ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

BOUNDARY CONTINUITY OF ROTATIONALLY SYMMETRIC PRESCRIBED MEAN CURVATURE HYPERSURFACES

MOZHGAN “NORA” ENTEKHABI AND KIRK E. LANCASTER

ABSTRACT. We examine the boundary behavior of variational solutions of Dirichlet problems for the prescribed mean curvature equation in smooth domains in R^n, $n \geq 3$, when the appropriate boundary curvature conditions are not satisfied, the Dirichlet data may be discontinuous and the Dirichlet problem has rotational symmetry. We establish the existence of the radial limits at points of the boundary and illustrate by example that the variational solution can be continuous on the closure of the domain even though the Dirichlet boundary data has no limit at some boundary points.

1. Introduction

The study of the geometry of fluid interfaces has generated interest for centuries (e.g. [22]), illustrated, for example, by the study of Plateau's problem (e.g. [19, Chap. V]). In this note, we wish to develop and study of higher dimensional prototypes of (generalized) nonparametric Plateau problems. Let $n \geq 2$, $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$ and $N_n f = \nabla \cdot Tf = \mathrm{div}(Tf)$ for $f \in C^2(\Omega)$ when Ω is a bounded open set in $\mathbb{R}^n$ with $C^{2,\lambda}$ boundary, for some $\lambda \in (0,1)$. Let $H \in C^{1,\lambda}(\mathbb{R}^{n+1})$ such that $H(\mathbf{x},z)$ is a non-decreasing function of $z \in \mathbb{R}$ for each $\mathbf{x} \in \Omega$ and nH satisfies the hypotheses of [9, Proposition 1.1]. Here and throughout the paper, we adopt the sign convention that the mean curvature of Ω is nonnegative when Ω is convex. We consider the following Dirichlet problem

$$N_n f = nH(\cdot, f) \quad \text{in} \quad \Omega \tag{1}$$

$$f = \phi \quad \text{on} \quad \partial\Omega \tag{2}$$

for $\phi \in L^\infty(\partial\Omega)$. The solvability of this problem depends on the mean curvature of the boundary of the domain Ω and the continuity (and smoothness) of the Dirichlet data ϕ.

When $n = 2$ and $H \equiv 0$, Bernstein ([1]) observed in 1912 that convexity of the domain was a sufficient condition for the existence of a minimal surface which is a classical solution (i.e. $f \in C^0(\overline{\Omega})$ and $f = \phi$ on $\partial\Omega$ when $\phi \in C^0(\partial\Omega)$) and a necessary condition for the existence of a classical solution for all $\phi \in C^0(\partial\Omega)$ ([19, §406]). In this case with $\phi \in C^0(\partial\Omega)$, a classical solution of (1)-(2) represents a nonparametric solution of the Plateau problem in the cylinder $\Omega \times \mathbb{R}$ which spans the graph of ϕ.

In general, a generalized (e.g. variational) solution f of (1)-(2) will satisfy $\lim_{\Omega \ni \mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \phi(\mathbf{x}_0)$ if $\mathbf{x}_0 \in \partial\Omega$, ϕ is continuous at $\mathbf{x}_0$ and $(n-1)H_{\partial\Omega}(\mathbf{x}_0) > n|H(\mathbf{x}_0, f(\mathbf{x}_0))|$, where $H_{\partial\Omega}(\mathbf{x})$ is the mean curvature of $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$ (e.g. [9, Theorem 3.2], [18]). If the appropriate curvature condition (i.e. $(n-1)H_{\partial\Omega}(\mathbf{x}) \geq n|H(\mathbf{x}, \phi(\mathbf{x}))|$ for all $\mathbf{x} \in \partial\Omega$) is not satisfied, then a classical solution will not exist

2020 *Mathematics Subject Classification.* Primary: 35J67; Secondary: 35J93, 53A10.
Key words and phrases. prescribed mean curvature, Dirichlet problem, radial limits.

for certain $\phi \in C^{\infty}(\partial\Omega)$ and one can ask about the boundary behavior of "generalized solutions" of (1)-(2). When $\phi \in C^{1,\lambda}(\partial\Omega)$ for some $\lambda \in (0,1)$, the remarkable paper [2] states "Our goal is to study the regularity of such a solution (of (1)-(2)) without imposing any curvature conditions for $\partial\Omega$," studies an associated integral $n-$current and establishes the regularity of the support of this current.

We shall examine some special cases where ϕ may not be in $C^0(\partial\Omega)$ and the curvature condition may not be satisfied and consider the $BV(\Omega)$ solution f of (1)-(2); that is, the function $f \in BV(\Omega)$ minimizes over $BV(\Omega)$ the functional

$$J(h) = \int_{\Omega} \sqrt{1+|Dh|^2} + \int\int_{\Omega}\int_0^{h(\mathbf{x},y)} nH(\mathbf{x},y,s)\, ds\, d\mathbf{x}dy + \int_{\partial\Omega} |h-\phi| dH_{n-1} \tag{3}$$

for $h \in BV(\Omega)$.

Suppose $\mathscr{C} = \mathscr{C}_M$ is a smooth, open subset of $\partial\Omega$ for which

$$H_{\mathscr{C}}(\mathbf{x}) < \frac{n}{n-1} \inf_{|z|\le M} |H(\mathbf{x},z)| \quad \text{for each } \mathbf{x} \in \mathscr{C}, \tag{4}$$

where $H_{\mathscr{C}}$ is the mean curvature of $\mathscr{C}$ and $M \ge 0$. Now [8, Corollary 14.13] (see also [11, 20]) implies that there exist $\phi \in C^{\infty}(\partial\Omega)$ such that (1)-(2) has no classical solution if $\mathscr{C} \neq \emptyset$ and $H(\mathbf{x},z) = H(\mathbf{x})$. Combining [9, Theorem 3.2] and [8, Theorem 14.12], we see that for each $M > 0$ and $\mathbf{x}_0 \in \mathscr{C}_M$, there exists $\phi \in C^{\infty}(\partial\Omega)$ with $\sup_{\partial\Omega}|\phi| < M$ such that $f(\mathbf{x}) = \phi(\mathbf{x})$ when $\mathbf{x} \in \partial\Omega$ and $(n-1)H_{\mathscr{C}}(\mathbf{x}) > n|H(\mathbf{x},\phi(\mathbf{x}))|$ and $\lim_{\Omega\ni\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) \neq \phi(\mathbf{x}_0)$, where f denotes the variational solution of (1)-(2). When $\phi \in C^{1,\lambda}(\partial\Omega)$ (and Ω need only be a bounded $C^{1,\lambda}$ domain), the boundary regularity of the variational solution is determined in [2, Theorem 4.2] (see also [17, 21]). What is the boundary behavior at $\mathbf{x}_0 \in \mathscr{C}$ of the variational solution of (1)-(2) when $\phi \notin C^{0,1}(\mathscr{C})$ or ϕ is discontinuous at $\mathbf{x}_0$?

In the two-dimensional case $\Omega \subset \mathbb{R}^2$ with $H(x,y,z)$ is independent of z for $(x,y) \in \Omega$, this behavior is investigated in [5, 6]. For simplicity of notation, we shall subsequently write (x,y) for points in $\mathbb{R}^2$ and $\mathbf{x}$ for points in $\mathbb{R}^{n-1}$.

