(\alpha,\sigma(1+2h-s))-\gamma(\alpha,\sigma(a+h-s))]|u(s)|ds\\
&\leq \frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,2\sigma h)\|u\|_{L^1(a,b)}.\end{aligned}$$

Consequently

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_{a+h}^{b-h}\int_a^x\left|(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}-(x-s)^{\alpha-1}e^{-\sigma(x-s)}\right||u(s)|dsdx\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}+\frac{\gamma(\alpha,2\sigma h)}{\sigma^\alpha}\right)\|u\|_{L^1(a,b)}.\end{aligned}\tag{3.2}$$

Therefore, by (3.1), (3.2) we obtain

$$\int_{a+h}^{b-h}|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x+h)-\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|dx\leq\frac{1}{\Gamma(\alpha)}\left(\frac{2\gamma(\alpha,\sigma h)}{\sigma^\alpha}+\frac{\gamma(\alpha,2\sigma h)}{\sigma^\alpha}\right).$$

Moreover, (2.1) implies that

$$\lim_{h\to0^+}\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}=\lim_{h\to0^+}\frac{\gamma(\alpha,2\sigma h)}{\sigma^\alpha}=0,$$

which implies that

$$\lim_{h\to0^+}\int_{a+h}^{b-h}|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x+h)-\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|dx=0$$

On the other hand, if $u\in B_1$, Theorem 2.1 yields that

$$\|\mathbb{I}_{a^+}^{\alpha,\sigma}u\|_{L^1(a,b)}\leq\frac{\gamma(\alpha,\sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}$$

from where we get

$$\int_a^b|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|dx=\lim_{h\to0^+}\int_{a+h}^{b-h}|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|dx,$$

so, for any $\epsilon>0$, there is $\delta>0$ such that:

$$\int_{(a,b)\setminus[a+h,b-h]}|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|dx<\epsilon\ \text{ whenever }\ 0<h<\delta.$$

Therefore, by [2, Theorem 1.95], $\mathbb{I}_{a^+}^{\alpha,\sigma}(B_1)$ is precompact in $L^1(a,b)$.

Now we consider the case $p\in(1,\infty)$. Let $\mathcal{H}\subset L^p(a,b)$ be a bounded set, that is

$$\|u\|_{L^p(a,b)}\leq C,\ \ \forall u\in\mathcal{H}\ \text{ and for some } C>0.$$

Now, for every $u\in\mathcal{H}$, Theorem 2.1 implies that

$$\|\mathbb{I}_{a^+}^{\alpha,\sigma}u\|_{L^p(a,b)}\leq\frac{\gamma(\alpha,\sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)}\leq C\frac{\gamma(\alpha,\sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}.$$

7 Jun 2023 20:15:30 PDT
230607-TorresLedesma Version 1 - Submitted to Rocky Mountain J. Math.

On the other hand, for $h>0$ we have

$$|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x+h)-\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)| \leq \frac{1}{\Gamma(\alpha)}\int_a^x \left|(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}-(x-s)^{\alpha-1}e^{-\sigma(x-s)}\right||u(s)|ds + \frac{1}{\Gamma(\alpha)}\int_x^{x+h}(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}|u(s)|ds.$$

By Hölder inequality we have

$$\begin{aligned}
&\int_x^{x+h}(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}|u(s)|ds\\
&\leq \left(\int_x^{x+h}(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}ds\right)^{1/q}\left(\int_x^{x+h}(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}|u(s)|^pds\right)^{1/p}\\
&\leq \left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}\right)^{1/q}\left(\int_x^{x+h}(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}|u(s)|^pds\right)^{1/p}
\end{aligned}$$

By other side, denote by $k(x,s)=(x-s)^{\alpha-1}e^{-\sigma(x-s)}$ then

$$\begin{aligned}
&\int_a^x \left|(x+h-s)^{\alpha-1}e^{-\sigma(x+h-s)}-(x-s)^{\alpha-1}e^{-\sigma(x-s)}\right||u(s)|ds = \int_a^x |k(x+h,s)-k(x,s)||u(s)|ds\\
&\leq \left(\int_a^x |k(x+h,s)-k(x,s)|ds\right)^{1/q}\left(\int_a^x |k(x+h,s)-k(x,s)||u(s)|^pds\right)^{1/p}\\
&= \left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}-\frac{\gamma(\alpha,\sigma(x-a+h))-\gamma(\alpha,\sigma(x-a))}{\sigma^\alpha}\right)^{1/q}\left(\int_a^x |k(x+h,s)-k(x,s)||u(s)|^pds\right)^{1/p}\\
&\leq \left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}\right)^{1/q}\left(\int_a^x |k(x+h,s)-k(x,s)||u(s)|^pds\right)^{1/p}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x+h)-\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|^p\\
&\leq \left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}\right)^{p/q}\left(\left(\int_x^{x+h}k(x+h,s)|u(s)|^pds\right)^{1/p}+\left(\int_a^x |k(x+h,s)-k(x,s)||u(s)|^pds\right)^{1/p}\right)^p\\
&\leq 2^p\left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}\right)^{\frac{p}{q}}\left(\int_x^{x+h}k(x+h,s)|u(s)|^pds+\int_a^x |k(x+h,s)-k(x,s)||u(s)|^pds\right).
\end{aligned}$$