In [5], we assume the curvature $H_{\mathscr{C}}$ of $\mathscr{C}$ satisfies

$$H_{\mathscr{C}}(x,y) < -2|H(x,y)| \quad \text{for each } (x,y) \in \mathscr{C}$$

and, in [6], we assume

$$H_{\mathscr{C}}(x,y) < 2|H(x,y)| \quad \text{for each } (x,y) \in \mathscr{C}.$$

The conclusion is that if $\phi \in L^{\infty}(\partial\Omega)$ and f is the variational solution of (1)-(2), then the radial limit

$$Rf(\theta,(x,y)) \stackrel{\text{def}}{=} \lim_{r\downarrow 0} f((x,y)+r(\cos\theta,\sin\theta)) \tag{5}$$

exists for each $(x,y) \in \mathscr{C}$ and each $\theta \in (\alpha(x,y),\beta(x,y))$, where $\beta(x,y) = \alpha(x,y)+\pi$, $\theta = \alpha(x,y)$ and $\theta = \beta(x,y)$ are the tangent rays to $\partial\Omega$ at (x,y) (in polar coordinates centered at (x,y)) and the tangent cone to Ω at (x,y) is $\{(x,y)+(r\cos\theta, r\sin\theta) : r \ge 0, \alpha(x,y) \le \theta \le \beta(x,y)\}$. In both [5] and [6], $Rf(\cdot,(x,y)) \in C^0(\alpha(x,y),\beta(x,y))$ and $Rf(\cdot,(x,y))$ behaves in one of the following ways:

(i) $Rf(\cdot,(x,y))$ is a constant function and the nontangential limit of f at (x,y) exists.

(ii) There exist $\alpha_1, \alpha_2 \in [\alpha(x,y), \beta(x,y)]$ with $\alpha_1 < \alpha_2$ such that

$$Rf(\theta) \text{ is } \begin{cases} \text{constant} & \text{for } \alpha(x,y) < \theta \leq \alpha_1 \\ \text{strictly monotonic} & \text{for } \alpha_1 \leq \theta \leq \alpha_2 \\ \text{constant} & \text{for } \alpha_2 \leq \theta < \beta(x,y). \end{cases}$$

An additional conclusion in [5] is that the tangential radial limits $Rf(\alpha(x,y),(x,y))$ and $Rf(\beta(x,y),(x,y))$ exist, $Rf(\cdot,(x,y)) \in C^0([\alpha(x,y),\beta(x,y)])$ and in case (i), f is continuous at (x,y), even if ϕ is discontinuous at (x,y) or has no limit at (x,y) (e.g. [5, Example 1.4]).

One would like to extend conclusions like those above to general smooth domains in $\mathbb{R}^n$. However the techniques required to do this are unclear and two dimensional surfaces are somewhat special; for example Bourni ([2, Theorem 4.11]) only considers ϕ with jump discontinuities in the two-dimensional case and Taylor ([23]) only considers two-dimensional capillary surfaces in smooth three-dimensional containers. In §2, we consider extensions of [5, 6] to rotationally symmetric Dirichlet problems in $\mathbb{R}^n$ and offer examples in §3; the study of such symmetric geometries occurs in other cases (e.g. [12, 13, 14]) and can act as prototypes for the general situation (e.g. [3]).

2. Rotationally Symmetric Dirichlet Problems

Consider a bounded, simply-connected open set (i.e. a bounded domain) $U \subset \mathbb{R}^2$ with $C^{2,\lambda}$ boundary such that $\overline{U} \subset \mathbb{R}^2_+$, where $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$. Let $\mu = \mu(x,y)$ be the interior unit normal to U at $(x,y) \in \partial U$. For a small $\delta = \delta(x,y) > 0$, $\partial U \cap B_\delta(x,y) \setminus \{(x,y)\}$ consists of disjoint, open arcs $\partial^- U(x,y)$ and $\partial^+ U(x,y)$ whose tangent rays approach the rays $\theta = \alpha(x,y)$ and $\theta = \beta(x,y)$ respectively, as the point (x,y) is approached and such that the interior directions from (x,y) into U are the rays $\vec{r}(\theta) = \{(x,y) + r(\cos\theta, \sin\theta) : 0 < r < \varepsilon(\theta)\}$ for $\theta \in (\alpha(x,y), \beta(x,y))$; here $\varepsilon(\cdot) : (\alpha(x,y), \beta(x,y)) \to (0,\delta)$ depends on U.

Fix $n \geq 3$ and define $\Omega \subset \mathbb{R}^n$ by

$$\Omega = \{(x\tau, y) \in \mathbb{R}^n : (x,y) \in U, \tau \in S^{n-2}\}. \tag{6}$$

Let $\hat{\mu} = \hat{\mu}(\mathbf{x},y)$ be the interior unit normal to Ω at $(\mathbf{x},y) \in \partial\Omega$. For $(\mathbf{x},y) \in \partial\Omega$, define $\partial^-\Omega(\mathbf{x},y) = \{(|\mathbf{x}|\tau, y) : \tau \in S^{n-2},\ (|\mathbf{x}|,y) \in \partial^- U(|\mathbf{x}|,y)\}$ and $\partial^+\Omega(\mathbf{x},y) = \{(|\mathbf{x}|\tau, y) : \tau \in S^{n-2},\ (|\mathbf{x}|,y) \in \partial^+ U(|\mathbf{x}|,y)\}$. If we set $\Gamma(\mathbf{x},y) = \{(|\mathbf{x}|\tau, y) : \tau \in S^{n-2}\}$ and $T_\delta(\mathbf{x},y) = \{(s\tau,t) \in \mathbb{R}^n : (s,t) \in B_\delta(|\mathbf{x}|,y), \tau \in S^{n-2}\}$, then $\partial\Omega \cap T_\delta(\mathbf{x},y) \setminus \Gamma(\mathbf{x},y) = \partial^+\Omega(\mathbf{x},y) \cup \partial^-\Omega(\mathbf{x},y)$.

Let C be a fixed, designated open subset of ∂U; we might consider C to be connected but this is not essential. Let us define a hypersurface $\mathscr{C}$ in $\mathbb{R}^n$ by

$$\mathscr{C} = \{(x\tau, y) : (x,y) \in C,\ \tau \in S^{n-2}\}. \tag{7}$$

For each $P = (\mathbf{x},y) \in \mathscr{C}$, we define

$$T_P = \{\omega \in S^{n-1} \ : \ \{P + r\omega : 0 < r < \varepsilon\} \subset \Omega \ \text{ for some } \ \varepsilon > 0\},$$

$T^o_P = \{\omega \in \overline{T_P} : \omega \ \text{ is not tangent to } \mathscr{C} \text{ at } P\}$ and $T^i_P = \{\omega \in T_P : \omega \ \text{ is not tangent to } \partial\Omega \text{ at } P\}$. Let

$$Rf(\omega,P) \stackrel{\text{def}}{=} \lim_{r \downarrow 0} f(P + r\omega) \quad \text{for each } \omega \in \overline{T_P} \text{ for which this limit exists.}$$

Set $\tau_0 = (1,0,\ldots,0) \in S^{n-2}$.

Let us assume that $\phi \in L^\infty(\partial\Omega)$ satisfies

$$\phi(\mathbf{x},y) = \phi(|\mathbf{x}|\tau_0,y) \tag{8}$$

for almost all $(\mathbf{x},y) \in \partial\Omega$ and H satisfies

$$H(\mathbf{x},y,z) = H(|\mathbf{x}|\tau_0,y,z) \tag{9}$$

for each $(\mathbf{x},y) \in \Omega$, $z \in \mathbb{R}$. If $f \in BV(\Omega)$ minimizes (3), then uniqueness (e.g. [7, Theorem 1]) implies that

$$f(\mathbf{x},y) = f(|\mathbf{x}|\tau_0,y) \quad \text{for } (\mathbf{x},y) \in \Omega. \tag{10}$$

For each $\tau \in S^{n-2}$, let $L_\tau \in SO(n)$ be the rotation about the x_n-axis which maps (τ,y) to (τ_0,y) for each $y \in \mathbb{R}$; we write $L_\tau(\omega) = (L_\tau(\omega)_1,\ldots,L_\tau(\omega)_n)$. We note using hyperspherical/polyspherical coordinates that

$$\int_\Omega \sqrt{1+|Df|^2} = \int_U \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sqrt{1+|Dg(r,y)|^2}\ r^{n-2}F\left(\vec{\theta}\right)\, d\theta_1\ldots d\theta_{n-2}drdy$$

$$= V_{n-2}\int_U \sqrt{1+|Dg(r,y)|^2}\ r^{n-2}drdy,$$

where $F(\vec{\theta}) = \Pi_{k=1}^{n-2}\sin^{n-2-k}(\theta_k)$ and $V_{n-2} = H_{n-2}(S^{n-2}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ is the surface area of the $(n-2)-$sphere. Similar calculations imply that (3) can be written as

$$\frac{J(f)}{V_{n-2}} = \int_U r^{n-2}\sqrt{1+|Dg(r,y)|^2}\, drdy + \int\int_U r^{n-2}\int_0^{g(r,y)} nH(r,y,s)\, ds\, drdy$$

$$+\int_{\partial U} r^{n-2}|g(r,y)-\phi(r,y)|drdy$$

We shall impose different conditions on C, and so $\mathscr{C}$, in the following.

2.1. ***For an*** $(x_0,y_0)\in C$, $\kappa(x_0,y_0) < n|H(x_0\tau_0,y_0,z)| - \frac{n-2}{x_0}$**.**

Let us suppose first that $(x_0,y_0) \in C$ and the curvature (with respect to μ) κ of C satisfies

$$\kappa(x_0,y_0) < n|H(x_0\tau_0,y_0,z)| - \frac{n-2}{x_0} \quad \text{for } z \in [-M,M] \tag{11}$$

and either

$$H(x_0\tau_0,y_0,z) < -\frac{n-2}{nx_0} \quad \text{for } |z| \le M, \tag{12}$$

or

$$H(x_0\tau_0,y_0,z) > \frac{n-2}{nx_0} \quad \text{for } |z| \le M, \tag{13}$$

where $M \ge 0$ will depend on the solution of (1)-(2) under consideration.

Theorem 1. *Let Ω, H, C, $\mathscr{C}$ and ϕ satisfy the conditions above. Let $f \in C^2(\Omega) \cap L^\infty(\Omega) \cap BV(\Omega)$ minimizes the functional (3) for $h \in BV(\Omega)$. Suppose κ satisfies (11) and either (12) or (13) when $M = \sup_\Omega |f|$. Then there exists a neighborhood $\mathscr{V} \subset \mathbb{R}^2$ of (x_0, y_0) such that for each $(x,y) \in C \cap \mathscr{V}$ and $\tau \in S^{n-2}$, $Rf(\omega, (x\tau, y))$ exists for each $\omega \in T^i_{(x\tau,y)}$.*

Define $g : U \to \mathbb{R}$ by $g(x,y) = f(x\tau_0, y)$. Then for each $(x,y) \in C \cap \mathscr{V}$, $\tau \in S^{n-2}$ and $\omega \in T^i_P$, we have

$$Rf(\omega, P) = Rf(L_\tau(\omega), P_0) = Rg(\theta, (x,y)) \overset{\text{def}}{=} \lim_{r \downarrow 0} g(x + r\cos\theta, y + r\sin\theta),$$

where $P = (x\tau, y) \in \mathscr{C}$, $P_0 = (x\tau_0, y) \in \mathscr{C}$, $\theta \in (\alpha(x,y), \beta(x,y))$ and

$$(\cos\theta, \sin\theta) = \frac{1}{\sqrt{(L_\tau(\omega)_1)^2 + (L_\tau(\omega_n)^2}} \left(L_\tau(\omega)_1, L_\tau(\omega)_n\right). \tag{14}$$

Further, $Rg(\theta, (x,y))$ behaves as in one of the following cases:

(a) *$Rg(\cdot, (x,y))$ is constant on $(\alpha(x,y), \beta(x,y))$ and g has a nontangential limit at (x,y).*

(b) *there exist $\theta_1, \theta_2 \in [\alpha(x,y), \beta(x,y)]$ with $\theta_1 < \theta_2$ such that*

$$Rg(\theta, (x,y)) \text{ is } \begin{cases} \text{constant} & \text{if } \alpha(x,y) < \theta \le \theta_1 \\ \text{strictly monotonic} & \text{if } \theta_1 \le \theta \le \theta_2 \\ \text{constant} & \text{if } \theta_2 \le \theta < \beta(x,y). \end{cases} \tag{15}$$

If case (a) holds, then f has a nontangential limit at $(x\tau, y)$ for each $\tau \in S^{n-2}$.

2.2. *For an* $(x_0, y_0) \in C$, $\kappa(x_0, y_0) < -n|H(x_0\tau_0, y_0, z)| - \frac{n-2}{x_0}$.

Let us suppose second that $(x_0, y_0) \in C$ and the curvature (with respect to μ) κ of C satisfies

$$\kappa(x_0, y_0) < -n|H(x_0\tau_0, y_0, z)| - \frac{n-2}{x_0} \quad \text{for } z \in [-M, M], \tag{16}$$

where $M \ge 0$ will depend on the solution of (1)-(2) under consideration.

Theorem 2. *Let Ω, H, C, $\mathscr{C}$ and ϕ satisfy the conditions above. Let $f \in C^2(\Omega) \cap L^\infty(\Omega)$ minimizes the functional $J(h)$ given in (3) for $h \in BV(\Omega)$ and define $g : U \to \mathbb{R}$ by $g(x,y) = f(x\tau_0, y)$. Suppose κ satisfies (16) when $M = \sup_\Omega |f|$. Then there exists a neighborhood $\mathscr{V} \subset \mathbb{R}^2$ of (x_0, y_0) such that for each $(x,y) \in C \cap \mathscr{V}$, the limits*

$$\lim_{\partial^- U(x,y) \ni (w,v) \to (x,y)} g(w,v) = z_-(x,y)$$

and

$$\lim_{\partial^+ U(x,y) \ni (w,v) \to (x,y)} g(w,v) = z_+(x,y)$$

exist. For each $(x,y) \in C \cap \mathscr{V}$, $\tau \in S^{n-2}$ and $\omega \in T^o_P$,

$$Rf(\omega, P) = Rf(L_\tau(\omega), P_0) = Rg(\theta, (x,y))$$

exists, where $P = (x\tau, y) \in \mathscr{C}$, $P_0 = (x\tau_0, y) \in \mathscr{C}$, $Rg(\alpha(x,y),(x,y)) = z_-(x,y)$, $Rg(\beta(x,y),(x,y)) = z_+(x,y)$ *and* $\theta \in [\alpha(x,y),\beta(x,y)]$ *satisfies (14). Further* $Rg(\cdot,(x,y)) \in C^0([\alpha(x,y),\beta(x,y)])$ *and* $Rg(\cdot,(x,y))$ *behaves as in one of the following cases:*

(a) $z_-(x,y) = z_+(x,y)$ *and* $g \in C^0(U \cup \{(x,y)\})$.

(b) $z_-(x,y) \neq z_+(x,y)$ *and there exist* $\theta_1, \theta_2 \in [\alpha(x,y),\beta(x,y)]$ *with* $\theta_1 < \theta_2$ *such that*

$$Rg(\theta,(x,y)) \text{ is } \begin{cases} z_-(\mathbf{x},y) & \text{if } \alpha(x,y) \leq \theta \leq \theta_1 \\ \text{strictly monotonic} & \text{if } \theta_1 \leq \theta \leq \theta_2 \\ z_+(x,y) & \text{if } \theta_2 \leq \theta \leq \beta(x,y). \end{cases} \tag{17}$$

If case (a) holds for $(x,y) \in C \cap \mathscr{V}$, *then* $f \in C^0(\Omega \cup \{(x\tau,y)\})$ *for each* $\tau \in S^{n-2}$.