Hence, by using Fubini's theorem we obtain

$$\begin{aligned}
&\int_a^b |\mathbb{I}_{a^+}^{\alpha,\sigma}u(x+h)-\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|^pdx\\
&\leq 2^p\left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}\right)^{\frac{p}{q}}\left(\int_a^b\int_x^{x+h}k(x+h,s)|u(s)|^pdsdx+\int_a^b\int_a^x |k(x+h,s)-k(x,s)||u(s)|^pdsdx\right)\\
&\leq 2^p\left(\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}\right)^{\frac{p}{q}+1}\|u\|_{L^p(a,b)}^p.
\end{aligned}$$

Note that, by (2.1)

$$\lim_{h\to 0^+}\frac{\gamma(\alpha,\sigma h)}{\sigma^\alpha}=0.$$

Therefore

$$\lim_{h\to 0^+}\sup_{u\in\mathcal{H}}\|\mathbb{I}^{\alpha,\sigma}_{a^+}u(\cdot+h)-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^p(a,b)}=0.$$

By Fréchet-Kolmogorov theorem [9, Theorem 1.3] we get the affirmation. □

Proof of Theorem 1.2. Letting $q\in[1,p^*_\alpha)$ and $\theta\in(0,1]$ such that

$$\frac{1}{q}=\frac{\theta}{1}+\frac{1-\theta}{p^*_\alpha},$$

then, by Hölder inequality we have

$$\begin{aligned}\|\mathbb{I}^{\alpha,\sigma}_{a^+}u(\cdot+h)-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^q(a+h,b-h)}&\leq\|\mathbb{I}^{\alpha,\sigma}_{a^+}u(\cdot+h)-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|^{\theta}_{L^1(a+h,b-h)}\|\mathbb{I}^{\alpha,\sigma}_{a^+}u(\cdot+h)-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|^{1-\theta}_{L^{p^*_\alpha}(a+h,b-h)}\\&\leq\|\mathbb{I}^{\alpha,\sigma}_{a^+}u(\cdot+h)-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|^{\theta}_{L^1(a+h,b-h)}\left(2\|\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^{p^*_\alpha}(a,b)}\right)^{1-\theta}.\end{aligned}$$

Consequently, Theorem 1.1 yields that

$$\lim_{h\to 0}\|\mathbb{I}^{\alpha,\sigma}_{a^+}u(\cdot+h)-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^q(a+h,b-h)}=0.$$

By other side, if $u\in B_p=\{v\in L^p(a,b):\|v\|_{L^p(a,b)}\leq 1\}$, Hölder inequality implies

$$\begin{aligned}\|\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^q((a,b)\setminus(a+h,b-h))}&\leq|(a,b)\setminus(a+h,b-h)|^{1-\frac{q}{p^*_\alpha}}\|\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^{p^*_\alpha}((a,b)\setminus(a+h,b-h))}\\&\leq C_{p^*_\alpha}|(a,b)\setminus(a+h,b-h)|^{1-\frac{q}{p^*_\alpha}}\to 0\quad\text{as}\quad h\to 0.\end{aligned}$$

Therefore, by [2, Theorem 1.95], $\mathbb{I}^{\alpha,\sigma}_{a^+}(B_p)$ is precompact in $L^q(a,b)$ for every $q\in[1,p^*_\alpha)$. □

To give the proof of Theorem 1.3, we need the following continuity property of the Tempered fractional integral with respect to the order of integration, that is:

Lemma 3.1. *Let $\alpha,\alpha_0\in(0,1)$ and $\sigma>0$. Then for any $u\in L^p(a,b)$ we have*

$$\lim_{\alpha\to\alpha_0}\|\mathbb{I}^{\alpha_0,\sigma}_{a^+}u-\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^p(a,b)}=0.$$

Proof. Note that

$$\begin{aligned}\mathbb{I}^{\alpha_0,\sigma}_{a^+}u(x)-\mathbb{I}^{\alpha,\sigma}_{a^+}u(x)&=\left(\frac{1}{\Gamma(\alpha_0)}-\frac{1}{\Gamma(\alpha)}\right)\int_a^x(x-s)^{\alpha_0-1}e^{-\sigma(x-s)}u(s)ds\\&+\frac{1}{\Gamma(\alpha)}\int_a^x[(x-s)^{\alpha_0-1}-(x-s)^{\alpha-1}]e^{-\sigma(x-s)}u(s)ds\\&=\Pi_1u(x)+\Pi_2u(x).\end{aligned}$$