3. Examples

Example 1. *Let* $a \in \left(\frac{1}{2},1\right)$, $b \in (0,1-a)$, $H \in \mathbb{R}$ *with* $-\frac{1}{b} < -3H - \frac{1}{a-b}$ *and* $H \in [0,1]$. *Set*

$$\Omega = \{(r\cos\theta, r\sin\theta, y) \in \mathbb{R}^3 : r^2+y^2 < 1, 0 \leq \theta < 2\pi, r \geq 0, (r-a)^2+y^2 > b^2\}$$

(see Figure 1). Define $\phi \in C^\infty\left(\mathbb{R}^3 \setminus Z_0\right)$ *by* $\phi(\mathbf{x},y) = (1-|\mathbf{x}|^2-y^2)\cos\left(\frac{1}{y}\right)$, *where* $Z_0 = \{(x_1,x_2,0) : (x_1,x_2) \in \mathbb{R}^2\}$. *Let* $f \in C^2(\Omega)$ *minimize the functional* $J(\cdot)$ *given in (3). Then* $f \in C^0(\overline{\Omega})$ *(but* f *is not equal to* ϕ *on portions of* $B_1(\mathscr{O}_3) \cap \partial\Omega$, *where* $\mathscr{O}_3 = (0,0,0)$*).*

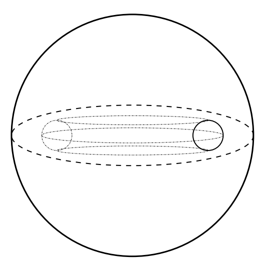

FIGURE 1. Ω

Example 2. *Let* $n=3$, $\varepsilon \in (0,1)$, $U = \{(x,y) \in \mathbb{R}^2 : (x-1-\varepsilon)^2+y^2 < 1\}$ *and* $\Omega = \{(x\cos\sigma, x\sin\sigma, y) \in \mathbb{R}^3 : (x,y) \in U, \sigma \in [0,2\pi]\}$. *Notice that* $\partial\Omega$ *is a torus and its mean curvature (with respect to* $\hat{\mu}$*) at* $((1+\varepsilon+\cos\chi)\cos\sigma, (1+\varepsilon+\cos\chi)\sin\sigma, \sin\chi)$ *is*

$$H_{\partial\Omega}((1+\varepsilon+\cos\chi)\cos\sigma, (1+\varepsilon+\cos\chi)\sin\sigma, \sin\chi) = \frac{1+\varepsilon+2\cos\chi}{2(1+\varepsilon+\cos\chi)}$$

for $0 \leq \sigma \leq 2\pi$, $-\frac{\pi}{2} \leq \chi \leq \frac{3\pi}{2}$. *Here* $M=1$, $\kappa \equiv 1$ *and*

$$\alpha(1+\varepsilon+\cos\chi, \sin\chi) = \frac{\pi}{2} - \chi \quad \text{and} \quad \beta(1+\varepsilon+\cos\chi, \sin\chi) = \frac{3\pi}{2} - \chi$$

if $\chi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *while*

$$\alpha(1+\varepsilon+\cos\chi, \sin\chi) = \chi - \frac{\pi}{2} \quad \text{and} \quad \beta(1+\varepsilon+\cos\chi, \sin\chi) = \chi + \frac{\pi}{2}$$

if $\chi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. *The isoperimetric inequality for Caccioppoli sets A in* $\mathbb{R}^3$ *says* $\mathscr{H}_3(A) \le \frac{\frac{4\pi}{3}}{(4\pi)^{\frac{3}{2}}}\left(\int |D\phi_A|\right)^{\frac{3}{2}}$ *or* $\int |D\phi_A| \ge 3\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} (\mathscr{H}_3(A))^{2/3}$. *Let A be a Caccioppoli set in* Ω *and write* $|A| \stackrel{\text{def}}{=} \mathscr{H}_3(A)$ *and* $|\partial A| \stackrel{\text{def}}{=} \int |D\phi_A|$. *Then* $|A| \le \mathscr{H}_3(\Omega) = 2\pi^2(1+\varepsilon) = \frac{3\pi(1+\varepsilon)}{2}\frac{4\pi}{3}$ *and so* $\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \ge \left(\frac{2}{3\pi(1+\varepsilon)}\right)^{\frac{1}{3}} |A|^{\frac{1}{3}}$ *with equality only if* $|A| = |\Omega|$. *Thus* $\int |D\phi_A| \ge 3\left(\frac{2}{3\pi(1+\varepsilon)}\right)^{\frac{1}{3}} |A|$. *Set* $H_1 = \left(\frac{2}{3\pi(1+\varepsilon)}\right)^{\frac{1}{3}}$, *so that*

$$|\int_A 3H_1 dx| = 3H_1|A| \le \int |D\phi_A|. \tag{18}$$

Notice that (18) is a strict inequality if $A \ne \emptyset, \Omega$ *and if we calculate, we obtain* $|\int_\Omega 3H_1 dx| = 6\pi^2\left(\frac{2}{3\pi(1+\varepsilon)}\right)^{\frac{1}{3}}(1+\varepsilon) = \left(\frac{9}{4\pi(1+\varepsilon)}\right)^{\frac{1}{3}} 4\pi^2(1+\varepsilon) < 4\pi^2(1+\varepsilon) = \int |D\phi_A|$. *Thus (18) is a strict inequality for all Caccioppoli sets A in* Ω *with* $|A| > 0$. *From* [10, Theorem 1.1], *we see that there is a function* $f \in C^2(\Omega)$ *which satifies (1) for each constant* $H \in (0, H_1)$.

Notice that there exists $\delta_0 \in (0,1)$ *such that for each* $\varepsilon \in (0, \delta_0]$,

$$H = \frac{1}{2+\varepsilon} + \frac{\varepsilon}{6} = \frac{\varepsilon^2 + 2\varepsilon + 6}{6(2+\varepsilon)} \le H_1;$$

this follows from the facts that $\lim_{\varepsilon\downarrow 0} \frac{1}{2+\varepsilon} + \frac{\varepsilon}{6} = \frac{1}{2}$ *and* $\lim_{\varepsilon\downarrow 0}\left(\frac{2}{3\pi(1+\varepsilon)}\right)^{\frac{1}{3}} = \left(\frac{2}{3\pi}\right)^{\frac{1}{3}}$ *is approximately* 0.5965. *Now condition (11) is*

$$1 < 3|H(1+\varepsilon+\cos\chi, \sin\chi)| - \frac{1}{1+\varepsilon+\cos\chi}.$$

Assume $\varepsilon \in (0, \delta_0)$ *and* $H = \frac{1}{2+\varepsilon} + \frac{\varepsilon}{6}$; *then* $1 < 3H - \frac{1}{1+\varepsilon+\cos\chi}$ *if and only if* $\chi \in \left(-\cos^{-1}\left(\frac{2-\varepsilon^2-\varepsilon^3}{2+\varepsilon^2}\right), \cos^{-1}\left(\frac{2-\varepsilon^2-\varepsilon^3}{2+\varepsilon^2}\right)\right)$.

Let us fix $\varepsilon \in (0, \delta_0)$ *and set* $H = \frac{1}{2+\varepsilon} + \frac{\varepsilon}{6}$ *and* $C = \{(1+\varepsilon+\cos\chi, \sin\chi) \in \mathbb{R}^2 : \cos(\chi) > \frac{2-\varepsilon^2-\varepsilon^3}{2+\varepsilon^2}\}$. *Let* $\phi(x_1, x_2, y) = \cos\left(\frac{1}{y}\right)$ *for* $y \ne 0$ *and let* $f \in C^2(\Omega)$ *minimize (3). Then we can apply Theorem 1 to see that the radial limits of f exist at each point of* $\mathscr{C}$. *Due to the symmetry of* ϕ *and* Ω, *case (a) of Theorem 1 holds, g has a nontangential limit at* $(\varepsilon, 0) \in C$ *and* $Rf(\cdot, (\varepsilon\cos\sigma, \varepsilon\sin\sigma, 0))$ *is a constant function for each* $\sigma \in [0, 2\pi)$.