Now we estimate the L^p norm of Π_1 and Π_2. Theorem 2.1 yields that

$$\begin{aligned}\|\Pi_1u\|_{L^p(a,b)}&=\left|1-\frac{\Gamma(\alpha_0)}{\Gamma(\alpha)}\right|\|\mathbb{I}^{\alpha_0,\sigma}_{a^+}u\|_{L^p(a,b)}\\&\leq\left|1-\frac{\Gamma(\alpha_0)}{\Gamma(\alpha)}\right|\frac{\gamma(\alpha_0,\sigma(b-a))}{\sigma^{\alpha_0}\Gamma(\alpha_0)}\|u\|_{L^p(a,b)}.\end{aligned}$$

Moreover, doing the change of variable $t = x - s$ and using generalized Minkowski inequality we get

$$\begin{aligned}\|\Pi_2 u\|_{L^p(a,b)} &= \left(\int_a^b |\Pi_2 u(x)|^p dx\right)^{1/p}\\ &= \left(\int_a^b \left|\frac{1}{\Gamma(\alpha)}\int_a^{x-a} t^{\alpha_0-1}(1-t^{\alpha-\alpha_0})e^{-\sigma t}u(x-t)dt\right|^p dx\right)^{1/p}\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^b \left(\int_0^{b-a} t^{\alpha_0-1}|1-t^{\alpha-\alpha_0}|e^{-\sigma t}|u(x-t)|dt\right)^p dx\right)^{1/p}\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{b-a} t^{\alpha_0-1}|1-t^{\alpha-\alpha_0}|e^{-\sigma t}\left(\int_a^b |u(x-s)|^p dx\right)^{1/p} dt\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{b-a} t^{\alpha_0-1}|1-t^{\alpha-\alpha_0}|e^{-\sigma t}dt\|u\|_{L^p(a,b)}\end{aligned}$$

Consequently, by the previous estimates we have

$$\|\mathbb{I}_{a^+}^{\alpha_0,\sigma}u - \mathbb{I}_{a^+}^{\alpha,\sigma}u\|_{L^p(a,b)} \leq \left(\left|1-\frac{\Gamma(\alpha_0)}{\Gamma(\alpha)}\right|\frac{\gamma(\alpha_0,\sigma(b-a))}{\sigma^{\alpha_0}\Gamma(\alpha_0)} + \frac{1}{\Gamma(\alpha)}\int_0^{b-a} t^{\alpha_0-1}|1-t^{\alpha-\alpha_0}|e^{-\sigma t}dt\right)\|u\|_{L^p(a,b)}.$$

Therefore, by the continuity of Γ

$$\lim_{\alpha\to\alpha_0}\|\mathbb{I}_{a^+}^{\alpha_0,\sigma}u - \mathbb{I}_{a^+}^{\alpha,\sigma}u\|_{L^p(a,b)} = 0, \quad \forall\sigma > 0.$$

□

Proof of Theorem 1.3. Let $\epsilon > 0$ and $\alpha_\epsilon = \alpha - \epsilon = \frac{1}{p} - \epsilon < \frac{1}{p}$. Then

$$p_{\alpha_\epsilon}^* = \frac{p}{1-\alpha_\epsilon p} = \frac{1}{\epsilon}.$$

By Theorem 1.2, the tempered fractional integral $\mathbb{I}_{a^+}^{\alpha_\epsilon,\sigma} : L^p(a,b) \to L^q(a,b)$ is compact for every $q \in [1, p_{\alpha_\epsilon}^*)$. Note that as $\epsilon \to 0$, $p_{\alpha_\epsilon}^* \to \infty$. Moreover, for any $u \in L^p(a,b)$ Lemma 3.1 yields that

$$\lim_{\epsilon\to 0}\mathbb{I}_{a^+}^{\alpha_\epsilon,\sigma}u(x) = \mathbb{I}_{a^+}^{\alpha,\sigma}u(x) \text{ a.e. } x \in (a,b).$$

Therefore

$$\mathbb{I}_{a^+}^{\alpha,\sigma} : L^p(a,b) \to L^q(a,b)$$

is compact for every $q \in [1,\infty)$. □

Proof of Theorem 1.4. Let $X \subset L^2(a,b)$ be a bounded set, that is

$$\|u\|_{L^2(a,b)} \leq C \quad \text{for all } u \in X \text{ and for some } C > 0.$$

According to the Arzelá-Ascoli compactness criterion, it suffices to show that $\mathbb{I}_{a^+}^{\alpha,\sigma}(X)$ is bounded and equicontinuous.

For every $u \in X$, by Theorem 2.2-(3) we have

$$\begin{aligned}\|\mathbb{I}_{a^+}^{\alpha,\sigma}u\|_\infty &\leq \frac{\gamma(2\alpha-1, 2\sigma(b-a))^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\|u\|_{L^2(a,b)}\\ &\leq C\frac{\gamma(2\alpha-1, 2\sigma(b-a))^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}.\end{aligned}$$

Hence, the set $\mathbb{I}^{\alpha,\sigma}_{a^+}(X)$ is bounded in $C[a,b]$.