4. Proofs

Proof of Theorem 1: First note that

$$f(\mathbf{x}, y) = f(L_\tau(\mathbf{x}, y)) \quad \text{for each } (\mathbf{x}, y) \in \Omega,\ \tau \in S^{n-2}. \tag{19}$$

Notice that $g : U \to \mathbb{R}$, given by $g(x, y) = f(x\tau_0, y)$, satisfies

$$N_n(f)(x, 0, \ldots, 0, y) = N_2(g)(x, y) + (n-2)\frac{g_x(x, y)}{x\sqrt{1+|\nabla g(x, y)|^2}} \tag{20}$$

and so the mean curvature $H_g(x,y)$ at $(x,y,g(x,y)) \in U \times \mathbb{R}$ of the graph of g satisfies

$$2H_g(x,y) \stackrel{\text{def}}{=} N_2(g)(x,y) = nH(x\tau_0, y, g(x,y)) - (n-2)\frac{g_x(x,y)}{x\sqrt{1+|\nabla g(x,y)|^2}}. \tag{21}$$

For $(x,y) \in U$, $nH(x\tau_0,y,g(x,y)) - \frac{n-2}{x} < 2H_g(x,y) < nH(x\tau_0,y,g(x,y)) + \frac{n-2}{x}$. Thus $|N_2(g)| \le n|H| + \sup_{(x,y)\in U} \frac{n-2}{x} < \infty$ shows $N_2(g)$ is bounded and the calculation on p. 170 of [16] shows that the area $A(S)$ of the graph of g over U is finite.

Since $c_0 = \inf\{r : (r,y) \in U \text{ for some } y \in \mathbb{R}\} > 0$, we see that

$$\int_U \sqrt{1+|Dg|^2} \le \frac{1}{c_0^{n-2} V_{n-2}} \int_\Omega \sqrt{1+|Df|^2}.$$

Since $|\int\int_\Omega \int_0^{f(\mathbf{x},y)} nH(\mathbf{x},y,s)\, ds\, d\mathbf{x}dy| + \int_{\partial\Omega} |f-\phi| dH_{n-1} < \infty$ and $J(f) < \infty$, we see again that the area $A(S)$ of the graph of g over U is finite.

It follows from (11) that there exists a neighborhood $\mathscr{V}_1 \subset \mathbb{R}^2$ of (x_0,y_0) such that

$$\kappa(x,y) < \inf_{z\in[-M,M]} |nH(x\tau_0,y,z)| - \frac{n-2}{x} \quad \text{for } (x,y) \in C \cap \mathscr{V}_1. \tag{22}$$

If (12) holds, then there exists a neighborhood $\mathscr{V}_2 \subset \mathbb{R}^2$ of (x_0,y_0) such that

$$\limsup_{U \ni (x,y) \to (x_1,y_1)} H_g(x,y) < 0 \quad \text{for all } (x_1,y_1) \in C \cap \mathscr{V}_2. \tag{23}$$

If (13) holds, then there exists a neighborhood $\mathscr{V}_2 \subset \mathbb{R}^2$ of (x_0,y_0) such that

$$\liminf_{U \ni (x,y) \to (x_1,y_1)} H_g(x,y) > 0 \quad \text{for all } (x_1,y_1) \in C \cap \mathscr{V}_2. \tag{24}$$

Set $\mathscr{V} = \mathscr{V}_1 \cap \mathscr{V}_2$. Let us fix $(x_1,y_1) \in C \cap \mathscr{V}$.

If g has a nontangential limit at (x_1,y_1), then $Rg(\theta,(x_1,y_1))$ exists for each $\theta \in (\alpha(x_1,y_1), \beta(x_1,y_1))$ and case (a) holds; that is, for each nontangential direction $(\cos\theta, \sin\theta)$ from (x_1,y_1) into U, the limit $Rg(\theta,(x_1,y_1))$ exists and these limits are all the same.

Let us now suppose that g does not have a nontangential limit at (x_1,y_1). Since we are only interested in interior radial limits (i.e. $\theta \in (\alpha(x_1,y_1), \beta(x_1,y_1))$, we may replace U by a subdomain U^* such that $\partial U^* \cap \partial U = \{(x_1,y_1)\}$, ∂U^* has the same tangent rays at (x_1,y_1) as does ∂U, $g \in C^0\left(\overline{U^*} \setminus \{(x_1,y_1)\}\right)$ and the curvature κ^* of ∂U^* satisfies

$$\kappa^*(x_1,y_1) < \inf_{z\in[-M,M]} |nH(x_1\tau_0,y_1,z)| - \frac{n-2}{x_1}. \tag{25}$$

Set $S_0 = \{(x,y,g(x,y)) : (x,y) \in U^*\}$.

Now we have not shown that g is the variational solution of the two dimensional version of (1)-(2) and so we cannot directly use the results of [6]. However, since we know that the area of S_0 is finite, we claim that the arguments used to prove [6, Theorem 2.3] and its conclusions continue to hold here.

Parametrizing the graph of g over U^* in isothermal coordinates, we see that the Dirichlet integral of the parametrization $Y : E \to \mathbb{R}^3$ is finite. We now argue as in the proof of [6, Theorem 2.3]. One detail we need to mention is that, when (12) (and (23)) holds, the upper Bernstein pair (U^+, ψ^+)

required in the later part of the proof will be assumed to satisfy $N_2\psi^+(x,y) \le nH(x\tau_0,y,-M) - \frac{n-2}{x}$ for $(x,y) \in U^+$ and

$$\lim_{U^+\ni(x,y)\to(x_2,y_2)} \frac{\nabla\psi^+(x,y)\cdot\hat{\nu}(x,y)}{\sqrt{1+|\nabla\psi^+(x,y)|^2}} = 1$$

for almost every $(x_2,y_2) \in \Gamma = B_\delta(x_1,y_1)\cap\partial U^+$ and, when (24) holds, the lower Bernstein pair (U^-,ψ^-) will be assumed to satisfy $N_2\psi^-(x,y) \ge nH(x\tau_0,y,M) + \frac{n-2}{x}$ for $(x,y) \in U^-$ and

$$\lim_{U^-\ni(x,y)\to(x_2,y_2)} \frac{\nabla\psi^-(x,y)\cdot\hat{\nu}(x,y)}{\sqrt{1+|\nabla\psi^-(x,y)|^2}} = -1$$

for almost every $(x_2,y_2) \in \Gamma = B_\delta(x_1,y_1)\cap\partial U^-$. It follows from this argument that $Rg(\theta,(x_1,y_1))$ exists for $\theta \in (\alpha(x_1,y_1),\beta(x_1,y_1))$ and $Rg(\cdot,(x_1,y_1)) \in C^0((\alpha(x_1,y_1),\beta(x_1,y_1)))$. Notice from [6] that if $0 < r(t) \to 0$ and $\theta(t) \to \theta \in (\alpha(x,y),\beta(x,y))$ as $t \downarrow 0$, then

$$\lim_{t\downarrow 0} g(x_1 + r(t)\cos\theta(t), y_1 + r(t)\sin\theta(t)) = Rg(\theta,(x_1,y_1)). \tag{26}$$