Moreover, by Theorem 2.2-(3), there is a positive constant K such that

$$|\mathbb{I}^{\alpha,\sigma}_{a^+}u(x) - \mathbb{I}^{\alpha,\sigma}_{a^+}u(y)| \leq K\|u\|_{L^2(a,b)}|x-y|^{\alpha-\frac{1}{2}}.$$

Thus $\mathbb{I}^{\alpha,\sigma}_{a^+}(X)$ is equicontinuous. □

4. Tempered fractional integrals with respect to functions: L^p boundedness

In this section we are going to prove Theorem 1.5 - Theorem 1.8. We start our analysis with the proof of Theorem 1.5.

Proof of Theorem 1.5. We divide the proof in three parts:
Case 1: $p = \infty$**.** Let $u \in L^\infty(a,b)$, then by doing the change of variable $t = \sigma(\psi(x) - \psi(s))$ we derive

$$\begin{aligned}
|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|ds\\
&\leq \frac{\|u\|_{L^\infty(a,b)}}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))e^{-\sigma(\psi(x)-\psi(s))}ds\\
&= \frac{\|u\|_{L^\infty(a,b)}}{\sigma^\alpha\Gamma(\alpha)}\int_0^{\sigma(\psi(x)-\psi(a))} t^{\alpha-1}e^{-t}dt\\
&= \frac{\gamma(\alpha,\sigma(\psi(x)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^\infty(a,b)}.
\end{aligned}$$

Since ψ and $\gamma(\alpha,\cdot)$ are increasing functions we get

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^\infty(a,b)} \leq \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^\infty(a,b)}.$$

Case 2: $p = 1$**.** Since ψ' is an increasing function, then the change of variable $t = \sigma(\psi(x)-\psi(s))$ and Fubini's Theorem yield that

$$\begin{aligned}
\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L1(a,b)} &\leq \frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|dsdx\\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x \psi'(x)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|dsdx\\
&= \frac{1}{\Gamma(\alpha)}\int_a^b |u(s)|\int_s^b \psi'(x)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}dxds\\
&= \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_a^b |u(s)|\gamma(\alpha,\sigma(\psi(b)-\psi(s)))ds.
\end{aligned}$$

Again by the monotony of ψ and $\gamma(\alpha,\cdot)$ we get

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^1(a,b)} \leq \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^1(a,b)}.$$

Case 3: $1 < p < \infty$**.** In this case, let $q > 0$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and denote by

$$\Psi(x,s) = \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}.$$

Hence, Hölder inequality, the monotony of ψ and $\gamma(\alpha,\cdot)$ yield that

$$\begin{aligned}|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x \Psi(x,s)|u(s)|ds\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \Psi(x,s)ds\right)^{1/q}\left(\int_a^x \Psi(x,s)|u(s)|^p\right)^{1/p}\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha}\right)^{1/q}\left(\int_a^x \Psi(x,s)|u(s)|^p ds\right)^{1/p}.\end{aligned}$$

Consequently, since ψ' is an increasing function, Fubini's Theorem yields that

$$\begin{aligned}&\int_a^b |\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)|^p dx\\ &\leq \frac{1}{\Gamma^p(\alpha)}\left(\frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha}\right)^{\frac{p}{q}}\int_a^b\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|^p dsdx\\ &\leq \frac{1}{\Gamma^p(\alpha)}\left(\frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha}\right)^{\frac{p}{q}}\int_a^b |u(s)|^p\int_s^b \psi'(x)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}dxds\\ &= \frac{1}{\Gamma^p(\alpha)}\left(\frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha}\right)^{\frac{p}{q}}\int_a^b \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(s)))}{\sigma^\alpha}|u(s)|^p ds.\end{aligned}$$

So, by the monotony of ψ and $\gamma(\alpha,\cdot)$ we obtain

$$\int_a^b |\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)|^p dx \leq \frac{1}{\Gamma^p(\alpha)}\left(\frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha}\right)^p \|u\|^p_{L^p(a,b)}.$$

Therefore

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^p(a,b)} \leq \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)}.$$

□

In our next result we consider the L^p boundedness of the tempered fractional integral $\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}$ when $\alpha\in(0,\frac{1}{p})$, more precisely we consider the proof of Theorem 1.6

Proof of Theorem 1.6. We divide the proof in two case:

Case 1: $p=1$ **and** $1\leq q<\frac{1}{1-\alpha}$. The case $q=1$ was proved in Theorem 1.5. Hence we consider the case $q\in(1,\frac{1}{1-\alpha})$. Let r the conjugate of q that is

$$\frac{1}{q}+\frac{1}{r}=1.$$

Then, Hölder inequality and the monotony of ψ' yield that

$$\begin{aligned}|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x (\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}(\psi'(s)|u(s)|)^{\frac{1}{q}}(\psi'(s)|u(s)|)^{\frac{1}{r}}ds\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \psi'(s)(\psi(x)-\psi(s))^{q(\alpha-1)}e^{-\sigma q(\psi(x)-\psi(s))}|u(s)|ds\right)^{1/q}\left(\int_a^x \psi'(s)|u(s)|ds\right)^{1/r}\\ &\leq \frac{(\psi'(b))^{\frac{1}{r}}\|u\|^{\frac{1}{r}}_{L^1(a,b)}}{\Gamma(\alpha)}\left(\int_a^x \psi'(s)(\psi(x)-\psi(s))^{q(\alpha-1)}e^{-\sigma q(\psi(x)-\psi(s))}|u(s)|ds\right)^{1/q}.\end{aligned}$$

Hence, by the monotony of ψ', $\gamma(\alpha,\cdot)$ and Fubini's theorem we get

$$\begin{aligned}\int_a^b |\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)|^q dx &\leq \frac{(\psi'(b))^{\frac{q}{r}}\|u\|^{\frac{q}{r}}_{L^1(a,b)}}{\Gamma^q(\alpha)}\int_a^b\int_a^x \psi'(s)(\psi(x)-\psi(s))^{q(\alpha-1)}e^{-\sigma q(\psi(x)-\psi(s))}|u(s)|dsdx\\ &\leq \frac{(\psi'(b))^{\frac{q}{r}}\|u\|^{\frac{q}{r}}_{L^1(a,b)}}{\Gamma^q(\alpha)}\int_a^b |u(s)|\int_s^b \psi'(x)(\psi(x)-\psi(s))^{q(\alpha-1)}e^{-\sigma q(\psi(x)-\psi(s))}dxds\\ &= \frac{(\psi'(b))^{\frac{q}{r}}\|u\|^{\frac{q}{r}}_{L^1(a,b)}}{\Gamma^q(\alpha)}\int_a^b \frac{\gamma(1-q(1-\alpha),\sigma q(\psi(b)-\psi(s)))}{(\sigma q)^{1-q(1-\alpha)}}|u(s)|ds\\ &\leq \frac{(\psi'(b))^{\frac{q}{r}}}{\sigma^{1-q(1-\alpha)}\Gamma^q(\alpha)}\gamma(1-q(1-\alpha),\sigma q(\psi(b)-\psi(a))))\|u\|^q_{L^1(\Omega)}.\end{aligned}$$

Therefore, since $q\in(1,\frac{1}{1-\alpha})$, hence $1-q(1-\alpha)>0$ then

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^q(a,b)}\leq \frac{(\psi'(b))^{\frac{1}{r}}}{\sigma^{\frac{1}{q}+\alpha-1}\Gamma(\alpha)}\left(\gamma(1-q(1-\alpha),\sigma q(\psi(b)-\psi(a)))\right)^{1/q}\|u\|_{L^1(a,b)}.$$

Case 2: $p\in\left(1,\frac{1}{\alpha}\right)$ **and** $q\in\left[p,\frac{p}{1-\alpha p}\right)$**.** If $q\in[1,p]$ the continuous embedding of $L^p(a,b)$ into $L^q(a,b)$ and Theorem 1.5 yield the desired result.

Suppose that $q\in\left(p,\frac{p}{1-\alpha p}\right)$. Let

$$\lambda=\frac{1}{p}-\frac{1}{q}<\alpha<1$$

and define

$$\frac{1}{\kappa}=1+\frac{1}{q}-\frac{1}{p}=1-\lambda.$$

Denote by $\Phi(x,s)=(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}$. Hence, by Hölder inequality we derive

$$\begin{aligned}|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)\Phi(x,s)|u(s)|ds\\ &= \frac{1}{\Gamma(\alpha)}\int_a^x \Phi(x,s)^{\frac{\kappa}{q}}(\psi'(s))^{\frac{1}{q}}|u(s)|^{\frac{p}{q}}\Phi(x,s)^{1-\frac{\kappa}{q}}(\psi'(s))^{1-\frac{1}{q}}|u(s)|^{p\left(\frac{1}{p}-\frac{1}{q}\right)}ds\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \Phi(x,s)^{\kappa}\psi'(s)|u(s)|^p ds\right)^{1/q}\left(\int_a^x \Phi(x,s)^{\frac{q-\kappa}{q-1}}\psi'(s)|u(s)|^{\frac{q-p}{q-1}}ds\right)^{\frac{q-1}{q}}\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \Phi(x,s)^{\kappa}\psi'(s)|u(s)|^p ds\right)^{1/q}\left(\int_a^x \Phi(x,s)^{\kappa}\psi'(s)ds\right)^{1-\frac{1}{p}}\left(\int_a^x \psi'(s)|u(s)|^p ds\right)^{\frac{1}{p}-\frac{1}{q}}\end{aligned}$$

Consequently, by the monotony of ψ' and $\gamma(\alpha,\cdot)$ we get

(4.1)
$$|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)|\leq \frac{(\psi'(b))^{\frac{1}{p}-\frac{1}{q}}}{\Gamma(\alpha)}\|u\|^{\frac{q-p}{q}}_{L^p(a,b)}\left(\frac{\gamma(1-\kappa(1-\alpha),\sigma\kappa(\psi(b)-\psi(a)))}{(\sigma\kappa)^{1-\kappa(1-\alpha)}}\right)^{1-\frac{1}{p}}\left(\int_a^x \Phi(x,s)^{\kappa}\psi'(s)|u(s)|^p ds\right)^{1/q}.$$

Hence, by (4.1), Fubini's theorem and the monotony of ψ', $\gamma(\alpha,\cdot)$ we obtain

$$\begin{aligned}
&\int_a^b |\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)|^q dx\\
&\leq \frac{(\psi'(b))^{\frac{q-p}{p}}}{\Gamma^q(\alpha)}\|u\|^{q-p}_{L^p(a,b)}\left(\frac{\gamma(1-\kappa(1-\alpha),\sigma\kappa(\psi(b)-\psi(a)))}{(\sigma\kappa)^{1-\kappa(1-\alpha)}}\right)^{\frac{q(p-1)}{p}}\int_a^b\int_a^x \Phi(x,s)^\kappa\psi'(s)|u(s)|^p ds ds\\
&\leq \frac{(\psi'(b))^{\frac{q-p}{p}}}{\Gamma^q(\alpha)}\left(\frac{\gamma(1-\kappa(1-\alpha),\sigma\kappa(\psi(b)-\psi(a)))}{(\sigma\kappa)^{1-\kappa(1-\alpha)}}\right)^{\frac{q}{\kappa}}\|u\|^q_{L^p(a,b)}.
\end{aligned}$$

Therefore

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^q(a,b)} \leq \frac{(\psi'(b))^{\frac{1}{p}-\frac{1}{q}}}{\Gamma(\alpha)}\left(\frac{\gamma(1-\kappa(1-\alpha),\sigma\kappa(\psi(b)-\psi(a)))}{(\sigma\kappa)^{1-\kappa(1-\alpha)}}\right)^{\frac{1}{\kappa}}\|u\|_{L^p(a,b)}.$$

□

In our next result we deal with the case $\alpha = \frac{1}{p}$.

Proof of Theorem 1.7. If $q\in[1,p)$ the result is a consequence of the continuous embedding of $L^p(a,b)$ into $L^q(a,b)$ and Theorem 2.1. Now we consider the case

$$\frac{1}{\alpha} = p \leq q < \infty.$$

Theorem 2.1 yields the case $q = p$, hence we just consider the case $\frac{1}{\alpha} < q < \infty$. In this case let us define

$$\kappa = \frac{1}{p}-\frac{1}{q} = \alpha - \frac{1}{q} \quad \text{and} \quad \frac{1}{\varrho} = 1-\kappa = 1+\frac{1}{q}-\frac{1}{p}$$

Hence

$$0<\kappa<\alpha<1 \quad \text{and} \quad \varrho < \frac{1}{1-\alpha}.$$

Denote by $\Phi(x,s) = (\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}$, hence by Hölder inequality and the monotony of ψ' and $\gamma(\alpha,\cdot)$ we derive

$$\begin{aligned}
|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x \Phi(x,s)^{\frac{\varrho}{q}}(\psi'(s))^{\frac{1}{q}}|u(s)|^{\frac{p}{q}}\Phi(x,s)^{\varrho\left(1-\frac{1}{p}\right)}(\psi'(s))^{\frac{q-1}{q}}|u(s)|^{\frac{q-p}{q}}ds\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \Phi(x,s)^\varrho\psi'(s)|u(s)|^p\right)^{1/q}\left(\int_a^x \Phi(x,s)^{\varrho\left(1-\frac{1}{p}\right)\frac{q}{q-1}}\psi'(s)|u(s)|^{\frac{q-p}{q-1}}ds\right)^{\frac{q-1}{q}}\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \Phi(x,s)^\varrho\psi'(s)|u(s)|^p ds\right)^{1/q}\left(\int_a^x \Phi(x,s)^\varrho\psi'(s)ds\right)^{\frac{p-1}{p}}\left(\int_a^x \psi'(s)|u(s)|^p ds\right)^{\frac{q-p}{pq}}\\
&\leq \frac{(\psi'(b))^{\frac{q-p}{pq}}}{\Gamma(\alpha)}\left(\frac{\gamma(\varrho(\alpha-1)+1,\sigma\varrho(\psi(b)-\psi(a)))}{(\sigma\varrho)^{\varrho(\alpha-1)+1}}\right)^{\frac{p-1}{p}}\left(\int_a^x \Phi(x,s)^\varrho\psi'(s)|u(s)|^p ds\right)^{\frac{1}{q}}\|u\|^{\frac{q-p}{q}}_{L^p(a,b)}.
\end{aligned}$$