We claim that if $(x,y) \in C\cap\mathscr{V}$, $\omega \in T^i_{(x\tau,y)}$ and $\theta \in (\alpha(x,y),\beta(x,y))$ satisfies $\sqrt{\omega_1^2+\omega_n^2}(\cos\theta,\sin\theta) = (\omega_1,\omega_n)$, then $Rf(\omega,(x\tau_0,y)) = Rg(\theta,(x,y))$.
Pf: Fix $(x,y) \in C\cap\mathscr{V}$ and $\omega \in T^i_{(x\tau,y)}$. Set $\omega' = (\omega_1,\dots,\omega_{n-1})$ and $\tau = \frac{1}{\|x\tau_0+r\omega'\|}(x\tau_0+r\omega')$. Notice that

$$L_\tau((x\tau_0,y)+r\omega) = (\sqrt{x^2+2rx\omega_1+r^2|\omega'|^2}\ \tau_0, y+r\omega_n) = ((x+r\omega_1+O(r^2))\tau_0, y+r\omega_n)$$

and

$$f((x\tau_0,y)+r\omega) = f(L_\tau((x\tau_0,y)+r\omega)) = g(x+r\omega_1+O(r^2), y+r\omega_n)$$

Thus

$$Rf(\omega,(x\tau_0,y)) = \lim_{r\downarrow 0} g(x+r\omega_1+O(r^2), y+r\omega_n) = Rg(\theta,(x,y)). \quad \square$$

The remainder of the claims in Theorem 1 follow from (19) and [6]. $\square$

Proof of Theorem 2: We shall adopt the notation and results of the previous proof. It follows from (16) that there exists a neighborhood $\mathscr{V} \subset \mathbb{R}^2$ of (x_0,y_0) such that

$$\kappa(x,y) \le -n|H(x\tau_0,y,z)| - \frac{n-2}{x} \quad \text{for } (x,y) \in C\cap\mathscr{V}, z \in [-M,M]$$

and so, for $(x,y) \in C\cap\mathscr{V}$,

$$\kappa(x,y) \le -n|\tilde{H}(x,y)| - \frac{n-2}{x} < -\left| n\tilde{H}(x,y) - \frac{n-2}{x}\frac{g_x(x,y)}{W(x,y)}\right| = -|2H_g(x,y)|.$$

where $W = \sqrt{1+|\nabla g|^2}$ and $\tilde{H}(x,y) = H(x\tau_0,y,g(x,y))$. It follows from the arguments in the proof of [5, Theorem 1.1] that for each $(x,y) \in C\cap\mathscr{V}$, the limits $z_1(x,y)$ and $z_2(x,y)$ exist, $Rg(\theta,(x,y))$ exists for each $\theta \in [\alpha(x,y),\beta(x,y)]$ and $Rg(\cdot,(x,y)) \in C^0([\alpha(x,y),\beta(x,y)])$. In addition, it follows from the proof of [5, Theorem 1.1] that $g \in C^0(U\cup\{(x,y)\})$ when $z_1(x,y) = z_2(x,y)$.

Fix $(x,y) \in C\cap\mathscr{V}$ and set $P_0 = (x\tau_0,y)$. Consider $\omega \in \overline{T_{P_0}}$. Notice that either (i) ω is tangent to $\mathscr{C}_{(x,y)} = \{(x\tau,y) : \tau \in S^{n-2}\}$ and $\omega_1 = \omega_n = 0$ or (ii) $(\omega_1,\omega_n) = \sqrt{\omega_1^2+\omega_n^2}(\cos\theta,\sin\theta) \ne (0,0)$ for some $\theta \in [\alpha(x,y), \beta(x,y)]$. Notice that in case (i), $\omega \notin T^o_{P_0}$ and in case (ii), $\omega \in T^o_{P_0}$. Using (19) and arguing as in the claim at the end of the previous proof, we see in case (ii) that

$$Rf(\omega, P_0) = Rg(\theta, (x,y)).$$

Remark 1. *When, for example, $\theta = \alpha(x,y)$ and $\omega = (t\cos\alpha(x,y), \omega'', t\sin\alpha(x,y))$ with $\omega'' = (\omega_2, \ldots, \omega_{n-1})$, $|\omega''| < 1$ and $t = \sqrt{1-|\omega''|^2} > 0$, we mean by the symbols Rg and Rf the limits*

$$Rg(\alpha(x,y),(x,y)) = \lim_{\partial^- U(x,y) \ni (w,v) \to (x,y)} g^*(w,v) \tag{27}$$

and

$$Rf(\omega, P_0) = \lim_{\partial^- \Omega(x\tau_0, y) \ni (\mathbf{w}, v) \to P_0} f^*(\mathbf{w}, v),$$

where g^* is the trace of g on ∂U and f^* is the trace of f on $\partial\Omega$. (When, for example, $\{(x + r\cos\alpha(x,y), y + r\sin\alpha(x,y)) : 0 < r < \delta\}$ is contained in U for some small $\delta > 0$ and the two limits (5) and (27) for $Rg(\alpha(x,y),(x,y))$ both exist, they agree. One feature of [5, Theorem 1] *is that (5) and (27) both exist and agree.)*

In case (i) with $\omega_1 = \omega_n = 0$, $(x\tau_0, y) + r\omega$ may not be in Ω for any $r > 0$ and we interpret Rf as

$$Rf(\omega, P_0) = \lim_{\mathscr{C}_{(x,y)} \ni (\mathbf{w}, v) \to P_0} f^*(\mathbf{w}, v). \tag{28}$$

When case (b) of Theorem 2 holds, $\lim_{\partial^- \Omega(x\tau_0,y) \ni (\mathbf{w},v) \to P_0} f^*(\mathbf{w},v) = z_-(x,y)$, $\lim_{\partial^+ \Omega(x\tau_0,y) \ni (\mathbf{w},v) \to P_0} f^*(\mathbf{w},v) = z_+(x,y)$ and (28) will not exist. When case (a) holds and g is continuous at (x,y), this together with (19) implies

$$\lim_{\Omega \ni (\mathbf{w},v) \to P_0} f(\mathbf{w},v) = g(x,y), \quad \lim_{\partial\Omega \ni (\mathbf{w},v) \to P_0} f^*(\mathbf{w},v) = g(x,y)$$

and f is continuous at P_0. The conclusions of Theorem 2 then follow using (19) as in the proof of Theorem 1. $\square$

Proof of Example 1: Notice first that $\phi = 0$ almost everywhere on $\partial B_1(\mathscr{O}_3)$, $f \in C^0(\overline{\Omega} \setminus \mathscr{T})$ and $f = 0$ on $\partial B_1(\mathscr{O}_3)$, where $\mathscr{T} = \{(r\cos\theta, r\sin\theta, y) \in \mathbb{R}^3 : 0 \le \theta < 2\pi, (r-a)^2 + y^2 = b^2\}$.

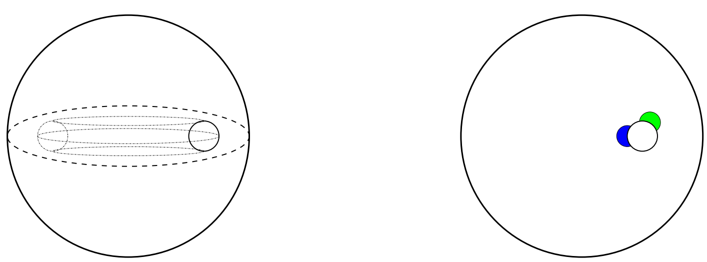

FIGURE 2. Ω (left) W in blue, $y_0 = 0$; W in green, $y_0 > 0$. (right)

Set $C = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + y^2 = b^2\}$ and fix $(x_0, y_0) \in C$. Since our interest is local (near (x_0,y_0)), let us set $U = \{(x,y) : x^2 + y^2 < 1,\ x > \frac{1}{2}(a-b),\ |y| < 2b\}$. Let $W = W(\delta) = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + y^2 > b^2, (x-x_0)^2 + (y-y_0)^2 < \delta^2\}$, where $0 < \delta < \min\{b, 1-a-b\}$ (see Figure 2). If

$y_0 \neq 0$, we may assume that δ is small enough that $y \neq 0$ for all $(x,y) \in \overline{W(\delta)}$ and so ϕ is continuous at $(x_0\tau_0, y_0)$. Notice that (16) holds because $\kappa = -\frac{1}{b}$ and $x_0 \geq a-b$.