Hence, by Fubini's theorem, the monotony of ψ' and $\gamma(\alpha,\cdot)$ we get

$$\begin{aligned}
&\int_a^b |\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)|^q dx\\
&\leq \frac{(\psi'(b))^{\frac{q-p}{p}}}{\Gamma^q(\alpha)}\|u\|^{q-p}_{L^p(a,b)}\left(\frac{\gamma(\varrho(\alpha-1)+1,\sigma\varrho(\psi(b)-\psi(a)))}{(\sigma\varrho)^{\varrho(\alpha-1)+1}}\right)^{\frac{q(p-1)}{p}}\int_a^b\int_a^x \Phi(x,s)^\varrho\psi'(s)|u(s)|^p dsdx\\
&\leq \frac{(\psi'(b))^{\frac{q-p}{p}}}{\Gamma^q(\alpha)}\left(\frac{\gamma(\varrho(\alpha-1)+1,\sigma\varrho(\psi(b)-\psi(a)))}{(\sigma\varrho)^{\varrho(\alpha-1)+1}}\right)^{q\varrho}\|u\|^q_{L^p(a,b)}
\end{aligned}$$

Therefore

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{L^q(a,b)} \leq \frac{(\psi'(b))^\kappa}{\Gamma(\alpha)}\left(\frac{\gamma(\varrho(\alpha-1)+1,\sigma\varrho(\psi(b)-\psi(a)))}{(\sigma\varrho)^{\varrho(\alpha-1)+1}}\right)^{\varrho}\|u\|_{L^p(a,b)}.$$

□

Finally we are going to show Theorem 1.8.

Proof of Theorem 1.8. Let $\Phi(x,s) = (\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}$, $a<x_1<x_2<b$ and $u\in L^2(a,b)$. Then, by Hölder inequality we derive

$$\begin{aligned}|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x_1)-\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x_2)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^{x_1}\psi'(s)|\Phi(x_1,s)-\Phi(x_2,s)||u(s)|ds + \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}\psi'(s)\Phi(x_2,s)|u(s)|ds\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_1}\psi'(s)|\Phi(x_1,s)-\Phi(x_2,s)|^2ds\right)^{1/2}\left(\int_a^{x_1}\psi'(s)|u(s)|^2ds\right)^{1/2}\\ &+\frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}\psi'(s)\Phi(x_2,s)^2ds\right)^{1/2}\left(\int_{x_1}^{x_2}\psi'(s)|u(s)|^2ds\right)^{1/2}.\end{aligned}$$

Note that, by doing the change of variable $t=2\sigma(\psi(x)-\psi(s))$ and (2.1) we get

$$\begin{aligned}\int_{x_1}^{x_2}\psi'(s)\Phi(x_2,s)^2ds &= \int_{x_1}^{x_2}\psi'(s)(\psi(x_2)-\psi(s))^{2(\alpha-1)}e^{-2\sigma(\psi(x)-\psi(s))}ds\\ &=\frac{1}{(2\sigma)^{2\alpha-1}}\gamma(2\alpha-1,2\sigma(\psi(x_2)-\psi(x_1)))\\ &\leq \frac{(\psi(x_2)-\psi(x_1))^{2\alpha-1}}{2\alpha-1}.\end{aligned}$$

Therefore, by our hypothesis there is a positive constant L such that

$$\int_{x_1}^{x_2}\psi'(s)\Phi(x_2,s)^2ds \leq \frac{L^{2\alpha-1}}{2\alpha-1}|x_2-x_1|^{2\alpha-1}. \tag{4.2}$$

By other side, doing the change of variable $t=\frac{\psi(x_1)-\psi(s)}{\psi(x_2)-\psi(x_1)}$ we obtain

$$\begin{aligned}&\int_a^{x_1}\psi'(s)|\Phi(x_1,s)-\Phi(x_2,s)|^2ds\\ &=\int_a^{x_1}\psi'(s)\left|(\psi(x_1)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x_1)-\psi(s))}-(\psi(x_2)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x_2)-\psi(s))}\right|^2ds\\ &=(\psi(x_2)-\psi(x_1))^{2\alpha-1}\int_0^{\frac{\psi(x_1)-\psi(a)}{\psi(x_2)-\psi(x_1)}}\left|t^{\alpha-1}e^{-\sigma t(\psi(x_2)-\psi(x_1))}-(1+t)^{\alpha-1}e^{-\sigma(1+t)(\psi(x_2)-\psi(x_1))}\right|^2dt.\end{aligned}$$

As in the proof of [14, Lemma 2.7] combining with [13, Theorem 2.6] we can show that the last integral is finite, that is, there is a positive constan C such that

$$\int_0^{\frac{\psi(x_1)-\psi(a)}{\psi(x_2)-\psi(x_1)}}\left|t^{\alpha-1}e^{-\sigma t(\psi(x_2)-\psi(x_1))}-(1+t)^{\alpha-1}e^{-\sigma(1+t)(\psi(x_2)-\psi(x_1))}\right|^2dt\leq C<\infty.$$

Hence, by using our hypothesis and (2.1) we get

$$\int_a^{x_1}\psi'(s)|\Phi(x_1,s)-\Phi(x_2,s)|^2ds\leq C|\psi(x_2)-\psi(x_1)|^{2\alpha-1}\leq CL^{2\alpha-1}|x_2-x_1|^{2\alpha-1}. \tag{4.3}$$

Consequently, by (4.2), (4.3) and since ψ' is monotone we have

(4.4)
$$|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x_1) - \mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x_2)| \leq \left(\frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\sqrt{C}L^{\alpha-\frac{1}{2}} + \frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\frac{L^{\alpha-\frac{1}{2}}}{\sqrt{2\alpha-1}} \right) |x_2-x_1|^{\alpha-\frac{1}{2}}\|u\|_{L^2(a,b)}.$$

On the other hand, for any $u \in L^2(a,b)$, the Hölder inequality yields that

$$\begin{aligned}
|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x \psi'(s)(\psi(x)-\psi(s))^{2(\alpha-1)}e^{-2\sigma(\psi(x)-\psi(s))}ds\right)^{1/2}\left(\int_a^x \psi'(s)|u(s)|^2 ds\right)^{1/2} \\
&\leq \frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\left(\frac{\gamma(2\alpha-1, 2\sigma(\psi(x)-\psi(a))}{(2\sigma)^{2\alpha-1}}\right)^{1/2}\|u\|_{L^2(a,b)}.
\end{aligned}$$

Hence, by the monotony of $\gamma(\alpha,\cdot)$ we get

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_\infty \leq \frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\left(\frac{\gamma(2\alpha-1, 2\sigma(\psi(b)-\psi(a)))}{(2\sigma)^{2\alpha-1}}\right)^{1/2}\|u\|_{L^2(a,b)}. \tag{4.5}$$

Therefore, combining (4.4) with (4.5) we have

$$\begin{aligned}
&\|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_{H^{\alpha-\frac{1}{2}}} = \|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u\|_\infty + \sup_{x_1\neq x_2\in[a,b]}\frac{|\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x_1)-\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x_2)|}{|x_2-x_1|^{\alpha-\frac{1}{2}}} \\
&\leq \left(\frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\left(\frac{\gamma(2\alpha-1,2\sigma(\psi(b)-\psi(a)))}{(2\sigma)^{2\alpha-1}}\right)^{1/2} + \frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\sqrt{C}L^{\alpha-\frac{1}{2}} + \frac{\sqrt{\psi'(b)}}{\Gamma(\alpha)}\frac{L^{\alpha-\frac{1}{2}}}{\sqrt{2\alpha-1}}\right)\|u\|_{L^2(a,b)},
\end{aligned}$$

which implies that $\mathbb{I}^{\alpha,\sigma}_{a^+,\psi} : L^2(a,b) \to H^{\alpha-\frac{1}{2}}[a,b]$ is bounded.

Finally, (2.1) yields that

$$e^{-2\sigma(\psi(x)-\psi(a))}\frac{(\psi(x)-\psi(a))^{2\alpha-1}}{2\alpha-1} \leq \frac{\gamma(2\alpha-1, 2\sigma(\psi(x)-\psi(a)))}{(2\sigma)^{2\alpha-1}} \leq \frac{(\psi(x)-\psi(a))^{2\alpha-1}}{2\alpha-1},$$

hence by the continuity of ψ and the Sandwich theorem we get

$$\lim_{x\to a^+}\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x) = 0.$$

This completes the proof of the Theorem. □

Remark 1. (1) *Note that in Theorem 1.8 we just consider the particular case $p=2$, the general case still is an open problem.*

(2) *In Theorems 1.5, 1.6, 1.7 and 1.8 we just consider the case of the left ψ-tempered fractional integral*

$$\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}u(s)ds.$$

In a similar way we can show that the results of the cited Theorem hold for the right ψ-tempered fractional integral defined as

$$\mathbb{I}^{\alpha,\sigma}_{b^-,\psi}u(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \psi'(s)(\psi(s)-\psi(x))^{\alpha-1}e^{-\sigma(\psi(s)-\psi(x))}u(s)ds$$

(3) *In Theorem 1.6 we showed that the ψ-tempered fractional integral $\mathbb{I}^{\alpha,\sigma}_{a^+,\psi}$ is bounded from $L^p(a,b)$ into $L^q(a,b)$ for every $q \in [1, p^*_\alpha)$, where $p^*_\alpha = \frac{p}{1-\alpha p}$. The case*

$$\mathbb{I}^{\alpha,\sigma}_{a^+,\psi} : L^p(a,b) \to L^{p^*_\alpha}(a,b)$$

still is an open problem.

CONTRIBUTION

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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