Consider first $y_0 = 0$. Since $\phi(\mathbf{x},y) = \phi(\mathbf{x},-y)$ for $(\mathbf{x},y) \in \partial\Omega \setminus Z_0$, f has this same symmetry; $f(\mathbf{x},y) = f(\mathbf{x},-y)$ for $(\mathbf{x},y) \in \Omega$. Thus $\phi^*(x,y)$ is an even function of y, $z_-(x_0,0) = z_+(x_0,0)$, case (a) of Theorem 2 holds and $f \in C^0(\Omega \cup \{(x_0\cos\theta, x_0\sin\theta, 0)\})$ for each $\theta \in [0,2\pi)$.

Consider second $y_0 \neq 0$. From Theorem 2 and, if necessary, by choosing $\delta > 0$ smaller (so that $W(\delta) \subset \mathscr{V}$), we see that for each $(x,y) \in W(\delta) \cap C$, the radial limits $Rg(\theta,(x,y))$ exist for each $\theta \in [\alpha(x,y), \beta(x,y)]$ and either case (a) holds or case (b) holds. If case (b) holds, then we can modify the argument in the proof of [5, Corollary 1.2] and obtain a contradiction. Set $z_1 = Rg(\alpha(x_0,y_0),(x_0,y_0))$, $z_2 = Rg(\beta(x_0,y_0),(x_0,y_0))$ and $z_3 = \phi(x_0,y_0)$. Since we assume case (b) holds, we have $z_1 \neq z_2$; we may assume that $z_1 < z_3$ and $z_1 < z_2$ and we may assume $\delta > 0$ is small enough that $\phi(x\tau_0,y) > (z_1+z_3)/2$ for $(x,y) \in \partial W(\delta) \cap C$. Then there exist $\alpha_1, \alpha_2 \in [\alpha(x_0,y_0), \beta(x_0,y_0)]$ with $\alpha_1 < \alpha_2$ such that

$$Rg(\theta,(x_0,y_0)) \text{ is } \begin{cases} \text{constant}(=z_1) & \text{for } \alpha(x_0,y_0) \leq \theta \leq \alpha_1 \\ \text{strictly increasing} & \text{for } \alpha_1 \leq \theta \leq \alpha_2 \\ \text{constant}(=z_2) & \text{for } \alpha_2 \leq \theta \leq \beta(x_0,y_0) \end{cases}$$

and, for $(x,y) \in C \cap W(\delta)$, $\tau \in S^1$ and $\omega \in T^o_{(x\tau,y)}$, $Rf(\omega,P) = Rf(L_\tau(\omega),P_0) = Rg(\theta,(x,y))$, where $P = (x\tau,y)$ $P_0 = (x\tau_0,y)$ and $L_\tau(\omega)$ and θ satisfy (14).

Let us adopt the terminology and arguments in the proof of [5, Corollary 1.2]. Let $z_0 \in (z_1, \min\{z_2,z_3\})$ with $z_0 < (z_1+z_3)/2$ and let $\theta_0 \in (\alpha(x_0,y_0), \beta(x_0,y_0))$ such that $Rg(\theta_0,(x_0,y_0)) = z_0$. Let $\theta_b \in (\theta_0, \beta(x_0,y_0))$ satisfy $z_0 < Rg(\theta_b,(x_0,y_0)) < (z_1+z_3)/2$. Set $T = \{(x_0 + r\cos\theta_b, y_0 + r\sin\theta_b) : r \in \mathbb{R}\}$. For each $R > 0$, let $C(R)$ be the circle of radius R which passes through (x_0,y_0), is tangent at (x_0,y_0) to the line T and intersects $\partial^- U(x_0,y_0)$ and let $V(R)$ be the open disk inside $C(R)$. Consider the torus $\mathscr{T} = \mathscr{T}(R) = \{(x\tau,y) \in \mathbb{R}^3 : (x,y) \in C(R), \tau \in S^1\}$ and let $\mathscr{S}(R) = \{(x\tau,y) \in \mathbb{R}^3 : (x,y) \in V(R), \tau \in S^1\}$ represent the solid torus. Choose $R > 0$ small enough that $V(R) \cap U \subset W(\delta)$ and that twice the minimum mean curvature of the torus $\mathscr{T}$ is greater than $3H$ (this is similar to the requirement that $2R|H(\mathbf{x})| \leq 1$ for all $\mathbf{x} \in B_{2R}(\mathbf{y})$ in the proof of [5, Corollary 1.2] and implies that the Dirichlet problem (1)-(2) is solvable (in $\mathscr{S}(R)$) for all continuous Dirichlet data on $\mathscr{T}(R)$). (See Figure 3 with $(x_0,y_0) = (a,b)$, $\alpha(a,b) = 0$, $\beta(a,b) = \pi$, E_0 (blue region), $C(R)$ (green), U (yellow & blue regions), $z_2 > z_3$, $z_a = \frac{1}{2}(z_3+z_1)$, and the various values of z labeled by their subscripts (e.g. z_0 is labeled by 0).)

Notice that $C(R) \cap \partial U = \{(x_0,y_0), (x_p,y_p)\}$ for some $(x_p,y_p) \in \partial^- U(x_0,y_0)$. Let $\psi \in C^2(\mathscr{T}(R))$ satisfy $\psi(x_0\tau, y_0) = z_0$ for $\tau \in S^1$, $\psi(\mathbf{x},y) = \psi(|\mathbf{x}|\tau_0, y)$ for $(\mathbf{x},y) \in \mathscr{T}(R)$, $\psi < f$ on $\mathscr{T}(R) \cap \Omega$ (recall $Rg(\theta_b,(x_0,y_0)) > Rg(\theta_0,(x_0,y_0)) = z_0$), $\sup_{\mathscr{T}(R)} \psi = z_0$ and $\psi(x_p\tau_0, y_p) < \liminf_{U \ni (x,y) \to (x_p,y_p)} g(x,y)$. Let $h \in C^2\left(\overline{\mathscr{S}(R)}\right)$ satisfy $N_3 h = 3H$ in $\mathscr{S}(R)$ and $h = \psi$ on $\mathscr{T}(R)$. Since $N_3(h) = 3H \geq 0$, $h \leq \sup_{\mathscr{T}(R)} \psi = z_0$. Since $\phi(x\tau_0,y) > (z_1+z_3)/2$ for $(x,y) \in \partial W(\delta) \cap C$, $h \leq z_0 < \phi$ on $\mathscr{S}(R) \cap \partial\Omega$.

Since h is a classical solution of the Dirichlet problem $Qh = 0$ in $\mathscr{S}(R)$ and $h = \psi$ on $\partial\mathscr{S}(R)$), where $Qk = N_3 k - 3H$, it is also the variational (i.e. $BV(\mathscr{S}(R))$) solution and minimizes

$$I(k) = \int_{\mathscr{S}(R)} \sqrt{1+|Dk|^2} + \int\int_{\mathscr{S}(R)} \int_0^{k(\mathbf{x},y)} 3H \, ds \, d\mathbf{x}dy + \int_{\mathscr{T}(R)} |k-\psi| dH_2$$

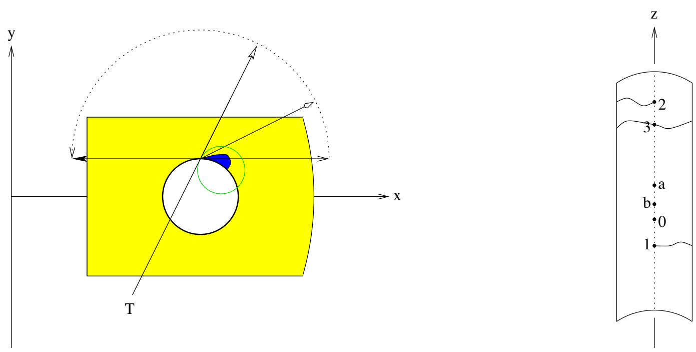

FIGURE 3. U, E_0 in blue (left) $z_1 < z_0 < z_b < z_a < z_3 < z_2$ (right)

over $BV(\mathscr{S}(R))$. Set $E = \{(\mathbf{x}, y) \in \Omega \cap \mathscr{S}(R) : h(\mathbf{x}, y) > f(\mathbf{x}, y)\}$ and $E_0 = \{(x, y) \in \mathbb{R}^2 : (x\tau_0, y) \in E, x > 0\}$.

Define $k \in C^0(\Omega) \cap BV(\Omega)$ by $k = f + (h - f)\chi_E$, where χ_E is the characteristic function of E. Since f minimizes J over $BV(\Omega)$, we have $J(f) \le J(k)$ or

$$\int_E \sqrt{1 + |Df|^2} + \int\int_E 3Hf \, d\mathbf{x}dy + \int_{\partial E \cap \partial \Omega} |f - \phi| dH_2$$

$$\le \int_E \sqrt{1 + |Dh|^2} + \int\int_E 3Hh \, d\mathbf{x}dy + \int_{\partial E \cap \partial \Omega} |h - \phi| dH_2.$$

Now define $l \in C^0(\mathscr{S}(R)) \cap BV(\mathscr{S}(R))$ by $l = h + (f - h)\chi_E$. Since h minimizes I over $BV(\mathscr{S}(R))$, we have $I(h) \le I(l)$ or

$$\int_E \sqrt{1 + |Dh|^2} + \int\int_E 3Hh \, d\mathbf{x}dy \le \int_E \sqrt{1 + |Df|^2} + \int\int_E 3Hf \, d\mathbf{x}dy + \int_{\partial E \cap \partial \Omega} |f - h| dH_2.$$

Combining these and using the facts that $|f - \phi| = \phi - f$, $|h - \phi| = \phi - h$, $|f - h| = h - f$ and $|f - h| + |h - \phi| = \phi - f = |f - \phi|$ on $\partial^- \Omega(a\tau_0, b)$, we see that

$$\int_E \sqrt{1 + |Df|^2} + \int\int_E 3Hf \, d\mathbf{x}dy + \int_{\partial E \cap \partial \Omega} |f - \phi| dH_2$$

$$= \int_E \sqrt{1 + |Dh|^2} + \int\int_E 3Hh \, d\mathbf{x}dy + \int_{\partial E \cap \partial \Omega} |h - \phi| dH_2$$

or $J(k) = J(f)$; that is, k also minimizes J and so $k = f$ on Ω; then $h = f$ on E. This is a contradiction and so case (b) cannot hold, $z_1 = z_2$ and case (a) holds. Therefore $f \in C^0(\overline{\Omega})$. □

5. Statements and Declarations

Neither author received funding for this project nor has any relevant financial or non-financial competing interest to report. The figures were produced by the authors using the open source program "xfig" running on open source linux systems.

References

1. S. N. Bernstein, *Sur les équations du calcul des variations*, Ann. Sci. Ec. Norm Super. **29** (1912), 431-485.
2. T. Bourni, $C^{1,\alpha}$ *Theory for the Prescribed Mean Curvature Equation with Dirichlet Data*, J. Geom. Anal. **21** (2011), 982–1035.
3. H. Deng, H. Liu and L. Tian, *Uniqueness of critical points of solutions to the mean curvature equation with Neumann and Robin boundary conditions*, J. Math. Anal. Appl. **477**, No. 2 (2019), 1072–1086.
4. A. Elcrat and K. E. Lancaster, *Boundary behavior of a nonparametric surface of prescribed mean curvature near a reentrant corner*, Trans. Amer. Math. Soc. **297** (1986), 645–650.
5. M. Entekhabi and K. E. Lancaster, *Boundary Continuity of Nonparametric Prescribed Mean Curvature Surfaces*, Taiwanese J. Math., **Vol. 24**, No. 2 (2020), 483–499.
6. M. Entekhabi and K. E. Lancaster, *Radial Limits of Nonparametric PMC Surfaces with Intermediate Boundary Curvature*, Taiwanese J. Math., **Vol. 25**, No. 3 (2021), 599–613.
7. C. Gerhardt, *Existence, Regularity, and Boundary Behaviour of Generalized Surfaces of Prescribed Mean Curvature*, Math. Z. **139** (1974), 173–198.
8. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer-Verlag, 1983.
9. E. Giusti, *Boundary Value Problems for non-parametric surfaces of prescribed mean curvature*, Ann. Sc. Norm. Super. Pisa. Cl. Sci. (4) **3** (1976), 501–548.
10. E. Giusti, *On the Equation of Surfaces of Prescribed Mean Curvature*, Invent. Math. **46** (1978), 111–138.
11. H. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math. **229** (1968), 170–187 https://doi.org/10.1515/crll.1968.229.170
12. A. Khanfer and K. Lancaster, *On a Minimal Hypersurface in* $\mathbb{R}^4$, J. Geom. Anal. **30**, No. 2 (2020), 2241–2252.
13. A. Khanfer and K. Lancaster, *Boundary Behavior of Rotationally Symmetric Prescribed Mean Curvature Hypersurfaces in* $\mathbb{R}^4$, J. Geom. Anal. **31**, No. 5 (2021), 4806–4815.
14. A. Khanfer and K. Lancaster, *Behavior of Prescribed Mean Curvature Hypersurfaces on Reentrant Ridges*, J Geom Anal 32, 268 (2022). https://doi.org/10.1007/s12220-022-01016-9
15. K. E. Lancaster, *Boundary behavior of a non-parametric minimal surface in* $\mathbb{R}^3$ *at a non-convex point*, Analysis (Berlin) **5** (1985), 61–69. Corrigendum: **6** (1986), 413.
16. K. E. Lancaster and D. Siegel, *Existence and Behavior of the Radial Limits of a Bounded Capillary Surface at a Corner*, Pacific J. Math. **Vol. 176**, No. 1 (1996), 165–194. *Correction to "Existence and Behavior of the Radial Limits of a Bounded Capillary Surface at a Corner"*, **Vol. 179**, No. 2 (1997), 397–402.
17. F H Lin, *Behaviour of nonparametric solutions and free boundary regularity*, Miniconference on Geometry and Partial Differential Equations, 2, Canberra, 1986, Proc. Centre Math. Appl. Austral. Nat. Univ. **12** (1987), pp. 96–116.
18. M. Miranda, *Dirichlet problem with* L^1 *data for the non-homogeneous minimal surface equation*, Indiana Univ. Math. J. **24**, No. 3 (1974), 227–241.
19. J. C. C. Nitsche, *Lectures on Minimal Surfaces: Volume 1*, Cambridge University Press, 1989.
20. J. Serrin, *The problem of Dirichlet for quasilinear equations with many independent variables*, Philos. Trans. Roy. Soc. A **264** (1969), 413–496.
21. L. Simon, *Boundary regularity for solutions of the non-parametric least area problem*, Ann. of Math. **103** (1976), 429–455.
22. B. Taylor, *Part of a Letter from Mr. Brook Taylor, F. R. S. to Dr. Hans Sloane R. S. Secr. Concerning the Ascent of Water between two Glass Planes*, Proc. Roy. Soc. London (1710).
23. J. Taylor, *Boundary regularity for solutions to various capillarity and free boundary problems*, Comm. Partial Differential Equations **2** (1977), 323–357.

DEPARTMENT OF MATHEMATICS, FLORIDA A & M UNIVERSITY, TALLAHASSEE, FL 32307
Email address: mozhgan.entekhabi@famu.edu

WICHITA KS 67226
Email address: kirk.e.lancaster@gmail.com